Precalculus: Geometry and Functions I

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Contents

Chapter 1

Ideas & Motivations

Welcome to Geometry & Functions 1 by me (Gudfit). The point of these notes is to cover everything I think is important as I build up to my current knowledge, while keeping it free and accessible for everyone from kids to adults.

I aim for each set of notes to be max 100 pages [1], as rigorous as possible, and far-reaching too. That means I'll cover the axioms and proofs of the most interesting stuff, plus I'll pull in other subjects we've already touched on to show how math builds on itself like funky Lego. These notes build on my existing informal logic and algebra I notes, and they're aimed at keeping the proofs, ideas, and build-up of geometry as informal as possible.

It'll be a mix of quick ideas and concepts, but in the appendix for each section, I'll go rigorous with the key axioms pulled from a bunch of books. For those theorems and ideas in the appendix, everything will be proved without algebra (since that's coming in the next book).

The original idea was a fully rigorous intro like Euclid's Elements, but that felt too grindy. Why slog through it when you can just read other people's notes, papers, or books? So this'll be more efficient, not totally deductive, assuming you've got some mathematical rigor. Either way, let's dive in and enjoy!

[1] I checked its less than 70 pages without diagrams

Chapter 2

The Building Blocks of Geometry

The geometry presented in this course deals mainly with figures such as points, lines, triangles, and circles, which we will study in a logical way. We begin by stating some basic properties. For the moment, we will be working with figures which lie in a plane. You can think of a plane as a flat surface which extends infinitely in all directions.

2.1 Lines, Segments, and Rays

We start with the most fundamental axiom of plane geometry.

***Axiom* 2.1.1.** Given two distinct points P and Q in the plane, there is one and only one line which goes through these points.

We denote this line by L_{PQ}. The line extends infinitely in both directions, as illustrated in Figure 2.1.

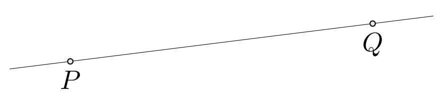


Figure 2.1: Line passing through points P and Q.

Definition 2.1.1. ***Line Segment***. The line segment, or segment, between points P and Q is the set of all points on the line L_{PQ} lying between P and Q, inclusive of P and Q. We denote this segment by $\overline{PQ}$.

Definition 2.1.2. ***Ray***. A ray is a part of a line that starts at a point and extends infinitely in one direction. The ray starting from P and passing through Q is denoted by $\boldsymbol{PQ}$. The starting point P is called the vertex.

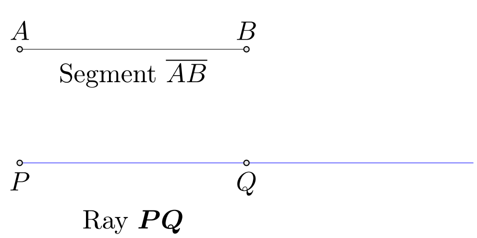


Figure 2.2: A line segment and a ray.

Distance The notion of distance is fundamental. We denote the distance between points P and Q by $d(P, Q)$ or sometimes $|PQ|$. It is a number satisfying the following properties, which we accept as axioms.

***Axiom* 2.1.2.** For any points P, Q, we have $d(P,Q) \geq 0$. Furthermore, $d(P,Q) = 0$ if and only if P = Q.

***Axiom* 2.1.3.** For any points P, Q we have $d(P,Q) = d(Q,P)$.

***Axiom* 2.1.4. *(Triangle Inequality)*.** Let P, Q, M be points. Then $d(P,M) \leq d(P,Q) + d(Q,M)$.

This final property illustrates that the length of one side of a triangle is never greater than the sum of the lengths of the other two sides.

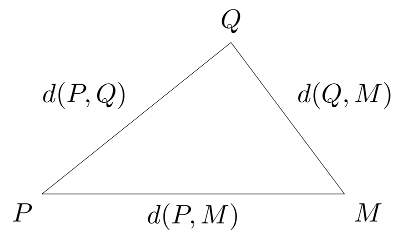


Figure 2.3: Illustration of the Triangle Inequality.

***Axiom* 2.1.5.** Let P, Q, M be points. We have $d(P,Q) + d(Q,M) = d(P,M)$ if and only if Q lies on the segment $\overline{PM}$.

Parallelism

Definition 2.1.3. *Parallel Lines*. Two lines K and L are parallel if either K = L, or K ≠ L and they do not intersect. We denote this by $K \parallel L$.

We accept the following properties of parallel lines as axioms.

***Axiom* 2.1.6.** Two lines in a plane which are not parallel meet in exactly one point.

***Axiom* 2.1.7.** Given a line L and a point P, there is one and only one line passing through P, parallel to L.

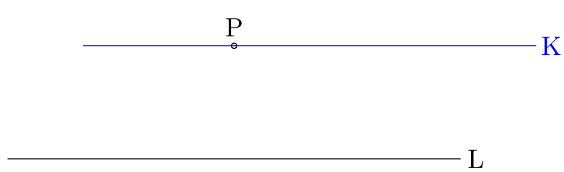


Figure 2.4: Line K passing through P is parallel to line L.

***Axiom* 2.1.8.** Let L_1, L_2, L_3 be three lines. If $L_1 \parallel L_2$ and $L_2 \parallel L_3$, then $L_1 \parallel L_3$.

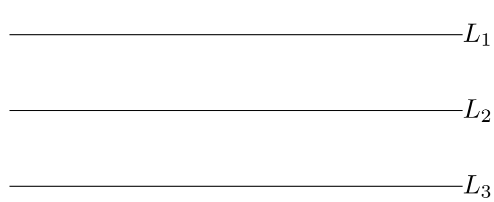


Figure 2.5: Transitivity of parallel lines.

Basic Figures

Definition 2.1.4. *Collinear Points*. Points are collinear if they lie on the same line.

Definition 2.1.5. *Triangle*. Let P, Q, and M be three non-collinear points. The triangle $\triangle PQM$ is the set consisting of the three line segments $\overline{PQ}$, $\overline{QM}$, and $\overline{PM}$.

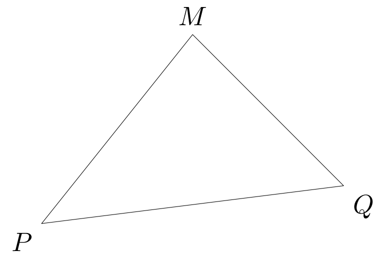


Figure 2.6: A triangle $\triangle PQM$.

2.1.1 Triangle Congruence Postulates

To compare geometric figures, we use the concept of congruence. Two figures are congruent if they have the same shape and size. For triangles, we accept the following fundamental postulates.

***Axiom* 2.1.9. *Side-Side-Side (SSS)*.** If the three sides of one triangle are equal in length to the three corresponding sides of another triangle, then the two triangles are congruent. We prove this later; see Theorem 6.9.5.

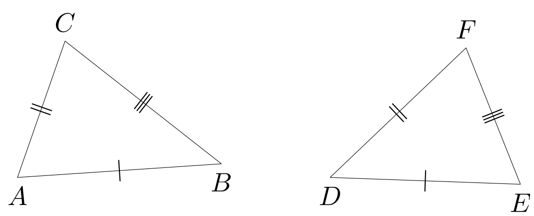


Figure 2.7: SSS: three corresponding sides equal.

Definition 2.1.6. ***Isosceles Triangle***. A triangle is isosceles if two of its sides have the same length.

Definition 2.1.7. ***Equilateral Triangle***. A triangle is equilateral if all three of its sides have the same length.

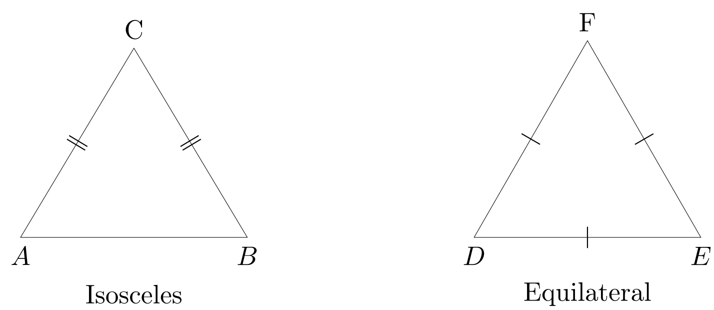


Figure 2.8: An isosceles and an equilateral triangle.

Definition 2.1.8. ***Circle and Disc***. Let r be a positive number and P be a point. The circle of centre P and radius r is the set of all points Q such that $d(P,Q) = r$. The disc of centre P and radius r is the set of all points Q such that $d(P,Q) \leq r$.

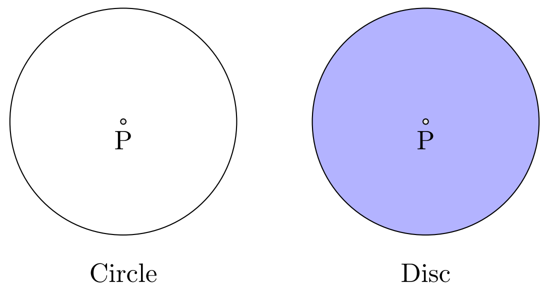


Figure 2.9: A circle and a disc with centre P.

2.2 Exercises

Part I: Fundamental Concepts and Definitions

1. For each of the following, state whether the object is a line, a line segment, or a ray. Draw a representation of each.

(a) The set of points between P and Q, inclusive.

(b) All points on the line L_{PQ}.

(c) The set of points on L_{PQ} starting at P and containing Q.

2. Let P, Q, and R be three distinct non-collinear points.

(a) How many distinct lines can be drawn that pass through at least two of these points? Name them.

(b) How many distinct line segments are determined by these three points? Name them.

3. ⋆ Let $n \geq 2$ be an integer. If you are given n points in a plane, no three of which are collinear, how many distinct lines are determined by these points? Justify your answer.

4. Explain the difference between the following pairs of notations. When are they the same object?

(a) $\overline{PQ}$ and $\overline{QP}$

(b) $\boldsymbol{PQ}$ and $\boldsymbol{QP}$

(c) L_{PQ} and L_{QP}

5. Let points A, B, C be on a line. If $d(A, B) = 7$ and $d(B, C) = 4$, what are the possible values for $d(A, C)$? Draw a diagram for each possibility and justify your answer using the axioms.

6. If $d(X, Y) = 0$, what can you conclude about the points X and Y? Which axiom supports your conclusion?

7. Three points P, Q, M satisfy $d(P, Q) = 5$, $d(Q, M) = 8$, and $d(P, M) = 13$. Are the points collinear? If so, which point lies between the other two? Justify your answer.

8. Three points A, B, C satisfy $d(A, B) = 6$, $d(B, C) = 9$, and $d(A, C) = 12$. Do these points form a triangle? Why or why not? Which axiom is central to your reasoning?

9. A point M is called a **midpoint** of the segment $\overline{AB}$ if M is on $\overline{AB}$ and $d(A, M) = d(M, B)$.

(a) If $d(A, B) = 10$, and M is the midpoint of $\overline{AB}$, find $d(A, M)$.

(b) ⋆ Using the distance axioms, prove that a line segment can have only one midpoint.

Remark. Assume there are two distinct midpoints, M_1 and M_2, and show this leads to a contradiction.

10. Let P be a point and $r > 0$ be a number. Is the circle of centre P and radius r a finite or infinite set of points? What about the disc? Explain your reasoning.

Part II: The Triangle Inequality

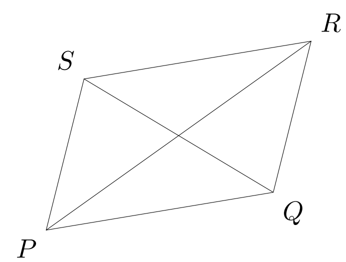
11. Two sides of a triangle have lengths 8 cm and 13 cm. What are the possible values for the length of the third side? Express your answer as an inequality.

12. Let A, B, C, D be four distinct points in a plane. Prove that $d(A, D) \leq d(A, B) + d(B, C) + d(C, D)$.

13. ⋆ **The Reverse Triangle Inequality.** Prove for any three points P, Q, M that $d(P, M) \geq |d(P, Q) - d(Q, M)|$.

Remark. Apply the triangle inequality to the triangle with vertices P, M, Q in three different ways.

14. A quadrilateral is a figure with four vertices P, Q, R, S and four sides $\overline{PQ}, \overline{QR}, \overline{RS}, \overline{SP}$. The segments $\overline{PR}$ and $\overline{QS}$ are its diagonals.



(a) Prove that the length of any diagonal is less than the sum of the lengths of any two adjacent sides. (For instance, show $d(P, R) < d(P, Q) + d(Q, R)$).

(b) Prove that the sum of the lengths of the diagonals is less than the perimeter of the quadrilateral (the sum of the lengths of its four sides).

15. Let $\triangle PQM$ be a triangle. Let X be a point on the segment $\overline{PM}$. Prove that $d(Q, X) < d(Q, P) + d(P, X)$.

16. Can a straight line segment intersect a triangle at exactly three distinct points? What about four? Justify your answers with diagrams and reasoning based on the axioms.

17. A traveller needs to go from town A to town C. Town B is not on the direct road between A and C. Using an axiom, explain why taking the detour through B ($A \to B \to C$) is always a longer journey than going directly from A to C.

18. Let P, Q, M be three points. Under what conditions could we have $d(P, M) > d(P, Q) + d(Q, M)$? Explain with reference to the axioms.

Part III: Parallel Lines

19. Let L_1, L_2, L_3, L_4 be four distinct lines. If $L_1 \parallel L_2$, $L_2 \parallel L_3$, and $L_3 \parallel L_4$, is it true that $L_1 \parallel L_4$? Which axiom supports this?

20. Let L and K be two distinct parallel lines. Let M be a line that is not parallel to L. Can M be parallel to K? Justify your reasoning using the axioms.

21. ⋆ Let L_1 and L_2 be two distinct parallel lines. Let M be a line that intersects L_1 at a point P. Prove that M must also intersect L_2.

Remark. Assume M does not intersect L_2. What does this imply about M and L_2? Use the parallel axiom (2.1.7) to find a contradiction.

22. A **parallelogram** is a quadrilateral where opposite sides lie on parallel lines. That is, for parallelogram PQRS, $L_{PQ} \parallel L_{RS}$ and $L_{QR} \parallel L_{SP}$. If you have three non-collinear points P, Q, R, how many distinct points S exist such that PQRS forms a parallelogram? Draw a diagram to illustrate the possibilities.

23. Is it possible for two distinct line segments to be parallel? What about two rays? Discuss what "parallel segments" or "parallel rays" might mean, based on the definition of parallel lines.

24. Consider the statement: "If two distinct lines L and M both intersect a third line K, then L and M must intersect each other." Is this statement true or false? Provide a diagram and an argument based on the axioms.

25. Could a triangle have two sides that are parallel? Explain your answer based on the definition of a triangle and parallel lines.

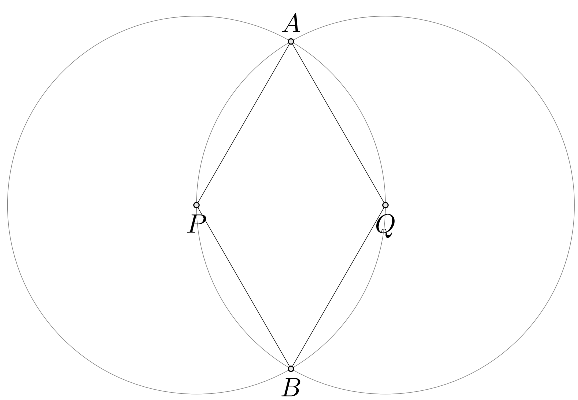
Part IV: Triangles, Circles, and Congruence

26. Let $\triangle ABC$ and $\triangle DEF$ be two triangles. If $d(A,B) = d(D,E) = 5$, $d(B,C) = d(E,F) = 7$, and $d(A,C) = d(D,F) = 9$, are the triangles congruent? Which postulate do you use?

27. Let $\triangle PQR$ be an equilateral triangle with a side length of 6 cm. Let $\triangle XYZ$ be a triangle with $d(X,Y) = d(Y,Z) = d(X,Z) = 6$ cm. Are the triangles congruent?

28. A **rhombus** is a quadrilateral with all four sides of equal length. Let PQRS be a rhombus. Consider the diagonal $\overline{PR}$. Prove that $\triangle PQR \cong \triangle RSP$.

29. Two circles, C_1 with centre P and radius r, and C_2 with centre Q and radius s, intersect at two distinct points A and B.



(a) What can you say about the lengths $d(P,A)$ and $d(P,B)$? What about $d(Q,A)$ and $d(Q,B)$?

(b) Prove that $\triangle PAQ \cong \triangle PBQ$.

30. Let $\triangle ABC$ be an isosceles triangle with $d(A,C) = d(B,C)$. Let M be the midpoint of the segment $\overline{AB}$. Prove that $\triangle AMC \cong \triangle BMC$.

31. Let C be a circle with centre P and radius r. Let A and B be two distinct points on the circle.

(a) Using the Triangle Inequality on $\triangle PAB$, prove that the distance between A and B is never greater than $2r$.

(b) According to the distance axioms, under what precise condition would $d(A,B) = 2r$?

32. Using only a straightedge (to draw lines) and a compass (to draw circles and copy distances), explain the steps to construct a triangle congruent to a given $\triangle ABC$. Which postulate guarantees your construction is correct?

33. Is it possible for an isosceles triangle to be congruent to an equilateral triangle? If so, under what conditions?

34. Let $\overline{AB}$ be a line segment. Describe the set of all points P such that $\triangle PAB$ is an isosceles triangle with $d(P,A) = d(P,B)$. How is this set of points related to circles?

35. The Kite. A "Kite" is a quadrilateral $PQRS$ where two pairs of adjacent sides are equal in length. Specifically, $d(P,Q) = d(P,S)$ and $d(R,Q) = d(R,S)$.

(a) Draw a diagram of a kite.

(b) Prove that $\triangle PQR \cong \triangle PSR$ using the postulates provided in this chapter.

Part V: Proofs and Deeper Explorations

36. A set of points $\mathcal{S}$ is called **convex** if for any two points $P, Q \in \mathcal{S}$, the entire line segment $\overline{PQ}$ is contained in $\mathcal{S}$.

(a) Prove that a line is a convex set.

(b) Prove that a ray is a convex set.

Remark. Use the definition of a ray and the line segment.

(c) Is a circle a convex set? Justify your answer.

(d) Is a triangle (the set of three segments) a convex set? Provide a reasoned argument.

37. $\star$ **The Perpendicular Bisector as a Locus.** Let A and B be two distinct points. Consider the set $\mathcal{L}$ of all points P such that $d(P,A) = d(P,B)$. We have seen that any point on the line passing through C and M in Exercise 30 has this property. The goal is to prove the converse. Let P be any point such that $d(P,A) = d(P,B)$. Let M be the midpoint of $\overline{AB}$.

(a) What kind of triangle is $\triangle PAB$?

(b) Use SSS to show $\triangle PMA \cong \triangle PMB$.

(c) It is a property we will prove later that this congruence implies that L_{PM} is perpendicular to L_{AB}. For now, what can you conclude about the set $\mathcal{L}$? Describe it in words.

38. $\star$ Prove that it is impossible to have two distinct lines L_1, L_2 and two distinct points P, Q such that L_1 passes through P and Q, and L_2 also passes through P and Q. Which axiom is this a direct restatement of?

39. Let L be a line. The line L divides the plane into two regions called **half-planes**. Two points P and Q are in the same half-plane if the segment $\overline{PQ}$ does not intersect L. They are in opposite half-planes if $\overline{PQ}$ does intersect L.

(a) Let P, Q, R be three points. If P and Q are in the same half-plane, and Q and R are in the same half-plane, prove that P and R must be in the same half-plane.

Remark. Assume $\overline{PR}$ intersects L and show this leads to a contradiction inside $\triangle PQR$.

(b) Is a half-plane a convex set?

40. Let $\triangle ABC$ be a triangle. The **interior** of the triangle is the set of points P that lie in the same half-plane as C with respect to line L_{AB}, AND in the same half-plane as B with respect to L_{AC}, AND in the same half-plane as A with respect to L_{BC}. Prove that the interior of a triangle is a convex set.

41. The exterior of a circle with centre P and radius r is the set of points Q such that $d(P,Q) > r$. Is the exterior of a circle a convex set? Provide a proof or a counterexample.

42. Prove that any equilateral triangle is also an isosceles triangle. Is the converse true?

43. Let P, Q, R be three distinct collinear points. Is it possible to find a point M not on their line such that $\triangle PQM \cong \triangle RQM$? Justify your answer.

44. $\star$ Let L and K be two lines intersecting at a point P. Let A be a point on L (A $\neq$ P) and B be a point on K (B $\neq$ P). Let C_A be the circle with centre P passing through A, and C_B be the circle with centre P passing through B. If another line M passes through P, intersecting C_A at A' and C_B at B', prove that $\triangle APB \cong \triangle A'PB'$ is not necessarily true using only SSS. What additional condition would be needed to prove congruence with SSS?

45. Let L be a line and P be a point not on the line.

(a) Is it possible to draw a circle with centre P that does not intersect L at all?

(b) Is it possible to draw a circle with centre P that intersects L at exactly two points?

(c) ⋆ Can a circle intersect a line at three distinct points? Use the axioms regarding lines to justify your answer.

46. Prove that the shortest distance between any two points is a straight line.

Remark. This may seem obvious, but it is a rephrasing of one of the axioms. Which one, and why?

47. Let $\triangle ABC$ be a triangle and let M be the midpoint of the side $\overline{BC}$. Prove that the length of the median $\overline{AM}$ satisfies the inequality:

$$d(A, M) < d(A, B) + d(B, M)$$

Does this imply that the median is shorter than the sum of the other two sides of the triangle?

48. ⋆ Let C_1 and C_2 be two non-intersecting discs. Let P be the centre of C_1 and Q be the centre of C_2. Prove that the distance between any point in C_1 and any point in C_2 is always greater than $d(P, Q) - r_1 - r_2$, where r_1 and r_2 are the radii.

49. Let's reconsider the parallel axiom. Imagine a geometry where, given a line L and a point P not on L, there are *no* lines through P parallel to L (this is Spherical Geometry).

(a) In such a world, what is the relationship between any two distinct lines?

(b) Does the axiom "Two lines in a plane which are not parallel meet in exactly one point" still make sense?

50. Let's imagine another geometry where, given a line L and a point P not on L, there are *infinitely many* lines through P parallel to L (this is Hyperbolic Geometry).

(a) In this world, if $L_1 \parallel L_3$ and $L_2 \parallel L_3$, and L_1, L_2 both pass through P, can we conclude $L_1 = L_2$?

(b) Does the transitivity axiom for parallel lines hold in general in this geometry? Explain your reasoning.

2.3 Angles

***Axiom* 2.3.1. *(Construction Primitives)*.** With straightedge and compass we may: (i) draw the unique line through two points (2.1.1); (ii) draw a circle with given centre and radius; (iii) mark intersections of lines/circles; (iv) through a point on a line construct the unique perpendicular to the line (2.3.2); (v) through a point construct the unique line parallel to a given line (2.1.7); (vi) copy an angle: given an angle A and a ray r with the same vertex side specified, construct a ray making an angle congruent to A with r.

Consider two rays $\boldsymbol{PQ}$ and $\boldsymbol{PM}$ starting from the same point P. These rays separate the plane into two regions, as shown in Figure 2.10.

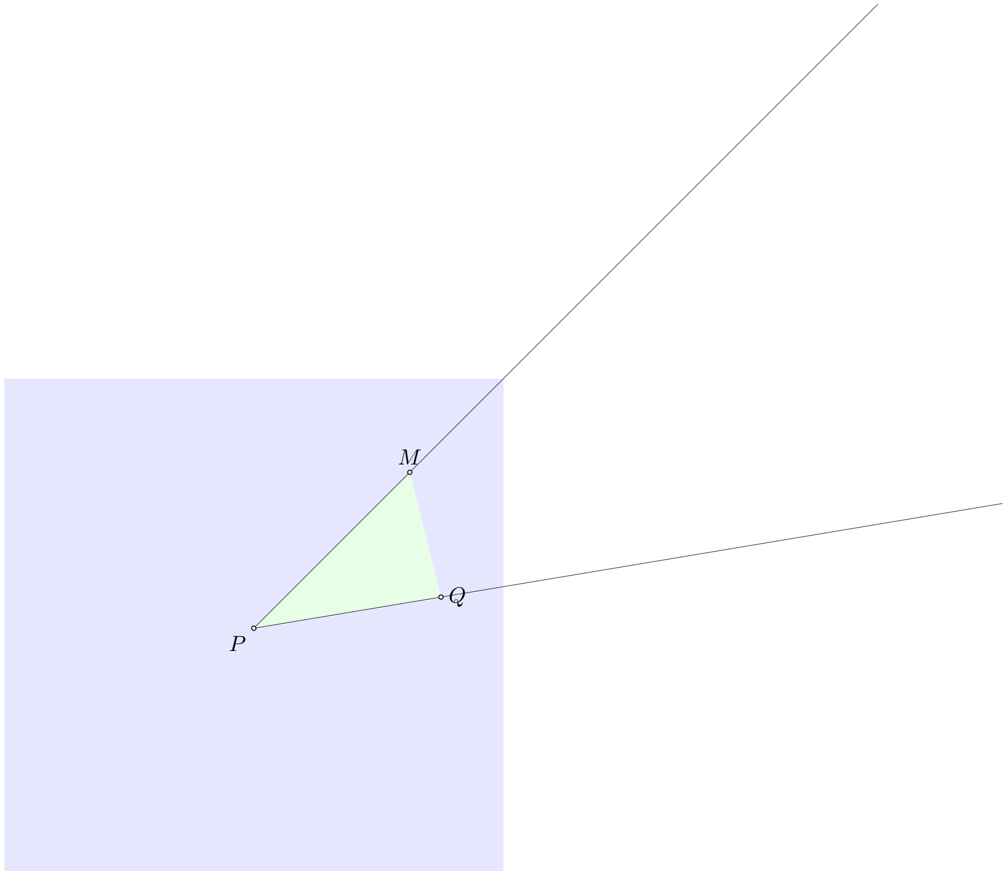


Figure 2.10: Two rays from a common vertex P divide the plane into two regions.

Definition 2.3.1. ***Angle***. An angle is a region of a plane bounded by two rays with a common vertex.

We use the notation $\angle QPM$ for an angle determined by the rays $\boldsymbol{PQ}$ and $\boldsymbol{PM}$. The context, or an arc as in Figure 2.11, specifies which of the two regions is meant.

Remark. On notation. We write $\boldsymbol{PQ}$ for the *ray* from P through Q, not to be confused with a free vector.

The order of the points in the notation often implies a direction, typically counter-clockwise, for measuring the angle.

Angle Measurement Just as we use numbers to measure distance, we use them to measure angles. The standard unit of measurement is the degree, such that a full rotation corresponds to 360 degrees (360°). Let A be an angle with vertex P, and let D be a disc centred at P. The part of the angle which lies in the disc is called a sector, S. The measure of A, denoted $m(A)$, is formally defined by the ratio of the area of the sector to the area of the disc.

$$m(A) \text{ in degrees} = 360 \times \frac{\text{area}(S)}{\text{area}(D)}$$

This definition relies on the properties that area is additive for non-overlapping regions and is invariant under congruence, ensuring the measure is well-defined.

Remark. On notation. The superscript circle ° is read "degrees." Thus 45° means forty-five degrees. When we write $\angle ABC = 45°$ we really mean $m(\angle ABC) = 45°$, but we often omit the explicit $m(\cdot)$ to lighten notation. One full turn is 360°, so $1° = \frac{1}{360}$ of a full turn.

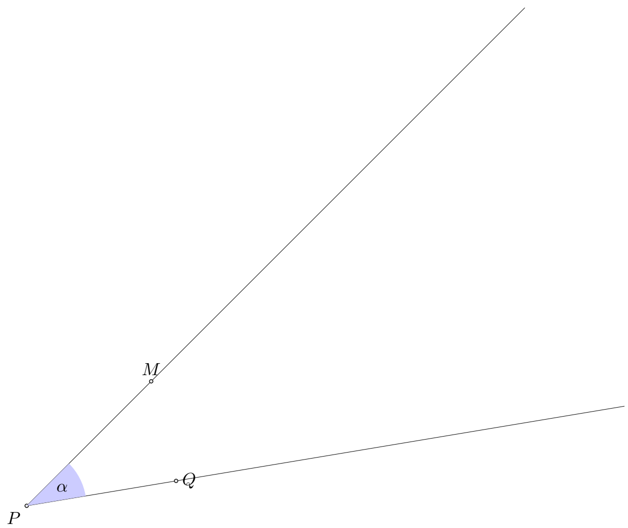


Figure 2.11: The angle $\angle QPM$.

Interlude: Why is area(**disc of radius** r) $= \pi r^2$**?** If we scale all lengths in the plane by a factor $s > 0$, then all areas scale by s^2. Hence the area $A(r)$ of a disc of radius r must satisfy $A(sr) = s^2 A(r)$ for every s. It follows that $A(r) = k\,r^2$ for some constant k independent of r. The constant k is exactly the area of the unit disc; by tradition we denote it by π. Therefore

$$\text{area(disc of radius } r) = \pi r^2.$$

Note. If you cut a circle into many equal sectors and alternate them tip-to-tail, you get an almost-rectangle whose height is r and whose base is about half the circumference. As the sectors get thinner, the base tends to $\frac{1}{2} \cdot 2\pi r = \pi r$, so the area tends to $r \cdot \pi r = \pi r^2$. (A fully rigorous proof can be given later via the method of exhaustion or calculus; but for now see Figure 2.12)

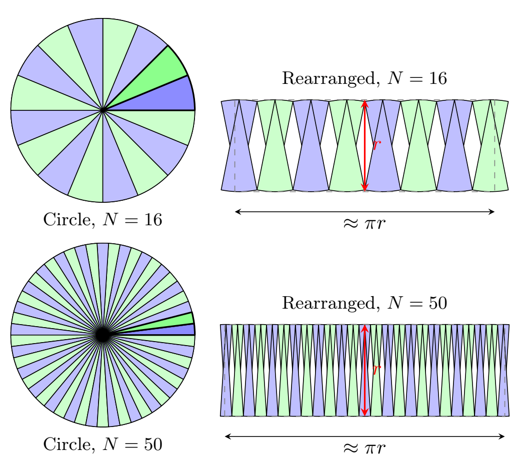


Figure 2.12: As N grows (bottom row vs. top), the rearranged shape approaches a true rectangle of base $\approx \pi r$ and height r, hence area πr^2.

Remark. For the inquisitive reader. In defining the number of degrees $m(A)$ of an angle A, we used a disc D centred at the vertex and took the ratio $\dfrac{\text{area}(S)}{\text{area}(D)}$, where S is the sector cut out by A in D. We did not specify the radius of D. It should be intuitively clear that if we change the disc (and the sector along with it) the quotient stays the same. We shall assume this independence for now, and return to it when we discuss area and similar figures in more detail.

***Proposition* 2.3.1.** *Area of a Disc.* Let D_r be the disc of radius $r > 0$. Then

$$\text{area}(D_r) = \pi\, r^2,$$

where $\pi := \text{area}(D_1)$ is the area of the unit disc.

Remark. Equivalently: scaling all lengths by r scales any area by r^2. We use 2.3.1 as the standing definition of π in what follows.

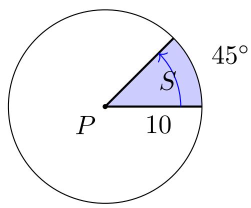
Example 2.3.1. (Calculating Sector Area). An angle measures 45°. If it is the angle of a sector within a disc of radius 10, what is the area of the sector?

The area of the disc is $\pi r^2 = \pi(10)^2 = 100\pi$. Using the formula for the measure of an angle:

$$45° = 360 \times \frac{\text{area}(S)}{100\pi}$$

Rearranging to solve for the area of the sector S:

$$\text{area}(S) = \frac{45}{360} \times 100\pi = \frac{1}{8} \times 100\pi = 12.5\pi$$



Comparing Angles (Inequalities) Let A and B be angles with measures $m(A) = x°$ and $m(B) = y°$, where $0° \le x \le 360°$ and $0° \le y \le 360°$. We write

$$A \le B \quad \text{iff} \quad x \le y, \qquad A < B \quad \text{iff} \quad x < y,$$

and similarly $A \ge B$ and $A > B$. For instance, an angle of 37° is smaller than an angle of 52°, so $37° < 52°$. These inequalities behave as expected: if $A \le B$ and C is any angle with non-negative measure, then $A + C \le B + C$; if $A \le B$, then their supplements satisfy $180° - A \ge 180° - B$.

Perpendicular Angle

Definition 2.3.2. ***Straight Angle.*** An angle whose rays form a straight line. It measures 180° (See Figure 2.13).

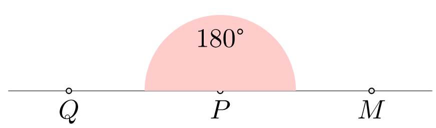


Figure 2.13: A straight angle.

Definition 2.3.3. ***Right Angle.*** An angle whose measure is half that of a straight angle. It measures 90°.

Definition 2.3.4. ***Perpendicular Lines.*** Two lines are perpendicular if their intersection forms a right angle. If line L_1 is perpendicular to line L_2, we write $L_1 \perp L_2$. This is shown in Figure 2.14.

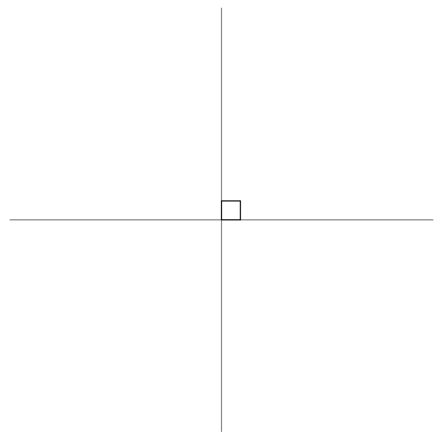


Figure 2.14: Perpendicular lines.

***Axiom* 2.3.2.** . Given a line L and a point P, there is one and only one line through P perpendicular to L.

Degenerate and Full Angles

Definition 2.3.5. ***Zero Angle***. If the two rays of an angle coincide, the interior region is the zero angle, with measure $0°$.

Definition 2.3.6. ***Full Angle***. When two rays coincide, the exterior region that wraps once around the vertex is the full angle, with measure $360°$.

(a) Zero angle. (b) Full angle.

Figure 2.15: Degenerate and full angles.

Definition 2.3.7. ***Linear Pair***. Two adjacent angles whose non-common sides are opposite rays.

Axiom* 2.3.3. *(Angle Additivity). If A and B are adjacent (non-overlapping) angles, then $m(A+B) = m(A)+m(B)$.

***Corollary* 2.3.1.** A linear pair of angles sums to a straight angle, $180°$.

Angle Relationships

Definition 2.3.8. ***Adjacent Angles***. Two angles are adjacent if they share a common vertex and a common ray, but do not overlap.

For adjacent angles $\angle MPR$ and $\angle RPQ$ as in Figure 2.16, it is clear that $m(\angle MPR) + m(\angle RPQ) = m(\angle MPQ)$.

Definition 2.3.9. ***Supplementary Angles***. Two angles are supplementary if their measures sum to $180°$.

2.3.1 Triangle Congruence Postulates P2

Building on 2.1.2 and the angle measure from Figure 2.11, we state the following fundamental postulate.

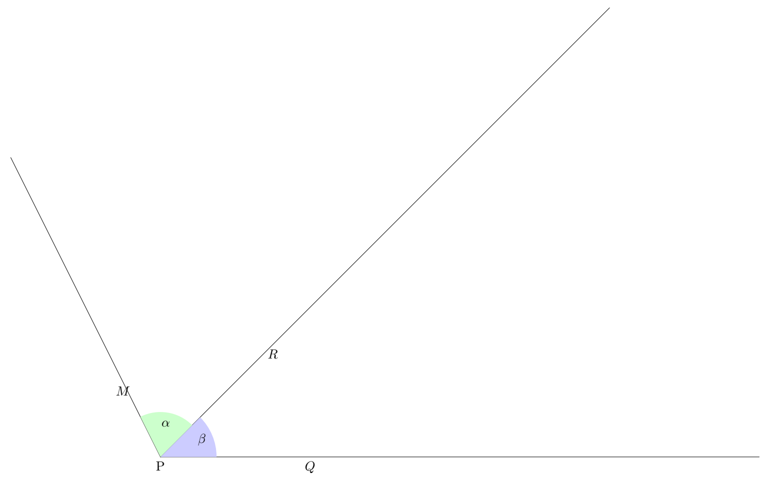


Figure 2.16: Adjacent angles α and β.

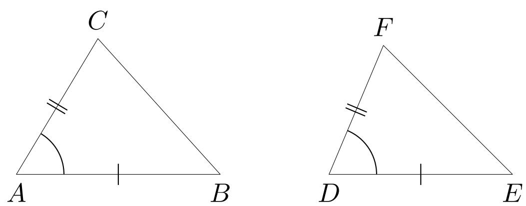


Figure 2.17: SAS: two sides and the *included* angle equal.

***Axiom* 2.3.4. *Side-Angle-Side (SAS)*.** If two sides and the included angle of one triangle are equal to the corresponding two sides and included angle of another triangle, then the triangles are congruent.

Remark. Congruent triangles have all corresponding sides and all corresponding angles equal.

***Theorem* 2.3.1. Isosceles Base Angles.** If a triangle has two equal sides, then the angles opposite those sides are equal.

Proof. Let $\triangle ABC$ be a triangle with $d(A,B) = d(A,C)$. Draw the bisector of $\angle BAC$ to meet $\overline{BC}$ at a point M. By the Side-Angle-Side axiom (2.3.4), $\triangle ABM \cong \triangle ACM$ because $d(A,B) = d(A,C)$, $m(\angle BAM) = m(\angle CAM)$ by construction, and the side $\overline{AM}$ is common. Therefore, the corresponding angles are equal, $m(\angle B) = m(\angle C)$. ∎

***Corollary* 2.3.2.** In an isosceles triangle, the segment from the apex to the base that is any one of: median, altitude, or angle bisector, is all three.

Angle Bisector

Definition 2.3.10. *Angle Bisector*. A ray that divides an angle into two adjacent angles of equal measure.

***Theorem* 2.3.2. Existence of an Angle Bisector.** Every angle has a bisector constructible with a straightedge and compass.

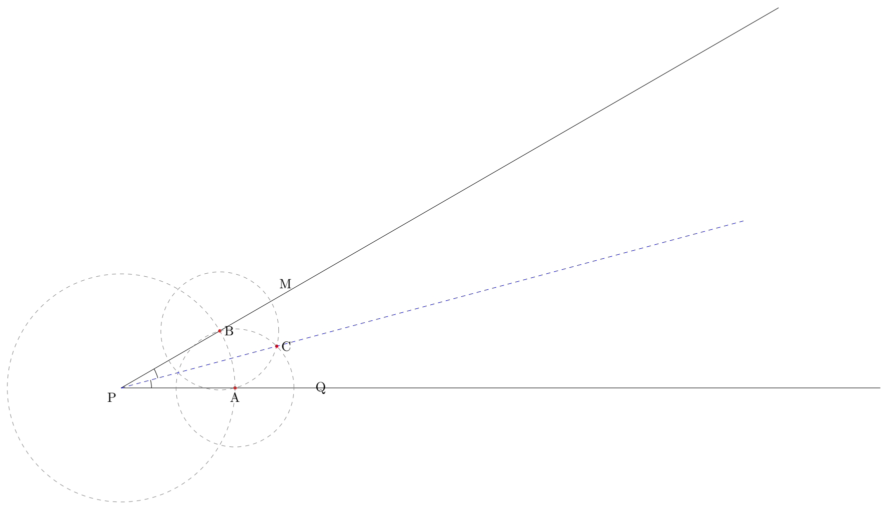


Figure 2.18: Classical bisection of $\angle QPM$.

Proof. Given $\angle QPM$, draw a circle with centre P, which meets the rays at points A and B respectively. Then $d(P, A) = d(P, B)$. With an equal radius $r > \frac{1}{2}d(A, B)$, draw circles centred at A and B. Let C be a point where these two circles intersect inside the angle. Then $d(A, C) = d(B, C) = r$. Consider the triangles $\triangle PAC$ and $\triangle PBC$. The side $\overline{PC}$ is common to both. We have established that the three corresponding sides are equal: $d(P, A) = d(P, B)$, $d(A, C) = d(B, C)$, and $d(P, C) = d(P, C)$. By the Side-Side-Side axiom (2.1.9), $\triangle PAC \cong \triangle PBC$. Since corresponding angles of congruent triangles are equal, we have $m(\angle APC) = m(\angle BPC)$. Therefore, the ray $\boldsymbol{PC}$ bisects $\angle QPM$. The construction is illustrated in Figure 2.18. ■

***Proposition* 2.3.2.** *Uniqueness of the Angle Bisector.* An angle has exactly one bisector.

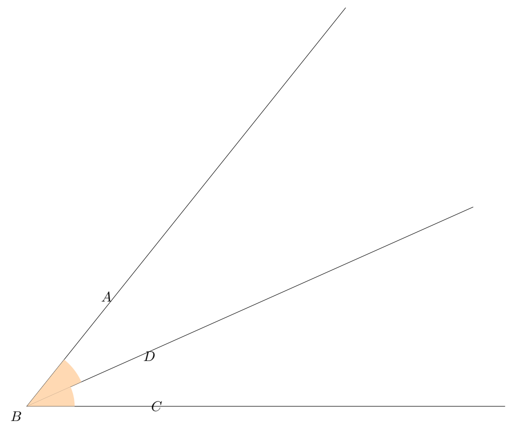
Proof. Assume, for contradiction, that an angle has two distinct bisecting rays, $\boldsymbol{PA}$ and $\boldsymbol{PB}$. One ray must lie within the angle formed by the other ray and a side of the original angle. By the principle of angle addition, this would imply that one of the "halves" is smaller than the other, contradicting the definition of an angle bisector. Therefore, the bisector must be unique. ■

***Corollary* 2.3.3.** Complements (and supplements) of equal angles are equal.

Example 2.3.2. (Angle Bisector). Ray $\boldsymbol{BD}$ is the angle bisector of $\angle ABC$. If $m(\angle ABC) = 82°$, what is $m(\angle ABD)$?

By definition, an angle bisector divides an angle into two equal parts. Therefore, $m(\angle ABD) = m(\angle DBC) = \frac{m(\angle ABC)}{2}$.

$$m(\angle ABD) = \frac{82°}{2} = 41°$$



Theorem **2.3.3. Vertical Angles Theorem**. When two lines intersect, the angles opposite each other, known as vertical angles, are equal in measure.

Proof. Let two lines intersect at a point P, forming angles α, β, γ, and δ as shown in Figure 2.19.

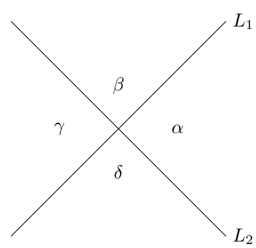


Figure 2.19: Vertical Angles

The angles α and β are adjacent and form a straight angle along the line L_1, thus they are supplementary.

$$m(\alpha) + m(\beta) = 180°$$

Similarly, angles β and γ are adjacent and form a straight angle along the line L_2.

$$m(\beta) + m(\gamma) = 180°$$

From these two equations, we have

$$m(\alpha) + m(\beta) = m(\beta) + m(\gamma)$$

Subtracting $m(\beta)$ from both sides gives

$$m(\alpha) = m(\gamma)$$

A similar argument shows that $m(\beta) = m(\delta)$. ■

Further Properties of Perpendicularity

Definition 2.3.11. ***Complementary Angles***. Two angles are complementary if their measures sum to 90°.

Combining the concepts of perpendicularity and bisection gives us a fundamental construction.

Definition 2.3.12. ***Midpoint***. The midpoint of a segment $\overline{AB}$ is the point M on the segment such that $d(A, M) = d(M, B)$. Its uniqueness follows from the axioms of distance.

Definition 2.3.13. ***Perpendicular Bisector***. The perpendicular bisector of a line segment is a line which is perpendicular to the segment and passes through its midpoint. This is shown in Figure 2.20.

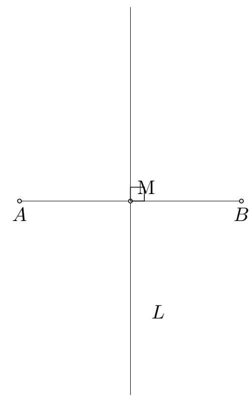


Figure 2.20: Line L is the perpendicular bisector of segment $\overline{AB}$.

Theorem **2.3.4. Perpendicular Bisector Theorem**. Any point on the perpendicular bisector of a segment is equidistant from the endpoints of the segment.

Proof. Let $\overline{AB}$ be a segment and let L be its perpendicular bisector, intersecting $\overline{AB}$ at its midpoint M. Let P be any point on L, as depicted in Figure 2.21. We want to show that $d(P, A) = d(P, B)$. Consider the triangles $\triangle PMA$ and $\triangle PMB$.

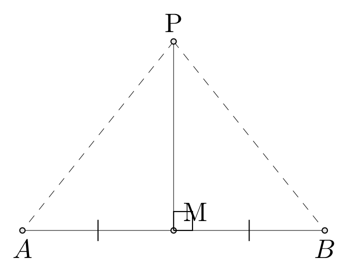


Figure 2.21: Proof of the Perpendicular Bisector Theorem.

By definition of a midpoint, $d(A, M) = d(M, B)$. The segment $\overline{PM}$ is common to both triangles. Since L is perpendicular to $\overline{AB}$, we have $m(\angle PMA) = m(\angle PMB) = 90°$. Therefore, by the Side-Angle-Side axiom (2.3.4), $\triangle PMA \cong \triangle PMB$. Corresponding sides of congruent triangles are equal in length, so $d(P, A) = d(P, B)$. ■

Properties of Right Triangles

A right triangle is a triangle in which one angle is a right angle. The sides of the triangle which determine the right angle are called the legs; the side opposite the right angle is called the hypotenuse, as shown in Figure 2.22. Many properties of arbitrary geometric figures can be reduced to an analysis of the properties of right triangles.

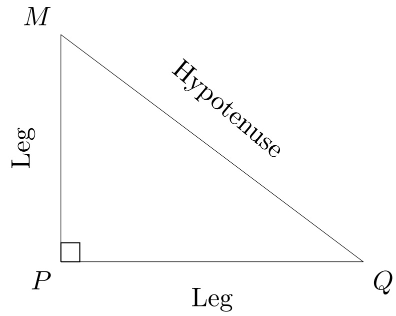


Figure 2.22: A right triangle with legs $\overline{PQ}$ and $\overline{PM}$, and hypotenuse $\overline{QM}$.

We formalise the congruence of right triangles in the next theorem.

***Theorem* 2.3.5. Leg-Leg Congruence**. If the legs of one right triangle are equal in length to the corresponding legs of another right triangle, then the triangles are congruent.

Proof. Let $\triangle ABC$ and $\triangle DEF$ be two right triangles with right angles at C and F respectively. Suppose $d(A,C) = d(D,F)$ and $d(B,C) = d(E,F)$. Since the included angles $\angle C$ and $\angle F$ are both right angles, their measures are equal, $m(\angle C) = m(\angle F) = 90°$. We have two sides and the included angle of $\triangle ABC$ equal to the corresponding parts of $\triangle DEF$. By the Side-Angle-Side axiom (2.3.4), it follows that $\triangle ABC \cong \triangle DEF$. ■

***Theorem* 2.3.6.** The two acute angles in a right triangle are complementary.

Proof. Let $\triangle ABC$ be a triangle with a right angle at C, so $m(\angle C) = 90°$. From the theorem on the sum of interior angles of a triangle (Sum of Interior Angles of a Triangle), we know that $m(\angle A)+m(\angle B)+m(\angle C) = 180°$. Substituting the known value for $m(\angle C)$ gives $m(\angle A) + m(\angle B) + 90° = 180°$. Subtracting 90° from both sides yields $m(\angle A) + m(\angle B) = 90°$. By definition, this means $\angle A$ and $\angle B$ are complementary. ■

***Theorem* 2.3.7. Exterior Angle Inequality**. An exterior angle of a triangle is greater than either of the interior opposite angles.

Proof. Consider $\triangle ABC$. Extend the side $\overline{BC}$ to a point D, forming the exterior angle $\angle ACD$, as shown in Figure 2.23. We will show $m(\angle ACD) > m(\angle BAC)$. Let M be the midpoint of $\overline{AC}$. Extend the segment $\overline{BM}$ to a point E such that $d(B,M) = d(M,E)$.

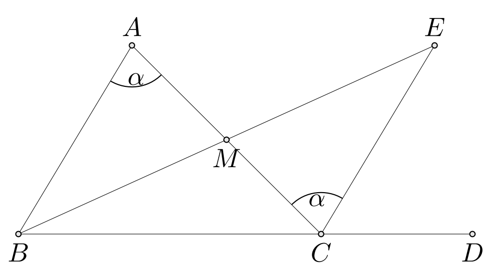


Figure 2.23: In the construction, $\triangle AMB \cong \triangle CME$, hence $\angle BAM = \angle ECM = \alpha$. Since E lies inside $\angle ACD$, one gets $m(\angle ACD) > m(\angle BAC)$.

Now consider $\triangle AMB$ and $\triangle CME$. By construction, $d(A,M) = d(C,M)$ and $d(B,M) = d(E,M)$. The angles $\angle AMB$ and $\angle CME$ are vertical angles, so $m(\angle AMB) = m(\angle CME)$. By the SAS axiom (2.3.4), $\triangle AMB \cong \triangle CME$. Therefore, their corresponding angles are equal, so $m(\angle BAM) = m(\angle ECM)$. The point E lies in the interior of $\angle ACD$. Therefore, $m(\angle ACD) = m(\angle ACE) + m(\angle ECD) = m(\angle BAM) + m(\angle ECD)$. Since angles have non-negative measure, this implies $m(\angle ACD) > m(\angle BAM)$, which is $m(\angle ACD) > m(\angle BAC)$. A similar construction shows that $m(\angle ACD) > m(\angle ABC)$. ■

***Theorem* 2.3.8. Angle-Side Monotonicity**. In any triangle, a larger angle is opposite a longer side.

Proof. This follows from the Exterior Angle Inequality (Figure 2.23) by a standard comparison argument. Given $\triangle ABC$, if $m(\angle B) > m(\angle C)$, we must show $d(A, C) > d(A, B)$. We can construct a ray $\boldsymbol{BD}$ inside $\angle B$ such that $m(\angle CBD) = m(\angle C)$, with D on $\overline{AC}$. Then $\triangle BDC$ is isosceles with $d(B, D) = d(D, C)$. In $\triangle ABD$, by the Triangle Inequality, $d(A, B) < d(A, D) + d(D, B)$. Substituting gives $d(A, B) < d(A, D) + d(D, C) = d(A, C)$. ■

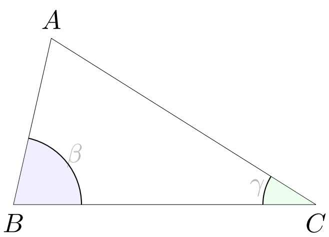


Figure 2.24: A larger angle (at B) faces the longer side (AC).

***Corollary* 2.3.4.** *The shortest distance from a point to a line is the length of the perpendicular segment from the point to the line.*

2.4 Transversals

Definition 2.4.1. *Transversal*. A line that intersects two or more coplanar lines at distinct points is called a transversal.

When a transversal intersects two lines, as in Figure 2.25, it forms eight angles. We classify certain pairs of these angles. Alternate interior angles are pairs on opposite sides of the transversal and between the two lines, such as $(\angle 3, \angle 6)$ and $(\angle 4, \angle 5)$. Corresponding angles are those in the same relative position at each intersection, such as $(\angle 1, \angle 5)$ and $(\angle 2, \angle 6)$. Consecutive interior angles are pairs on the same side of the transversal and between the two lines, namely $(\angle 3, \angle 5)$ and $(\angle 4, \angle 6)$.

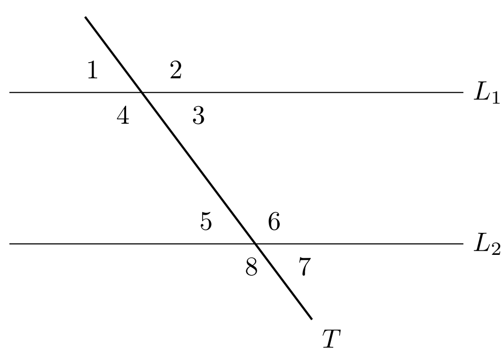


Figure 2.25: Angles formed by a transversal.

The properties of these angles are directly linked to the parallel axiom.

***Theorem* 2.4.1. Converse of Parallel Lines Theorem**. If two lines are intersected by a transversal such that the alternate interior angles are equal, or corresponding angles are equal, then the lines are parallel.

Proof. Let lines L_1 and L_2 be intersected by transversal K at points P and Q respectively. First, assume a pair of alternate interior angles are equal. Let γ be the angle vertically opposite one of these, so that γ

and the other angle are corresponding angles. By the Vertical Angles Theorem, the corresponding angles are also equal.

Now, assume corresponding angles α and β are equal, as in Figure 2.26, and suppose for contradiction that L_1 and L_2 are not parallel. They must then intersect at some point C, forming $\triangle PQC$. In this triangle, α is an exterior angle and β is an interior opposite angle. By the Exterior Angle Inequality theorem, an exterior angle must be strictly greater than an interior opposite angle, so $m(\alpha) > m(\beta)$. This contradicts our premise that $m(\alpha) = m(\beta)$. Thus, the assumption that the lines intersect must be false. Therefore, $L_1 \parallel L_2$.

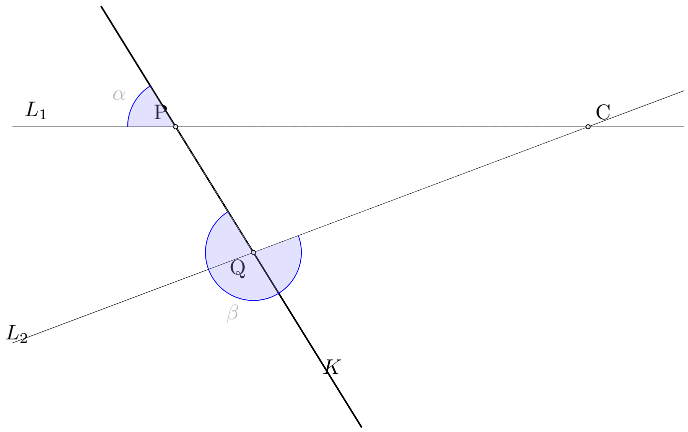


Figure 2.26: Converse proof setup: assume corresponding angles α and β are equal. If L_1 and L_2 met at C, then in $\triangle PQC$, α would be an exterior angle and β an interior opposite angle — contradicting the exterior–interior inequality.

■

***Theorem* 2.4.2. Transversal Angle Equalities**. Given line K intersecting parallel lines L_1 and L_2, corresponding angles have the same measure, and alternate interior angles have the same measure.

Proof. Let K be a transversal intersecting parallel lines L_1 and L_2 at points P and Q. Let α be an angle at P and β be the corresponding angle at Q. Assume for contradiction that $m(\alpha) \neq m(\beta)$. By our axioms, we can construct a line L_3 through Q that forms a corresponding angle with K equal in measure to α, as shown in Figure 2.27.

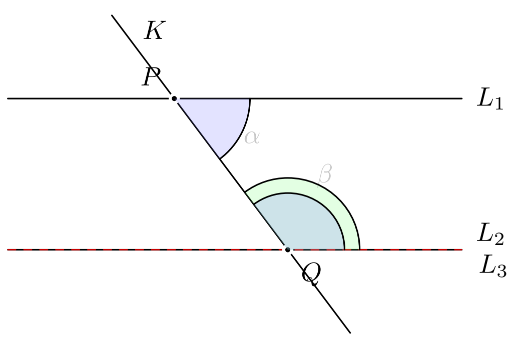


Figure 2.27: Through Q construct L_3 so that the corresponding angle with K equals α; by the converse, $L_3 \parallel L_1$. If also $L_2 \parallel L_1$, uniqueness of parallels forces $\beta = \alpha$.

By the preceding theorem, because their corresponding angles are equal, L_3 must be parallel to L_1. We are given that L_2 is also parallel to L_1. Since $m(\alpha) \neq m(\beta)$, the lines L_2 and L_3 are distinct. We now have two distinct lines through Q, both parallel to L_1, which contradicts the parallel axiom (2.1.7). Our initial assumption must be false, so $m(\alpha) = m(\beta)$. That alternate interior angles are equal follows, since an alternate interior angle is vertically opposite a corresponding angle. ■

***Corollary* 2.4.1.** If two parallel lines are intersected by a transversal, then consecutive interior angles are supplementary.

Example 2.4.1. (Angle Chase with a Transversal). In Figure 2.28, lines L_1 and L_2 are parallel ($L_1 \parallel L_2$). If $m(\angle 1) = 65°$, find $m(\angle 5)$ and $m(\angle 6)$.

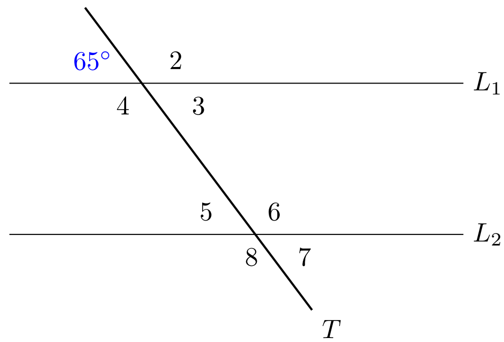


Figure 2.28: Quick angle chase.

By the corresponding angles theorem, $m(\angle 5) = m(\angle 1) = 65°$. Angles $\angle 5$ and $\angle 6$ form a linear pair on the line L_2, so they are supplementary. Thus, $m(\angle 6) = 180° - m(\angle 5) = 180° - 65° = 115°$.

Triangles and Parallel Lines

***Theorem* 2.4.3. Sum of Interior Angles of a Triangle**. For any triangle $\triangle ABC$, the sum of the measures of its interior angles is 180°. That is, $m(\angle A) + m(\angle B) + m(\angle C) = 180°$.

Proof. Construct a line L through vertex A parallel to the opposite side $\overline{BC}$, as shown in Figure 2.29. This is possible by the parallel axiom. The line L forms a straight angle at A, measuring 180°. This angle is composed of three adjacent angles: $\angle PAB$, $\angle BAC$, and $\angle CAQ$. Because L is parallel to $\overline{BC}$, we can use the properties of transversals. The line $\overline{AB}$ is a transversal, so $m(\angle PAB) = m(\angle B)$ (alternate interior angles are equal). Similarly, $\overline{AC}$ is a transversal, so $m(\angle CAQ) = m(\angle C)$ (alternate interior angles are equal). The sum of angles along the straight line L at vertex A is:

$$m(\angle PAB) + m(\angle BAC) + m(\angle CAQ) = 180°$$

Substituting the equal angles gives:

$$m(\angle B) + m(\angle A) + m(\angle C) = 180°$$

■

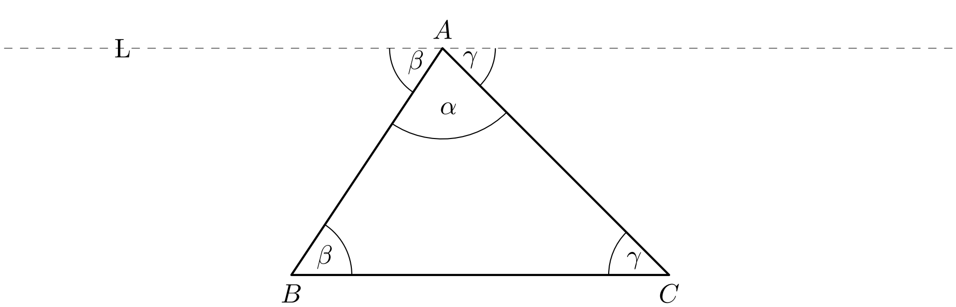


Figure 2.29: Parallel line through a vertex demonstrates the triangle angle sum.

***Corollary* 2.4.2.** *Exterior Angle Theorem.* An exterior angle of a triangle equals the sum of the two interior opposite angles.

Example 2.4.2. (Numerical Triangle). In $\triangle ABC$, suppose $m(\angle A) = 41°$ and $m(\angle B) = 72°$. Then $m(\angle C) = 180° - (41° + 72°) = 180° - 113° = 67°$.

2.5 Exercises

Part I: Fundamental Concepts and Calculations

1. Classify each of the following angles as acute, right, obtuse, straight, or reflex (greater than 180°).

(a) 89°
(b) 180°
(c) 91°
(d) 179°
(e) 90°
(f) 270°

2. Find the complementary and supplementary angles for each of the following angle measures. If one does not exist, explain why.

(a) 30°
(b) 45°
(c) 90°
(d) 120°
(e) $x°$, where $0 < x < 90$.

3. Two lines intersect as in Figure 2.19. If $m(\alpha) = 42°$, find the measures of β, γ, and δ. Justify each step of your reasoning.

4. In Figure 2.16, let $\angle MPQ$ be a straight angle. If $m(\angle MPR) = (2x+10)°$ and $m(\angle RPQ) = (3x-30)°$, find the value of x and the measure of each angle.

5. Ray ***PC*** is the bisector of $\angle APB$.

(a) If $m(\angle APB) = 78°$, what is $m(\angle APC)$?
(b) If $m(\angle BPC) = 25.5°$, what is $m(\angle APB)$?

6. In a triangle $\triangle PQR$, $m(\angle P) = 35°$ and $m(\angle Q) = 96°$. Find $m(\angle R)$.

7. In an isosceles triangle $\triangle ABC$, the two equal sides are $\overline{AC}$ and $\overline{BC}$.

(a) If the vertex angle $m(\angle C) = 50°$, find the measures of the base angles $\angle A$ and $\angle B$.
(b) If a base angle $m(\angle A) = 50°$, find the measures of $\angle B$ and $\angle C$.

8. An exterior angle at vertex C of $\triangle ABC$ measures 110°. If the interior angle $m(\angle B) = 75°$, find the measure of the interior angle $\angle A$.

9. Can a triangle have two right angles? Can it have two obtuse angles? Explain your reasoning using the theorem on the sum of interior angles of a triangle.

10. Two angles of a triangle are complementary. What kind of triangle must it be?

Part II: Area, Sectors, and Angle Measurement

11. A pizza with a radius of 20 cm is cut into 10 equal slices.

(a) What is the angle of each slice in degrees?

(b) What is the area of one slice of pizza?

12. A searchlight illuminates a sector with an area of 50π square metres. If the range (radius) of the searchlight is 30 metres, what is the measure of the angle of the sector?

13. Two discs, D_1 and D_2, have radii r_1 and r_2 respectively. Two sectors, S_1 in D_1 and S_2 in D_2, have the same angle measure α. Using the definition of angle measure given in the text, prove that

$$\frac{\text{area}(S_1)}{r_1^2} = \frac{\text{area}(S_2)}{r_2^2} = \frac{\pi\alpha}{360}.$$

14. What is the angle, in degrees, between the hour hand and the minute hand of a clock at the following times?

(a) 3:00

(b) 6:00

(c) 4:30

(d) ⋆ 2:20

15. Consider the reasoning used to justify that the area of a disc is πr^2, as shown in Figure 2.12. Why is the base of the resulting shape approximately πr and not $2\pi r$?

16. ⋆ A **radian** is another unit for measuring angles. A full circle is 2π radians. The angle of a sector in radians is defined as the ratio of the arc length to the radius, $\theta = \frac{s}{r}$.

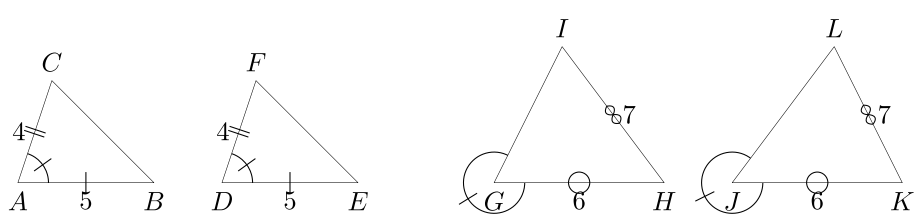
(a) Prove that this definition of angle measure is independent of the radius of the circle chosen.

Remark. If you scale the radius by a factor k, what happens to the arc length of the sector?

(b) Show that the area of a sector is given by $A = \frac{1}{2}r^2\theta$ when θ is in radians.

Part III: Congruence, Constructions, and Proofs

17. For each pair of triangles below, determine if there is enough information to conclude they are congruent by SSS or SAS. If so, state the congruence and the postulate used. If not, explain what additional information would be needed.



18. In a circle with centre O, two radii $\overline{OA}$ and $\overline{OB}$ are drawn. If M is the midpoint of the segment $\overline{AB}$, prove that $\triangle OMA \cong \triangle OMB$. What can you conclude about the line L_{OM} and the segment $\overline{AB}$?

19. ⋆ **Proof of ASA Congruence.** Let $\triangle ABC$ and $\triangle DEF$ be two triangles such that $m(\angle A) = m(\angle D)$, $d(A, B) = d(D, E)$, and $m(\angle B) = m(\angle E)$. Prove that $\triangle ABC \cong \triangle DEF$.

Remark. Place point D on A and ray $\boldsymbol{DE}$ on ray $\boldsymbol{AB}$. Where must E lie? Use the angle copying axiom to argue where ray $\boldsymbol{DF}$ must lie relative to $\boldsymbol{AC}$. Conclude that C and F must coincide.

20. ⋆ Using the result from the previous exercise (ASA), prove the **Converse of the Isosceles Triangle Theorem**: If two angles of a triangle are equal, then the sides opposite those angles are equal.

21. A quadrilateral with two pairs of equal-length sides that are adjacent to each other is called a **kite**. Let ABCD be a kite with $d(A, B) = d(A, D)$ and $d(C, B) = d(C, D)$. Prove that the diagonal $\overline{AC}$ is the perpendicular bisector of the diagonal $\overline{BD}$.

22. Let $\boldsymbol{BD}$ be the bisector of $\angle ABC$. Let P be a point on $\boldsymbol{BD}$. Let L be the line through P perpendicular to $\boldsymbol{BA}$, intersecting it at M, and let K be the line through P perpendicular to $\boldsymbol{BC}$, intersecting it at N. Prove that $d(P, M) = d(P, N)$.

Remark. This requires proving AAS congruence first, or using the sum of angles in a triangle.

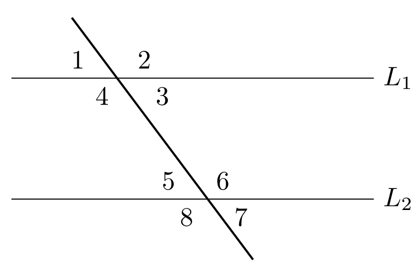
23. Describe the steps to construct an angle of 45° using only a straightedge and compass. Justify your construction.

24. Let $\overline{AB}$ be a line segment. Explain how to construct its perpendicular bisector using only a compass and straightedge. Use SSS congruence to prove that your construction is valid.

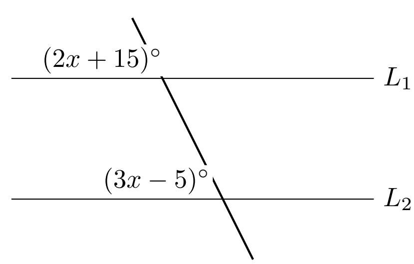
25. ⋆ Prove that the angle bisectors of two supplementary angles that form a linear pair are perpendicular.

Part IV: Parallel Lines and Transversals

26. In the diagram, $L_1 \parallel L_2$. Find the measures of all numbered angles if $m(\angle 1) = 118°$.



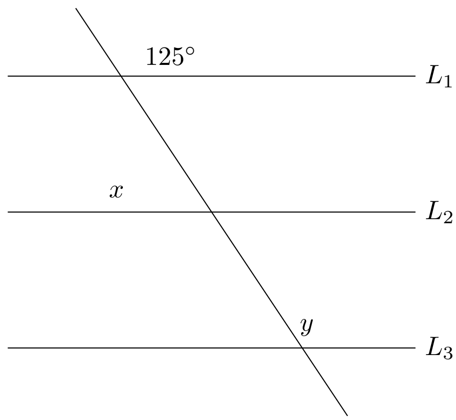
27. In the diagram, $L_1 \parallel L_2$. The angle measures are given as algebraic expressions. Find the value of x and the measure of each marked angle.



28. Prove the corollary mentioned in the text: If two parallel lines are intersected by a transversal, then consecutive interior angles are supplementary.

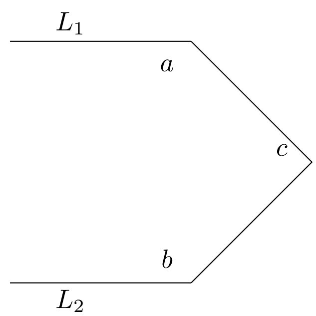
29. In a parallelogram ABCD, $L_{AB} \parallel L_{DC}$ and $L_{AD} \parallel L_{BC}$. Prove that opposite angles are equal (i.e., $m(\angle A) = m(\angle C)$ and $m(\angle B) = m(\angle D)$).

30. The three lines L_1, L_2, L_3 in the diagram are parallel. Find the values of x and y.

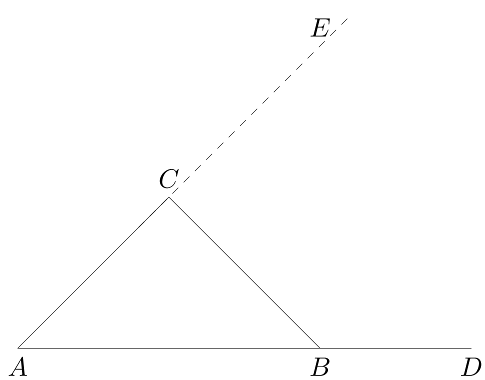


31. ⋆ In the figure, $L_1 \parallel L_2$. Prove that $m(\angle c) = m(\angle a) + m(\angle b)$.

Remark. Draw a third line parallel to L_1 and L_2 passing through the vertex of angle c.



32. Using the theorems on transversals, provide an alternative proof that the sum of the angles in a triangle is 180° by extending one of the sides of the triangle and drawing a parallel line, as shown.



33. Is the converse of the Vertical Angles Theorem true? That is, if you have four angles around a point P arranged as in Figure 2.19, and you know $m(\alpha) = m(\gamma)$, does this guarantee that the lines forming the angles are straight? Provide a proof or a counterexample.

34. Using the Angle-Side Monotonicity theorem, prove that the shortest distance from a point P to a line L is the length of the perpendicular segment from P to L.

Part V: Advanced Proofs and Explorations

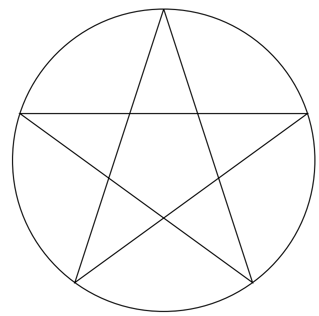
35. ⋆ **Hypotenuse-Leg (HL) Congruence.** Prove that if the hypotenuse and one leg of a right triangle are equal to the corresponding parts of another right triangle, then the two triangles are congruent.

Remark. Let the triangles be $\triangle ABC$ and $\triangle DEF$, with right angles at C and F. Assume $d(A,B) = d(D,E)$ and $d(A,C) = d(D,F)$. Construct a third triangle $\triangle GFE$ adjacent to $\triangle DEF$ such that $\triangle GFE \cong \triangle ABC$. Show that $\triangle DFG$ is isosceles and use this to prove $\triangle DEF \cong \triangle GFE$.

36. ⋆⋆ **Angle Sum of a Polygon.** Find a formula for the sum of the interior angles of a convex polygon with n sides ($n \geq 3$). Prove your formula is correct.

Remark. Divide the polygon into triangles by drawing all diagonals from a single vertex.

37. Find the sum of the measures of the angles at the five points of a five-pointed star inscribed in a circle.



38. Let P be a point in the interior of $\triangle ABC$. Prove that $d(P,B) + d(P,C) < d(A,B) + d(A,C)$.

39. ⋆ Let P be a point in the interior of $\triangle ABC$. Prove that $m(\angle BPC) > m(\angle BAC)$.

Remark. Extend the segment $\overline{BP}$ until it intersects side $\overline{AC}$ at a point D. Apply the Exterior Angle Inequality twice.

40. **Converse of the Perpendicular Bisector Theorem.** We proved in the text that any point on the perpendicular bisector of a segment is equidistant from the endpoints. Now prove the converse: Let A and B be two distinct points. If a point P satisfies $d(P,A) = d(P,B)$, prove that P must lie on the perpendicular bisector of the segment $\overline{AB}$.

Remark. Construct the midpoint M of $\overline{AB}$ and use SSS congruence on $\triangle PAM$ and $\triangle PBM$.

41. A line L is tangent to a circle with centre O at a point P. We accept that the radius $\overline{OP}$ is perpendicular to the tangent line L. If another line through P intersects the circle at Q, prove that $m(\angle QPL) = \frac{1}{2}m(\angle QOP)$.

42. ⋆⋆ **Morley's Trisector Theorem.** A famous (and very difficult to prove) theorem states that in any triangle, the three points of intersection of the adjacent angle trisectors form an equilateral triangle. While the proof is beyond our scope, let's explore a simpler case. In an equilateral triangle $\triangle ABC$, draw the angle trisectors. Show that the triangle formed by their intersections is indeed equilateral.

43. In $\triangle ABC$, let the angle bisectors of the interior angles $\angle B$ and $\angle C$ meet at a point I. Prove that $m(\angle BIC) = 90° + \frac{1}{2}m(\angle A)$.

44. Let the bisectors of the exterior angles at vertices B and C of $\triangle ABC$ meet at a point E. Prove that ray $\boldsymbol{AE}$ bisects the interior angle $\angle A$.

45. Consider a world where the parallel axiom is false. Specifically, assume the axiom of **Hyperbolic Geometry**: "Given a line L and a point P not on L, there exist at least two distinct lines through P parallel to L." In such a world, prove that the sum of the angles in any triangle must be less than 180°.

Remark. Use the proof setup from Figure 2.29. What goes wrong, and what can you conclude?

46. Now consider a world where the parallel axiom is replaced by the axiom of **Spherical Geometry**: "Given a line L and a point P, any line through P intersects L." (There are no parallel lines). What can you deduce about the sum of the angles in a triangle in this geometry? Explain why the proof of the Exterior Angle Inequality fails on a sphere.

Chapter 3

The Pythagorean Theorem

This chapter introduces one of the most celebrated results in all of mathematics, which relates the lengths of the sides of a right-angled triangle. To prove it, we first need a precise way to measure the area of a triangle.

3.1 The Area of a Triangle

We begin with the area of a rectangle, a figure we take as intuitively understood.

Definition 3.1.1. ***Rectangle***. A rectangle is a four-sided figure whose adjacent sides are perpendicular. It follows from the properties of parallel lines that opposite sides are parallel and have equal length.

We accept as a basic principle that the area of a rectangle is the product of the lengths of two adjacent sides. If these lengths are a and b, the area is ab. A square is a rectangle where $a = b$, so its area is a^2.

A diagonal divides any rectangle into two right-angled triangles. Consider the rectangle $PQNM$ in Figure 3.1. The diagonal $\overline{QM}$ divides it into $\triangle PQM$ and $\triangle NMQ$.

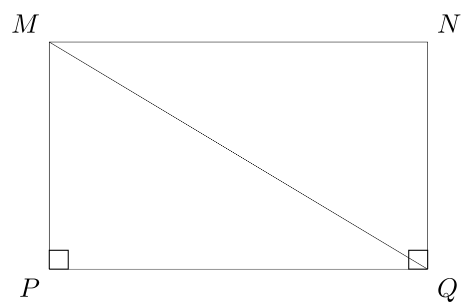


Figure 3.1: A rectangle divided into two right-angled triangles.

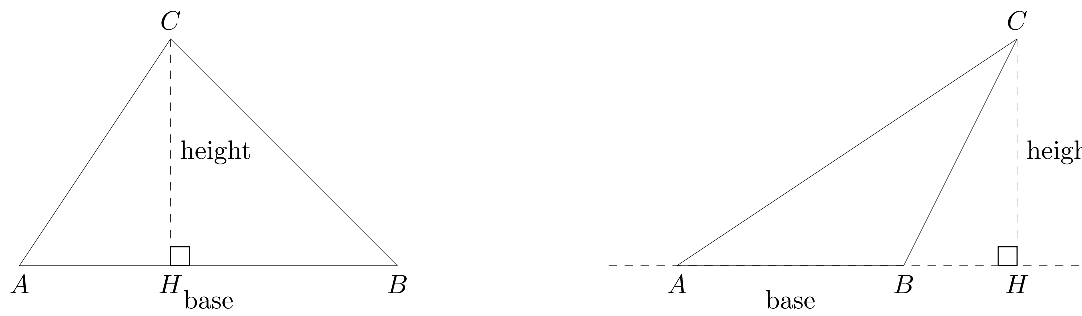
The triangles $\triangle PQM$ and $\triangle NMQ$ are congruent. We have $d(P, Q) = d(N, M)$ and $d(P, M) = d(N, Q)$ because they are opposite sides of a rectangle. The diagonal $\overline{QM}$ is common to both. By the SSS axiom (2.1.9), $\triangle PQM \cong \triangle NMQ$. Congruent figures have equal area, so the area of each triangle must be half the area of the rectangle.

***Theorem* 3.1.1. Area of a Right-Angled Triangle**. The area of a right-angled triangle is one-half the product of the lengths of its two legs.

Proof. Let the legs have lengths a and b. These legs can form the adjacent sides of a rectangle of area ab. The hypotenuse of the triangle is the diagonal of this rectangle. As shown, the triangle's area is half that of the rectangle, which is $\frac{1}{2}ab$. ■

To find the area of a general triangle, we introduce the concept of an altitude.

Definition 3.1.2. ***Altitude and Base***. An altitude of a triangle is the perpendicular line segment from a vertex to the line containing the opposite side. The length of the altitude is the height, and the opposite side is called the base.



(a) Altitude falls inside the triangle. (b) Altitude falls outside the triangle.

Figure 3.2: Altitudes of a triangle.

***Theorem* 3.1.2. Area of a Triangle**. The area of any triangle is one-half the product of the length of a base and its corresponding height. If a base has length b and the corresponding height is h, the area is $\frac{1}{2}bh$.

Proof. There are two cases, depending on whether the altitude meets the base segment or its extension, as illustrated in Figure 3.2. **Case 1:** The altitude falls inside the triangle (Figure 3.2a). Let the triangle be $\triangle ABC$, with base $\overline{AB}$ of length b. The altitude from C to $\overline{AB}$ is $\overline{CH}$ of length h. This altitude divides $\triangle ABC$ into two right-angled triangles, $\triangle AHC$ and $\triangle BHC$. Let $d(A, H) = x$ and $d(B, H) = y$, so $b = x+y$. The total area is the sum of the areas of the two smaller triangles:

$$\text{Area}(\triangle ABC) = \text{Area}(\triangle AHC) + \text{Area}(\triangle BHC) = \frac{1}{2}xh + \frac{1}{2}yh = \frac{1}{2}(x + y)h = \frac{1}{2}bh.$$

Case 2: The altitude falls outside the triangle (Figure 3.2b). Let the base be $\overline{AB}$ of length b, and let the altitude from C be $\overline{CH}$ of length h, where H is on the extension of $\overline{AB}$. The area of $\triangle ABC$ is the area of the large right-angled triangle $\triangle AHC$ minus the area of $\triangle BHC$. Let $d(A, H) = x$ and $d(B, H) = y$, so $b = x - y$.

$$\text{Area}(\triangle ABC) = \text{Area}(\triangle AHC) - \text{Area}(\triangle BHC) = \frac{1}{2}xh - \frac{1}{2}yh = \frac{1}{2}(x - y)h = \frac{1}{2}bh.$$

In both cases, the formula holds. ■

Area of a Trapezoid

Definition 3.1.3. ***Trapezoid***. A trapezoid is a four-sided figure with at least one pair of parallel sides. These parallel sides are called the bases, and the perpendicular distance between them is the height.

***Theorem* 3.1.3. Area of a Trapezoid**. The area of a trapezoid with parallel bases of lengths b_1 and b_2 and height h is given by the formula: Area $= \frac{1}{2}(b_1 + b_2)h$.

Proof. Let the trapezoid be $PQRS$ with parallel bases $\overline{PQ}$ (length b_1) and $\overline{SR}$ (length b_2). Draw the diagonal $\overline{PR}$, which divides the trapezoid into two triangles, $\triangle PQR$ and $\triangle PSR$, as shown in Figure 3.3.

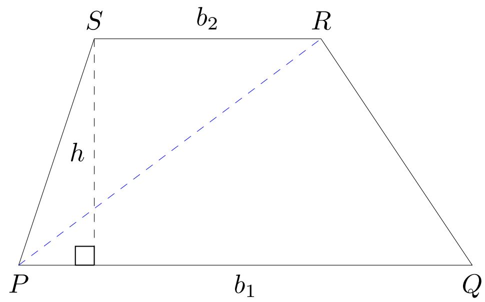


Figure 3.3: Dividing a trapezoid into two triangles.

The total area of the trapezoid is the sum of the areas of these two triangles. The area of $\triangle PQR$ with base $\overline{PQ}$ is $\frac{1}{2}b_1h$. The area of $\triangle PSR$ with base $\overline{SR}$ is $\frac{1}{2}b_2h$. (The height for $\triangle PSR$ relative to base $\overline{SR}$ is the same perpendicular distance h between the parallel lines). Summing these areas gives:

$$\text{Area} = \text{Area}(\triangle PQR) + \text{Area}(\triangle PSR) = \frac{1}{2}b_1h + \frac{1}{2}b_2h = \frac{1}{2}(b_1 + b_2)h.$$

■

3.2 The Pythagorean Theorem

We now state and prove the main theorem of this chapter.

***Theorem* 3.2.1. Pythagoras' Theorem**. Let $\triangle PQM$ be a right-angled triangle with the right angle at P. Let $a = d(P, M)$ and $b = d(P, Q)$ be the lengths of the legs, and let $c = d(Q, M)$ be the length of the hypotenuse. Then

$$a^2 + b^2 = c^2.$$

Proof. We construct a square with side length $a + b$. On each side of this large square, we mark a point that divides the side into segments of length a and b. By connecting these points, we form four right-angled triangles and a central quadrilateral, as shown in Figure 3.4.

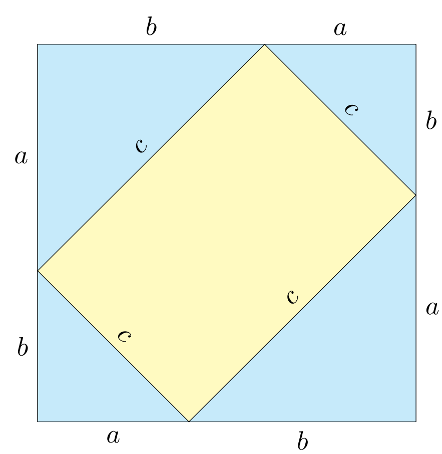


Figure 3.4: Proof of Pythagoras' Theorem.

Each of the four outer triangles has legs of length a and b. By the Leg-Leg Congruence theorem, these four triangles are congruent. Therefore, their hypotenuses are all equal in length, which we call c. This shows the inner quadrilateral has four equal sides. Let α and β be the two acute angles of one of the right-angled triangles. We know from the theorem on the sum of angles in a triangle (Sum of Interior Angles of a Triangle) that $\alpha + \beta + 90° = 180°$, which implies $\alpha + \beta = 90°$. At each vertex of the inner quadrilateral, three angles meet along a straight line: α, β, and the interior angle of the quadrilateral, let's call it γ. The sum is 180°. Thus, $\alpha+\beta+\gamma = 180°$. Since $\alpha+\beta = 90°$, it follows that $\gamma = 90°$. All four angles of the inner quadrilateral are right angles, so it is a square with side length c.

We now compute the area of the large square in two ways. First, its side length is $a + b$, so its area is $(a+b)^2 = a^2+2ab+b^2$. Second, the area is the sum of the areas of the four congruent right-angled triangles and the central square. The area of each triangle is $\frac{1}{2}ab$, and the area of the central square is c^2. The total area is $4 \times (\frac{1}{2}ab) + c^2 = 2ab + c^2$. Equating the two expressions for the area gives:

$$a^2 + 2ab + b^2 = 2ab + c^2$$

Subtracting $2ab$ from both sides yields the result:

$$a^2 + b^2 = c^2$$

■

Example 3.2.1. (Applying the Theorem). (a) Find the length of the diagonal of a square whose sides have length 1. The diagonal is the hypotenuse of a right-angled triangle with both legs of length 1. Let the diagonal be d.

$$1^2 + 1^2 = d^2 \implies 1 + 1 = d^2 \implies d^2 = 2 \implies d = \sqrt{2}.$$

(b) One leg of a right-angled triangle has length 8 cm, and the hypotenuse has length 17 cm. Find the length of the other leg. Let the unknown length be b. By Pythagoras' theorem:

$$8^2 + b^2 = 17^2$$
$$64 + b^2 = 289$$
$$b^2 = 289 - 64 = 225$$
$$b = \sqrt{225} = 15 \text{ cm}.$$

The converse of the theorem is also true and is a useful tool for identifying right-angled triangles.

***Theorem* 3.2.2. Converse of Pythagoras' Theorem**. If the lengths of the sides of a triangle, a, b, and c, satisfy the relation $a^2+b^2 = c^2$, then the triangle is a right-angled triangle, and the right angle is opposite the side of length c.

Proof. Given $\triangle ABC$ with side lengths a, b, c such that $a^2 + b^2 = c^2$. Construct another triangle, $\triangle A'B'C'$, with a right angle at C', and legs of length $d(A', C') = b$ and $d(B', C') = a$. By Pythagoras' theorem, the hypotenuse c' of $\triangle A'B'C'$ satisfies $(c')^2 = a^2+b^2$. Since we are given $c^2 = a^2+b^2$, we have $(c')^2 = c^2$, which implies $c' = c$. Now, $\triangle ABC$ and $\triangle A'B'C'$ have corresponding sides of equal length ($a = a, b = b, c = c'$). By the SSS axiom (2.1.9), the triangles are congruent, $\triangle ABC \cong \triangle A'B'C'$. Therefore, their corresponding angles are equal. Since $\angle C'$ is a right angle, $\angle C$ must also be a right angle. ■

Pythagoras' theorem provides a powerful algebraic method to prove geometric properties related to distance.

As established in 2.3.4, the perpendicular from a point to a line realises the minimal distance.

3.2.1 Pythagoras in Three Dimensions

The Pythagorean theorem can be applied sequentially to determine diagonal lengths in three-dimensional figures, such as a rectangular prism (a box).

***Theorem* 3.2.3. Diagonal of a Rectangular Prism**. For a rectangular prism with side lengths l, w, and h, the length of the space diagonal, d, from one corner to the opposite corner is given by $d^2 = l^2 + w^2 + h^2$.

Proof. Consider a rectangular prism with vertices as labelled in Figure 3.5. We wish to find the length of the space diagonal $\overline{AG}$.

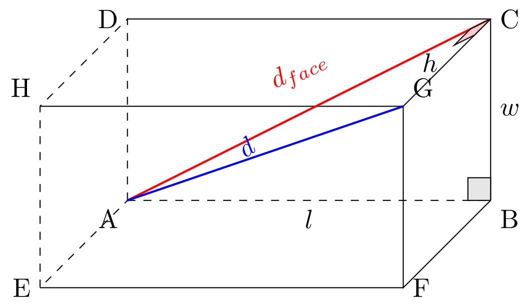


Figure 3.5: Space diagonal of a rectangular prism.

First, consider the right-angled triangle $\triangle ABC$ on the base of the prism. The diagonal of this base, $\overline{AC}$, is the hypotenuse. Let its length be d_{face}. By Pythagoras' theorem:

$$d_{face}^2 = d(A, B)^2 + d(B, C)^2 = l^2 + w^2.$$

Now, consider the triangle $\triangle ACG$. The side $\overline{CG}$ is perpendicular to the base plane, and therefore perpendicular to $\overline{AC}$. This makes $\triangle ACG$ a right-angled triangle with the right angle at C. The space diagonal $\overline{AG}$ is its hypotenuse. Applying Pythagoras' theorem again:

$$d^2 = d(A, C)^2 + d(C, G)^2 = d_{face}^2 + h^2.$$

Substituting the expression for d_{face}^2 from the first step gives:

$$d^2 = (l^2 + w^2) + h^2.$$

Thus, the square of the space diagonal's length is the sum of the squares of the three side lengths. ■

3.3 Exercises

Part I: Area Calculations

1. Calculate the area of the following figures.

(a) A rectangle with adjacent sides of length 8 cm and 5 cm.

(b) A right-angled triangle with legs of length 7 m and 10 m.

(c) The triangle $\triangle ABC$ from Figure 3.2a, if the base $|AB| = 12$ and the height $|CH| = 5$.

(d) A trapezoid with parallel bases of length 9 cm and 15 cm, and a height of 6 cm.

2. A triangle has an area of 54 square units.

(a) If its base is 12 units long, what is its corresponding height?

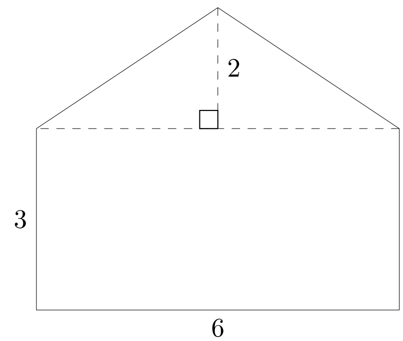
(b) If its height is 9 units, what is the length of the corresponding base?

3. A **parallelogram** is a quadrilateral with two pairs of parallel sides. Prove that the area of a parallelogram is given by the formula $A = bh$, where b is the length of a base and h is the corresponding perpendicular height.

Remark. Divide the parallelogram into a trapezoid and a triangle, or into two triangles.

4. Let $\triangle ABC$ be a triangle. Let h_a be the altitude to side a (of length $|BC|$), and h_b be the altitude to side b (of length $|AC|$). Prove that $a \cdot h_a = b \cdot h_b$. What does this relationship signify?

5. Find the area of the composite figure shown, which is formed from a rectangle and a triangle.



6. Consider the two cases in the proof of the area of a triangle (Theorem 3.1.2). What happens in the third case, where the altitude from C falls directly on one of the other vertices (e.g., vertex A)? Show that the formula $A = \frac{1}{2}bh$ still holds.

7. A trapezoid has an area of 120 square units and a height of 10 units. If one of its parallel bases is 15 units long, what is the length of the other parallel base?

8. Prove that the median of a triangle (a line segment joining a vertex to the midpoint of the opposite side) divides the triangle into two triangles of equal area.

Part II: The Pythagorean Theorem and its Converse

9. For each right-angled triangle, find the length of the unknown side x.

(a) Legs are 5 and 12. Find the hypotenuse.
(b) One leg is 9, hypotenuse is 15. Find the other leg.
(c) Legs are 1 and 2. Find the hypotenuse.
(d) One leg is 7, hypotenuse is 10. Find the other leg.

10. Determine whether the following sets of side lengths can form a right-angled triangle.

(a) $\{8, 15, 17\}$
(b) $\{7, 24, 25\}$
(c) $\{10, 12, 15\}$
(d) $\{1, \sqrt{3}, 2\}$

11. A **Pythagorean triple** is a set of three positive integers $\{a, b, c\}$ such that $a^2 + b^2 = c^2$.

(a) If you multiply each number in a Pythagorean triple by the same integer k, is the new triple also a Pythagorean triple? Justify your answer algebraically.
(b) A primitive Pythagorean triple is one where a, b, c have no common factors other than 1. Euclid's formula states that for any two positive integers $m > n$, the numbers $a = m^2 - n^2$, $b = 2mn$, and $c = m^2 + n^2$ form a Pythagorean triple. Prove this.

12. Find the length of the diagonal of a rectangle whose sides are 10 cm and 24 cm.

13. A 13-metre ladder is placed against a vertical wall. The base of the ladder is 5 metres from the base of the wall. How high up the wall does the ladder reach?

14. Two ships leave a port at the same time. One sails due south at 15 km/h, and the other sails due east at 20 km/h. How far apart are the ships after two hours?

15. $\star$ **Classifying Triangles.** The converse of the Pythagorean theorem can be extended. For a triangle with side lengths a, b, c where c is the longest side:

- If $a^2 + b^2 > c^2$, the triangle is acute.
- If $a^2 + b^2 < c^2$, the triangle is obtuse.

Classify the triangle with the given side lengths as acute, right, or obtuse.

(a) $\{5, 8, 10\}$

(b) $\{9, 12, 15\}$

(c) $\{10, 11, 12\}$

(d) $\{4, 5, \sqrt{42}\}$

16. In the proof of Pythagoras' theorem presented in Figure 3.4, a key step is proving that the inner quadrilateral is a square. Provide a detailed argument for why one of its interior angles, γ, must be 90°. Which theorem from the previous chapter is essential for this step?

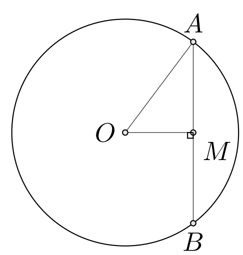
Part III: Geometric Applications

17. Find the height of an equilateral triangle with side length s. Using this, prove that the area of an equilateral triangle is $A = \frac{s^2\sqrt{3}}{4}$.

18. Find the area of an isosceles triangle with two sides of length 13 cm and a base of length 10 cm.

19. A rhombus has diagonals of length 16 cm and 30 cm. (Recall that the diagonals of a rhombus are perpendicular bisectors of each other). Find the perimeter of the rhombus.

20. In the figure, a circle with centre O has a radius of 10 cm. A chord $\overline{AB}$ has a length of 16 cm. Find the distance from the centre of the circle to the chord ($|OM|$).



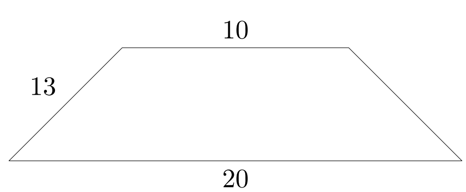
21. Consider a coordinate plane. The distance between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by the formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. Explain how this formula is a direct application of the Pythagorean theorem.

Remark. Draw a right-angled triangle with the segment $\overline{P_1P_2}$ as its hypotenuse.

22. Using the distance formula from the previous exercise, determine if the points A(1,1), B(5,3), and C(4,5) are the vertices of a right-angled triangle.

23. Find the area of the triangle with vertices P(-2, 1), Q(4, 1), and R(2, 4).

24. ⋆ Find the area of the isosceles trapezoid shown, with parallel bases of length 10 and 20, and non-parallel sides of length 13.

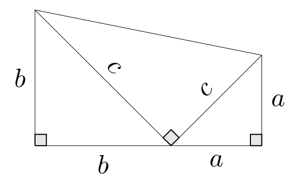


25. A triangle has side lengths 13, 20, and 21. Is it a right-angled triangle? Find its area.

Remark. Let the base be the side of length 21. Let the altitude from the opposite vertex be h. This altitude divides the base into segments of length x and $21 - x$. Set up two Pythagorean equations and solve for h.

Part IV: Proofs and Generalisations

26. ⋆ **Garfield's Proof.** In the 1870s, James A. Garfield, who later became President of the United States, discovered a proof of the Pythagorean theorem using a trapezoid. The trapezoid is formed from two congruent right-angled triangles and an isosceles right-angled triangle, as shown below.



(a) Argue that the figure ABCD is a trapezoid with parallel sides $\overline{AD}$ and $\overline{BC}$.

(b) Write an expression for the area of the trapezoid using the formula from Theorem 3.1.2 (treating $\overline{AB}$ as the height).

(c) Write another expression for the total area by summing the areas of the three triangles. (Note: You must implicitly accept or prove that the central triangle is a right-angled triangle).

(d) Equate the two expressions to prove that $a^2 + b^2 = c^2$.

27. In a right-angled triangle, let h be the length of the altitude drawn to the hypotenuse. Let p and q be the lengths of the segments into which the altitude divides the hypotenuse. Prove the following relations, known as the geometric mean theorems:

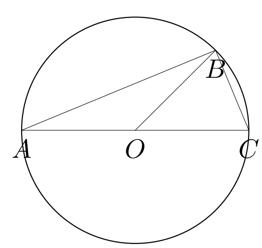
(a) $h^2 = pq$

(b) $a^2 = cp$ and $b^2 = cq$ (where a, b are legs and $c = p + q$ is the hypotenuse).

28. Prove that in a parallelogram, the sum of the squares of the lengths of the diagonals is equal to the sum of the squares of the lengths of the four sides.

29. ⋆ **Thales' Theorem.** Prove that any triangle inscribed in a semicircle must be a right-angled triangle.

Remark. Let $\triangle ABC$ be inscribed in a circle with diameter $\overline{AC}$ and centre O. Draw the radius $\overline{OB}$. Consider the two isosceles triangles $\triangle OAB$ and $\triangle OCB$. Sum the angles of $\triangle ABC$.



30. A median of a triangle is a line segment from a vertex to the midpoint of the opposite side. **Apollonius's Theorem** states that for a triangle $\triangle ABC$, if $\overline{AD}$ is a median to side $\overline{BC}$, then $|AB|^2+|AC|^2 = 2(|AD|^2 + |BD|^2)$. Prove this theorem.

Remark. Drop a perpendicular from A to BC and apply the Pythagorean theorem to multiple right-angled triangles.

31. Prove that if the altitudes of a triangle satisfy $h_a^2 + h_b^2 = h_c^2$, this does not imply the triangle is right-angled.

Remark. From the area formula, let Area $= \mathcal{A}$. Then $\mathcal{A} = \frac{1}{2}ah_a = \frac{1}{2}bh_b = \frac{1}{2}ch_c$. Express h_a, h_b, h_c in terms of $\mathcal{A}$ and the sides, substitute into the equation, and see if it simplifies to $a^2 + b^2 = c^2$.

Part V: Three-Dimensional Applications and Challenges

32. Find the length of the space diagonal of a cube with a side length of 5 cm.

33. A rectangular room measures 4 metres in length, 3 metres in width, and 2.5 metres in height. Find the length of the longest straight rod that can fit inside the room.

34. A square pyramid has a base with sides of length 10 cm and a slant height (the height of the triangular faces) of 13 cm. Find the perpendicular height of the pyramid.

35. ⋆ An ant is on one corner of a cube with a side length of 1 metre. What is the shortest distance the ant can walk along the surface of the cube to reach the diametrically opposite corner?

Remark. Unfold the cube into a 2D net and find the straight-line distance.

36. A right circular cone has a radius of 6 cm and a perpendicular height of 8 cm. What is its slant height?

37. Let's place a coordinate system in 3D space. Generalise the distance formula to find the distance between two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$. Justify your formula using the theorem for the diagonal of a rectangular prism.

38. ⋆ **De Gua's Theorem (a special case).** Consider a tetrahedron with a vertex O and three mutually perpendicular edges $\overline{OA}$, $\overline{OB}$, and $\overline{OC}$ (a corner of a box). Let the areas of the faces $\triangle OAB$, $\triangle OAC$, and $\triangle OBC$ be A_{OAB}, A_{OAC}, and A_{OBC} respectively. Let the area of the "hypotenuse" face $\triangle ABC$ be A_{ABC}. Prove that

$$(A_{OAB})^2 + (A_{OAC})^2 + (A_{OBC})^2 = (A_{ABC})^2.$$

39. A rope is wound tightly and symmetrically around a cylindrical pole of height 12 metres and circumference 4 metres. The rope makes exactly 3 full turns from the bottom to the top. What is the length of the rope?

Remark. Imagine "unrolling" the cylinder into a rectangle.

Chapter 4

Coordinates and Functions

To study geometric transformations with algebraic precision, we require a method to describe the position of points numerically. This is achieved by imposing a coordinate system on the plane. This framework will allow us to define geometric figures and transformations as sets of points and functions, respectively.

4.1 The Real Number Line, Absolute Value, and Intervals

We begin by associating each real number with a unique point on a line. We select a point for 0, the origin, and a point to its right for 1. This establishes a scale and orientation. A number x corresponds to a point whose distance from the origin is given by its absolute value, $|x|$. The point lies to the right of the origin if $x > 0$ and to the left if $x < 0$. This geometric representation is called the real number line.

Definition 4.1.1. ***Absolute Value***. The absolute value of a real number x, denoted $|x|$, is defined as:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

This correspondence preserves order: if $a < b$, the point for a lies to the left of the point for b. The distance between the points corresponding to a and b is $|b - a|$.

Subsets of the real line that correspond to line segments are called intervals. We use specific notation for different types of intervals, illustrated in Figure 4.1.

Definition 4.1.2. ***Intervals***. Let a and b be real numbers with $a < b$.

- The open interval (a, b) is the set $\{x \in \mathbb{R} \mid a < x < b\}$.
- The closed interval $[a, b]$ is the set $\{x \in \mathbb{R} \mid a \leq x \leq b\}$.

We also define half-open intervals such as $[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$ and infinite intervals such as $(a, \infty) = \{x \in \mathbb{R} \mid x > a\}$. The set of all real numbers, $\mathbb{R}$, can be written as the interval $(-\infty, \infty)$.

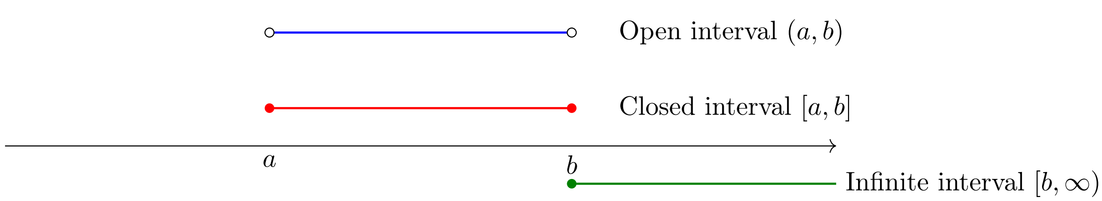


Figure 4.1: Different types of intervals on the real number line.

4.2 The Coordinate Plane

We establish a coordinate system by selecting two perpendicular lines in the plane, which we call the coordinate axes. Their point of intersection is defined as the origin, denoted O. Typically, one axis is horizontal (the x-axis) and the other is vertical (the y-axis). Each axis is a number line, with the origin corresponding to the number 0 on both.

Note. The origin has coordinates $(0, 0)$.

Any point P in the plane can be uniquely identified by an ordered pair of real numbers (x, y). The first number, x, is the x-coordinate (or abscissa), found by drawing a vertical line from P to the x-axis. The second number, y, is the y-coordinate (or ordinate), found by drawing a horizontal line from P to the y-axis. This system is known as the Cartesian coordinate system, and the plane equipped with it is the coordinate plane, denoted $\mathbb{R}^2$.

Remark. Formally, the coordinate plane is the Cartesian product of the set of real numbers with itself, $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) \mid x, y \in \mathbb{R}\}$. One must distinguish the ordered pair (a, b), a point in $\mathbb{R}^2$, from the open interval (a, b), a subset of $\mathbb{R}$. The Cartesian product of two intervals defines a rectangular region in the plane, as shown in Figure 4.2.

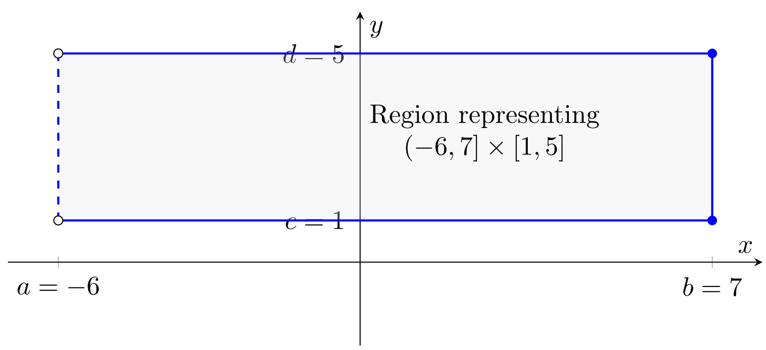


Figure 4.2: The Cartesian product of $(-6, 7]$ and $[1, 5]$.

The axes divide the plane into four regions called quadrants, numbered counter-clockwise from the upper right, as shown in Figure 4.3. The sign of the coordinates determines the quadrant in which a point lies.

- Quadrant I: $x > 0, y > 0$
- Quadrant II: $x < 0, y > 0$
- Quadrant III: $x < 0, y < 0$
- Quadrant IV: $x > 0, y < 0$

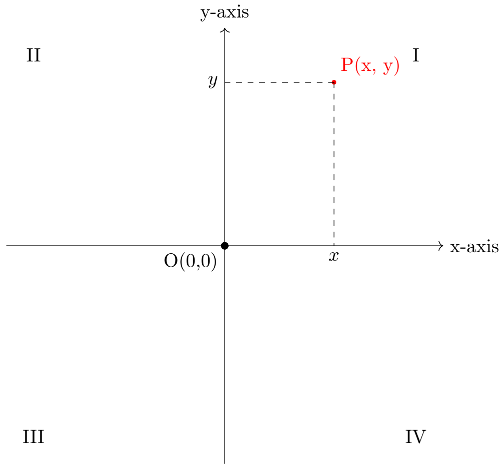


Figure 4.3: The Cartesian coordinate plane.

4.2.1 Distance Between Points

The coordinate system allows us to calculate the distance between any two points using an algebraic formula derived from the Pythagorean theorem.

Distance on a Line

As established, the distance between two points with coordinates x_1 and x_2 on the real number line is given by the absolute value of their difference, $|x_2 - x_1|$.

Remark. Recall that for any real number a, $|a| = \sqrt{a^2}$, so the distance can also be expressed as $\sqrt{(x_2 - x_1)^2}$.

***Corollary* 4.2.1.** $|-a| = |a|$.
***Corollary* 4.2.2.** $|a|^2 = a^2$.

Example 4.2.1. (Distance on a Number Line). (a) Find the distance of the point -7 from the origin. The distance is given by $|-7|$. Since $-7 < 0$, the definition of absolute value gives $|-7| = -(-7) = 7$. The distance is 7 units.

(b) Find the distance between the points -5 and 3 on the number line. Let $x_1 = -5$ and $x_2 = 3$. The distance is $|x_2 - x_1| = |3 - (-5)| = |3 + 5| = |8| = 8$.

Remark. calculating in the reverse order gives the same result: $|x_1 - x_2| = |-5 - 3| = |-8| = 8$.

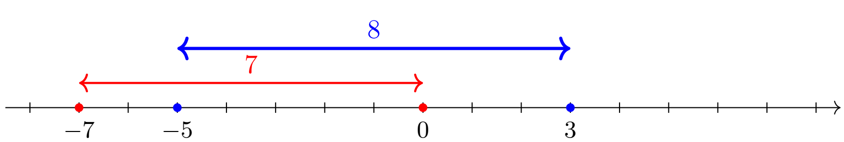


Figure 4.4: Visualising distance on a number line: $|-7| = 7$ and $|3 - (-5)| = 8$.

Distance in the Plane

To find the distance between two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in the plane, we construct a right-angled triangle whose hypotenuse is the segment $\overline{P_1P_2}$, as shown in Figure 4.5. The third vertex of this triangle, M, has coordinates (x_2, y_1).

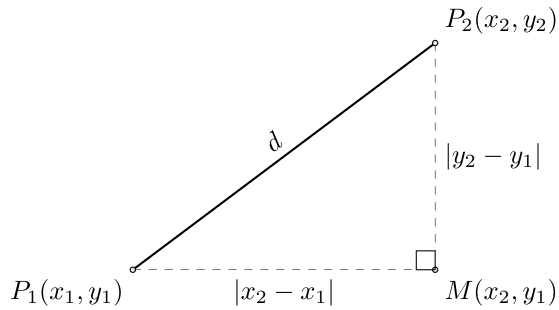


Figure 4.5: Derivation of the distance formula.

The segments $\overline{P_1M}$ and $\overline{P_2M}$ are the legs of the right-angled triangle $\triangle P_1MP_2$. Their lengths are the distances along the horizontal and vertical directions:

$$d(P_1, M) = |x_2 - x_1|$$

$$d(P_2, M) = |y_2 - y_1|$$

Let $d = d(P_1, P_2)$ be the length of the hypotenuse. By Pythagoras' theorem:

$$\begin{aligned} d^2 = (d(P_1, M))^2 + (d(P_2, M))^2 &= |x_2 - x_1|^2 + |y_2 - y_1|^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2. \end{aligned}$$

Taking the square root gives the distance formula.

***Theorem* 4.2.1. Distance Formula**. The distance d between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in the plane is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Example 4.2.2. (Calculating Distance). Find the distance between the points $A(-2, 5)$ and $B(4, -3)$. Using the distance formula:

$$\begin{aligned} d(A, B) &= \sqrt{(4 - (-2))^2 + (-3 - 5)^2} \\ &= \sqrt{(4 + 2)^2 + (-8)^2} \\ &= \sqrt{6^2 + (-8)^2} \\ &= \sqrt{36 + 64} \\ &= \sqrt{100} = 10. \end{aligned}$$

Distance in Three-Dimensional Space

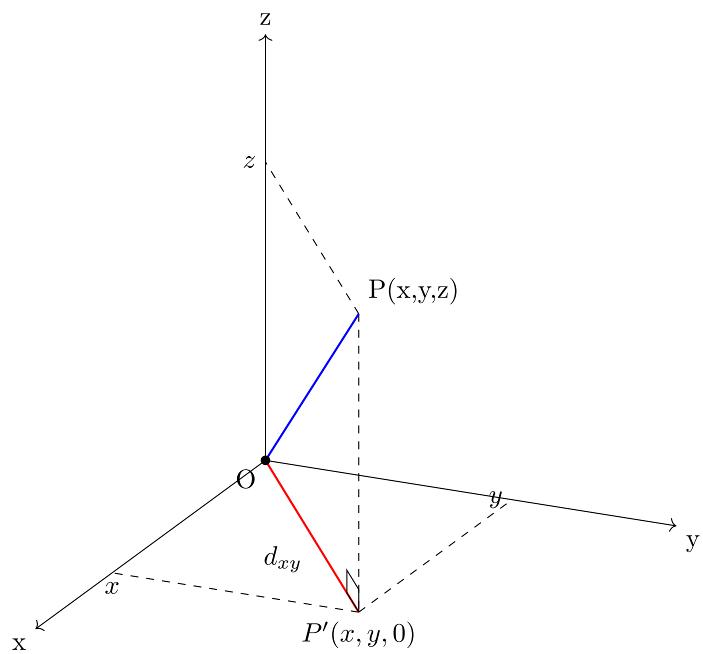


Figure 4.6: Distance from the origin in three-dimensional space.

The concept of coordinates extends naturally to three-dimensional space. We use three mutually perpendicular axes (x, y, and z), and a point P is represented by a triple of numbers (x, y, z). The distance formula can be derived by a double application of the Pythagorean theorem.

Consider the distance from the origin $O(0, 0, 0)$ to a point $P(x, y, z)$. Let P' be the projection of P onto the xy-plane, so $P' = (x, y, 0)$, as shown in Figure 4.6. The distance of P' from the origin, d_{xy}, is found using the planar distance formula: $d_{xy}^2 = x^2 + y^2$. Now, $\triangle OP'P$ is a right-angled triangle with the right angle at P'. The legs are $\overline{OP'}$ (length d_{xy}) and $\overline{P'P}$ (length $|z|$). The distance $d = d(O, P)$ is the hypotenuse. By Pythagoras' theorem:

$$d^2 = d_{xy}^2 + |z|^2 = (x^2 + y^2) + z^2 = x^2 + y^2 + z^2.$$

This result extends to the distance between any two points.

***Theorem* 4.2.2. Distance Formula in 3D**. The distance d between two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ in space is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

A Glimpse into Three Dimensions We now connect the co-ordinate picture to classical perpendicularity in three dimensions.

Definition 4.2.1. *Line Perpendicular to a Plane*. A line is perpendicular to a plane if it intersects the plane and is perpendicular to every line in the plane that passes through the point of intersection.

***Theorem* 4.2.3.** A line is perpendicular to a plane if and only if it is perpendicular to two intersecting lines in the plane at their point of intersection.

This theorem provides a practical test for perpendicularity in three dimensions. We also introduce related terminology. A slant is a line that intersects a plane but is not perpendicular to it. The foot is the point of intersection of a line with a plane. The projection of a segment $\overline{AC}$ onto a plane P is the segment $\overline{BC}$, where B is the foot of the perpendicular from A to the plane.

***Theorem* 4.2.4. Theorem of the Three Perpendiculars**. A line drawn in a plane through the foot of a slant, perpendicular to the slant's projection, is also perpendicular to the slant itself.

Proof. Let $\overline{AC}$ be a slant to plane P with foot C. Let $\overline{BC}$ be the projection of $\overline{AC}$ onto P, which means $\overline{AB} \perp P$. Let a line DE be drawn in plane P through C such that $DE \perp \overline{BC}$. We wish to prove that $DE \perp \overline{AC}$, as illustrated in Figure 4.7.

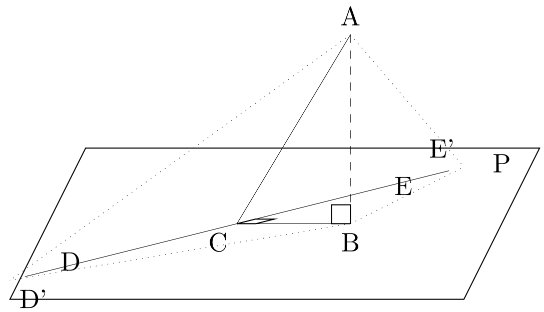


Figure 4.7: The Three Perpendiculars

On the line DE, choose points D' and E' such that $d(C, D') = d(C, E')$. Since C is the midpoint of $\overline{D'E'}$ and $\overline{BC} \perp \overline{D'E'}$, any point on the line containing $\overline{BC}$ is equidistant from D' and E'. Thus, $d(B, D') = d(B, E')$. Now consider the slants from A to the plane P. Since they have equal projections ($\overline{BD'}$ and $\overline{BE'}$), the slants themselves must be equal: $d(A, D') = d(A, E')$. (This step relies on the Pythagorean theorem, which will be established later; for now we accept it as geometrically evident from the symmetry of the construction.) This makes $\triangle AD'E'$ an isosceles triangle. The line segment $\overline{AC}$ is the median to the base $\overline{D'E'}$ (since C is the midpoint). In an isosceles triangle, the median to the base is also the altitude. Therefore, $\overline{AC} \perp \overline{D'E'}$, which means $\overline{AC} \perp DE$. ■

4.3 Exercises

Part I: The Number Line, Intervals, and Absolute Value

1. Evaluate the following absolute value expressions.

(a) $|-15|$
(b) $|10 - 4|$
(c) $|4 - 10|$
(d) $|-3 - 8|$
(e) $|\pi - 3|$
(f) $|3 - \pi|$

2. Find the distance between the following pairs of points on the real number line.

(a) 5 and 17
(b) -3 and 8
(c) -12 and -4
(d) -2.5 and 6.5

3. Write each of the following sets using interval notation and draw it on a number line.

(a) $\{x \in \mathbb{R} \mid -2 < x \leq 5\}$
(b) The set of all real numbers greater than or equal to -3.

(c) The set of all real numbers whose distance from 0 is less than 4.
(d) The set of all real numbers whose distance from 2 is at most 3.

4. Solve the following equations for x.

(a) $|x| = 7$
(b) $|x - 3| = 5$
(c) $|2x + 1| = 9$
(d) $|x^2 - 10| = 6$

5. Solve the following inequalities for x and express the solution set using interval notation.

(a) $|x| < 4$
(b) $|x| \geq 2$
(c) $|x - 5| \leq 1$
(d) $|3x - 2| > 7$

6. A point M is the midpoint of the interval $[a, b]$. Express the coordinate of M in terms of a and b. Prove that for any point x in $[a, b]$, $|x - M| \leq \frac{|b-a|}{2}$.

7. Let a and b be real numbers. Prove the following properties of absolute value using its definition.

(a) $|-a| = |a|$
(b) $|ab| = |a||b|$
(c) **The Triangle Inequality.** $|a + b| \leq |a| + |b|$.

Remark. Consider the cases where a and b have the same sign and where they have different signs.

8. Using the triangle inequality, prove the **Reverse Triangle Inequality**: $|a - b| \geq ||a| - |b||$.

9. Find the set of all points x on the number line such that the sum of the distances from x to -2 and from x to 4 is equal to 6. What if the sum is equal to 10? What if the sum is equal to 5?

Part II: The Coordinate Plane and the Distance Formula

10. Plot the following points on a Cartesian plane and state the quadrant in which each point lies: A(3, 4), B(-2, 5), C(-4, -1), D(1, -3), E(0, 5), F(-2, 0).

11. Sketch the region in the plane represented by the Cartesian product $[-2, 3) \times [1, 4]$. Indicate which parts of the boundary are included and which are not.

12. Calculate the distance between the following pairs of points.

(a) (1, 2) and (4, 6)
(b) (-3, 5) and (2, -7)
(c) (0, 0) and (-5, 12)
(d) (a, b) and $(-a, -b)$

13. Find the perimeter of the triangle with vertices A(1, 1), B(7, 1), and C(4, 5).

14. Use the distance formula and the converse of Pythagoras' theorem to determine if the triangle with vertices P(-2, 4), Q(2, 1), and R(5, 5) is a right-angled triangle.

15. Show that the points A(-1, 3), B(3, 11), and C(5, 15) are collinear by using the distance formula.

Remark. Three points are collinear if the sum of the lengths of two shorter segments equals the length of the longest segment.

16. A triangle has vertices D(0,0), E(6,0), and F(3,4).

(a) Show that the triangle is isosceles.

(b) Calculate the area of the triangle.

17. Find a point on the y-axis that is equidistant from the points A(-4, 2) and B(3, 1).

18. Show that the points P(1, -2), Q(7, -2), R(7, 4), and S(1, 4) are the vertices of a rectangle. Prove this by showing that opposite sides are equal in length and the diagonals are equal in length.

19. ⋆ Find the area of the triangle with vertices A(2, 3), B(8, 5), and C(4, 9).

Remark. Enclose the triangle in a rectangle and subtract the areas of the surrounding right-angled triangles.

20. A point (x, y) is such that its distance from $(0, 0)$ is twice its distance from $(6, 0)$. Find an equation relating x and y.

Part III: Introducing Geometric Figures Algebraically

21. Deriving the Midpoint Formula. Let $M(x_m, y_m)$ be the midpoint of the segment connecting $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. Draw vertical and horizontal lines from P_1, M, and P_2 to the axes to form two right-angled triangles. Using Angle-Angle-Side (AAS) congruence, prove that these two triangles are congruent. Conclude that M must be halfway between the x-coordinates and halfway between the y-coordinates, and derive the formula:

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

22. Find the midpoint of the line segment with the given endpoints.

(a) (2, 5) and (8, 1)

(b) (-4, 3) and (2, -3)

23. The midpoint of a segment $\overline{AB}$ is M(3, -2). If point A has coordinates (-1, 4), find the coordinates of point B.

24. A circle is defined as the set of all points in a plane that are a fixed distance (the radius, r) from a fixed point (the centre, (h, k)). Use the distance formula to show that the equation of a circle is:

$$(x - h)^2 + (y - k)^2 = r^2$$

25. Write the standard equation of the circle with the given centre and radius.

(a) Centre (0, 0), radius 5

(b) Centre (2, -3), radius 4

(c) Centre (-1, 0), radius $\sqrt{7}$

26. A circle has a diameter with endpoints A(-2, 5) and B(4, -1). Find the equation of the circle.

Remark. First find the centre, then find the radius.

27. Find the centre and radius of the circle with the equation $x^2 + y^2 - 6x + 8y - 11 = 0$.

Remark. Complete the square for both the x and y terms.

28. Find the points of intersection of the circle $(x - 1)^2 + y^2 = 25$ and the line $y = x + 2$.

29. Recall that the perpendicular bisector of a segment $\overline{AB}$ is the set of all points P such that $d(P, A) = d(P, B)$. Let A = (1, 2) and B = (5, 8).

(a) Let P = (x, y). Write an equation by setting $d(P, A)^2 = d(P, B)^2$.
(b) Simplify this equation to show that it represents a line. This line is the perpendicular bisector.

30. The vertices of a triangle are D(0,0), E(4,0), and F(0,3). Find the equation of the circle that passes through all three vertices (the circumcircle).

31. ⋆ Let k be a positive constant. Describe the set of points (x, y) such that the distance from (x, y) to the line $x = -1$ is equal to k times the distance from (x, y) to the point $(1, 0)$. What shape is it if $k = 1$?

Part IV: Three-Dimensional Geometry

32. Plot the point P(3, 4, 5) in a three-dimensional coordinate system.

33. Find the distance between the points A(1, 2, 3) and B(4, 6, 15).

34. Show that the points P(1, 1, 3), Q(3, 4, 2), and R(5, 7, 1) are collinear.

35. Determine if the triangle with vertices A(3, 0, 2), B(4, 3, 0), and C(8, 1, -1) is a right-angled triangle.

36. A sphere is the set of all points in space equidistant from a centre point (h, k, l). Derive the equation of a sphere with radius r.

37. Find the equation of the sphere with centre (2, -1, 3) and radius 6.

38. Find the centre and radius of the sphere given by the equation $x^2 + y^2 + z^2 + 2x - 4y + 6z - 2 = 0$.

39. Find the length of the diagonal of a rectangular prism with vertices at (0,0,0) and (5, 3, 2). Compare your result with the formula derived in the previous chapter.

40. A point P lies on the z-axis. Find its coordinates if it is equidistant from A(1, 5, 7) and B(5, 1, -4).

41. Let A = (4, 0, 3), B = (4, 0, 0), and C = (0, 0, 0). Consider the line DE in the xy-plane defined by the equation $y = 2x$.

(a) $\overline{AB}$ is a slant/perpendicular to the xy-plane. Which is it?
(b) What is the projection of the slant $\overline{AC}$ onto the xy-plane?
(c) Is the line DE perpendicular to the projection of $\overline{AC}$?
(d) Using coordinates, verify whether DE is perpendicular to the slant $\overline{AC}$ itself.

Part V: Proofs and Deeper Connections

42. Show that the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ satisfies the three axioms for distance from Chapter 1.

(a) $d(P_1, P_2) \geq 0$, with equality if and only if $P_1 = P_2$.
(b) $d(P_1, P_2) = d(P_2, P_1)$.
(c) The Triangle Inequality: $d(P_1, P_3) \leq d(P_1, P_2) + d(P_2, P_3)$.

Remark. Proving this part is challenging and relies on the triangle inequality for vectors, but you can demonstrate it for a specific numerical example.

43. Use coordinate geometry to prove that the diagonals of a rectangle are equal in length.

Remark. Place the vertices of the rectangle at (0,0), (a,0), (0,b), and (a,b).

44. Use coordinate geometry to prove that the midpoint of the hypotenuse of a right-angled triangle is equidistant from the three vertices.

45. Use coordinate geometry to prove that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and half its length.

46. A line is defined by the equation $Ax + By + C = 0$. The distance from a point (x_0, y_0) to this line is given by the formula $d = \frac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}}$.

(a) Find the distance from the point (5, 2) to the line $3x - 4y + 8 = 0$.

(b) ⋆ The line passing through points A(1,1) and B(8,2) has the equation $x - 7y + 6 = 0$. Find the height of the triangle with vertices A, B, and C(3,6) corresponding to the base $\overline{AB}$.

47. ⋆ **Heron's Formula.** The area of a triangle with side lengths a, b, c is given by $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{a+b+c}{2}$ is the semi-perimeter. Use the distance formula and Heron's formula to find the area of the triangle with vertices P(-1, 2), Q(5, 4), R(3, -2). Compare your answer to the box method from Exercise 19.

48. A regular octahedron has vertices at $(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)$.

(a) Find the length of an edge of the octahedron.

(b) Find the distance between two opposite vertices.

(c) Find the area of one of its triangular faces.

49. ⋆ Let A = (a, 0, 0), B = (0, b, 0), and C = (0, 0, c) be three points on the coordinate axes. Prove that the area of the triangle $\triangle ABC$ is $\frac{1}{2}\sqrt{a^2b^2 + b^2c^2 + c^2a^2}$. Connect this result to De Gua's Theorem from the previous exercise set.

4.4 Functions

Informally, a function is a rule that for each allowed input, assigns exactly one output. It can be visualised as a machine, as shown in Figure 4.8: an input is provided, the machine applies its rule, and a single output is produced.

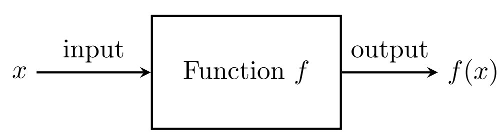


Figure 4.8: A function represented as a machine.

The notation $f(x)$ is read "f of x" and represents the output value of the function f for the input x. The rule itself is denoted by f.

Four Ways to Represent a Function

A single function can be described in several ways, and it is crucial to be able to move between these representations.

Definition 4.4.1. ***Representations***. A function can be described:

1. Verbally.
2. Algebraically (by a formula).
3. Numerically (by a table of values).
4. Visually (by a graph).

Example 4.4.1. (Representations of a Function). Consider the function described verbally as "take a number, square it, and subtract one." Its algebraic representation is the formula $f(x) = x^2 - 1$. A numerical representation can be a table listing some input-output pairs.

x	$f(x)$
-2	3
-1	0
0	-1
1	0
2	3

The visual representation is its graph, which is the set of all points (x, x^2-1) in the coordinate plane, shown in Figure 4.9.

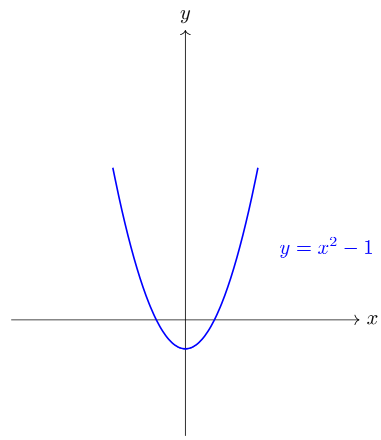
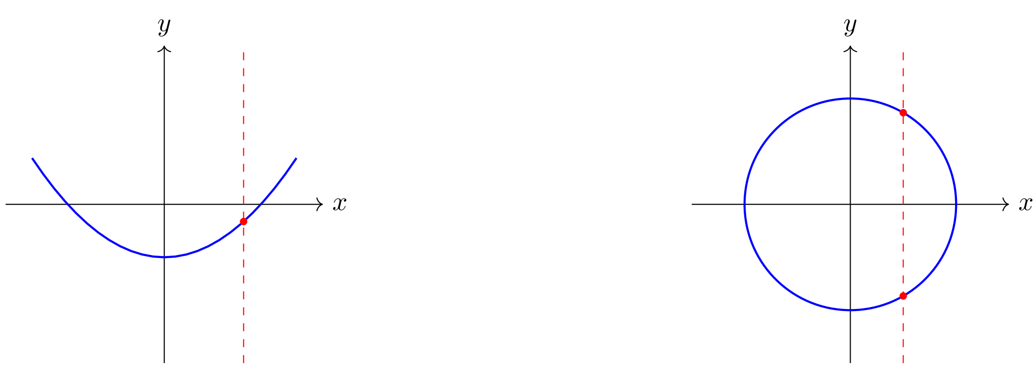


Figure 4.9: The graph of the function $f(x) = x^2 - 1$.

Remark. For a rule to be a function, for any input x, there is only one output $f(x)$. This uniqueness requirement leads to a simple visual test to determine if a curve in the plane is the graph of a function.

***Theorem* 4.4.1. The Vertical Line Test**. A curve in the coordinate plane is the graph of a function if and only if no vertical line intersects the curve more than once. Figure 4.10 illustrates this test.



(a) Passes test: Is a function.

(b) Fails test: Not a function.

Figure 4.10: The Vertical Line Test distinguishes functions from other relations.

Definition 4.4.2. ***Domain and Range***. The set of all allowed inputs for a function is called its domain. The set of all resulting outputs is its range.

For a rule to be a function, it must be unambiguous. The rule "take a positive number x and find a number y such that $y^2 = x$" is not a function, because for an input of $x = 4$, the output could be either 2 or -2. However, if we specify that we want only the principal (non-negative) root, the rule $f(x) = \sqrt{x}$ is a function. Its domain is $[0, \infty)$ and for any input x in the domain, the output $\sqrt{x}$ is uniquely determined. Notice that different inputs (e.g., 2 and -2) can lead to the same output (4) in a valid function like $f(x) = x^2$.

The Natural Domain

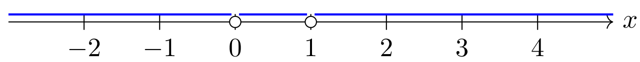
When a function is defined by a formula without an explicit domain, we adhere to the convention that its domain is the largest set of real numbers for which the rule is defined. This is the natural domain. In determining the natural domain, we must avoid operations that are undefined in the real number system.

1. Division by zero is undefined.
2. The square root of a negative number is not a real number.

Example 4.4.2. (Finding Domains). (a) Find the domain of $f(x) = \frac{1}{x^2-x}$. The function is undefined when the denominator is zero.

$$x^2 - x = x(x-1) = 0 \implies x = 0 \text{ or } x = 1$$

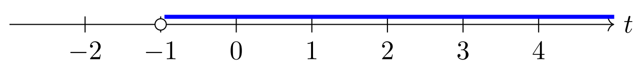
The domain is all real numbers except 0 and 1, written $\{x \in \mathbb{R} \mid x \neq 0, x \neq 1\}$.



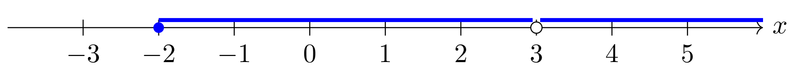
(b) Find the domain of $g(t) = \frac{t}{\sqrt{t+1}}$. The expression under the square root must be non-negative, $t + 1 \geq 0$. Since the square root is in the denominator, it cannot be zero, so we require a stricter condition:

$$t + 1 > 0 \implies t > -1$$

The domain is $\{t \in \mathbb{R} \mid t > -1\}$, or in interval notation, $(-1, \infty)$.



(c) Find the domain of $h(x) = \frac{\sqrt{x+2}}{x-3}$. Two conditions must be met. The term under the square root must be non-negative: $x + 2 \geq 0 \implies x \geq -2$. The denominator cannot be zero: $x - 3 \neq 0 \implies x \neq 3$. Combining these, the domain is $[-2, 3) \cup (3, \infty)$.



(d) Find the domain of $k(x) = \sqrt{5 - \sqrt{x-1}}$. This function involves nested roots. First, the inner root requires $x - 1 \geq 0 \implies x \geq 1$. Second, the outer root requires $5 - \sqrt{x-1} \geq 0$. This implies $5 \geq \sqrt{x-1}$, and since both sides are non-negative, we can square them: $25 \geq x - 1 \implies 26 \geq x$. Both conditions must hold, so the domain is the closed interval $[1, 26]$.

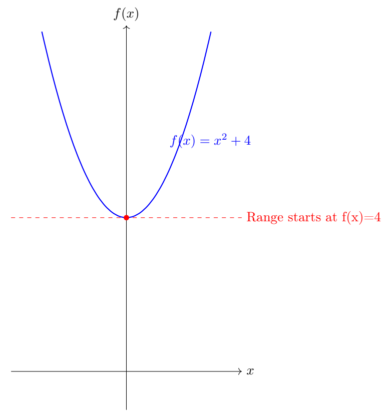
Codomain

While the domain concerns the inputs, the range concerns the outputs. When discussing functions, it is useful to specify a target set for the outputs that is not necessarily the exact range. This is called a codomain. A codomain is any set that contains the range. For $f(x) = x^2 + 4$, both $\mathbb{R}$ and $[0, \infty)$ are valid codomains, but $[4, \infty)$ is the range.

Example 4.4.3. (Finding the Range). Find the range of the function $f(x) = x^2 + 4$. The domain is $\mathbb{R}$. To find the range, we consider the possible output values. For any real number x, $x^2 \geq 0$. The minimum value occurs at $x = 0$:

$$f(0) = 0^2 + 4 = 4$$

For any other value of x, x^2 is positive, so $f(x) > 4$. The set of all possible outputs is every real number greater than or equal to 4. The range is the interval $[4, \infty)$.



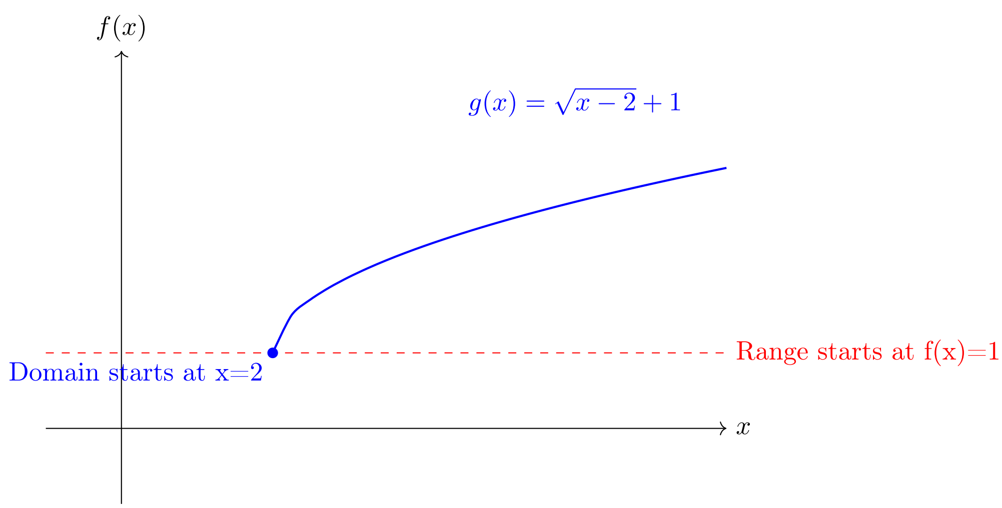
Example 4.4.4. (Finding the Range of a Radical Function). Find the domain and range of the function $g(x) = \sqrt{x-2} + 1$. The domain requires $x - 2 \geq 0 \implies x \geq 2$, so the domain is $[2, \infty)$. To find the range, we note that the principal square root $\sqrt{\cdot}$ produces non-negative results.

$$\sqrt{x-2} \geq 0 \quad \text{for all } x \text{ in the domain.}$$

Adding 1 to both sides gives:

$$\sqrt{x-2} + 1 \geq 1$$

The output, $g(x)$, is always greater than or equal to 1. The range is the interval $[1, \infty)$.



Piecewise-Defined Functions

Definition 4.4.3. ***Piecewise-Defined Function***. A function defined by different formulas on different parts of its domain.

The absolute value function, $f(x) = |x|$, is a primary example.

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

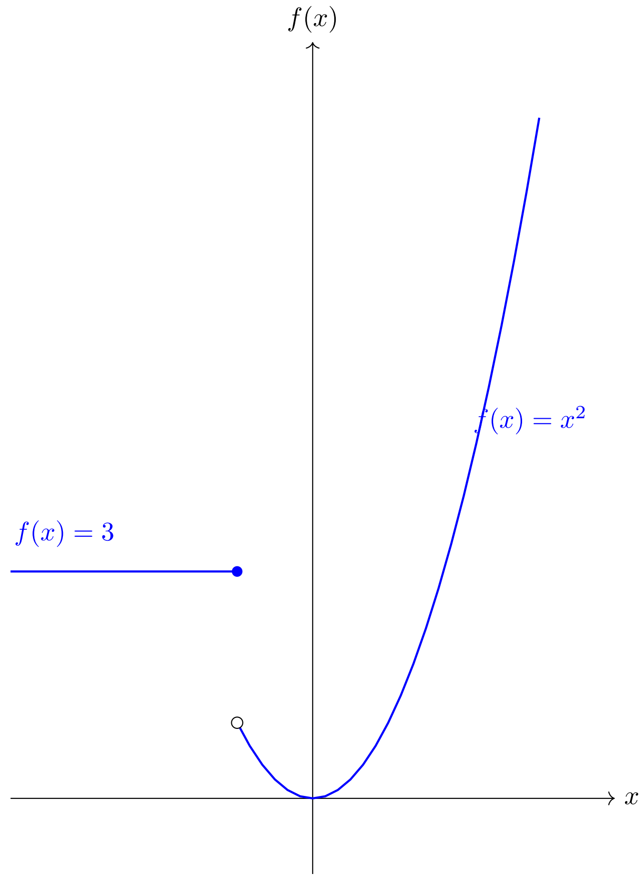
Example 4.4.5. (A Piecewise Function). Analyse the function $g(x)$ defined by:

$$g(x) = \begin{cases} 3 & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$$

To evaluate $g(x)$, we determine which part of the domain the input x belongs to.

- For $x = -5$, since $-5 \leq -1$, we use the first rule: $g(-5) = 3$.
- For $x = 2$, since $2 > -1$, we use the second rule: $g(2) = 2^2 = 4$.

The graph consists of a horizontal ray and part of a parabola.



The filled circle at $(-1, 3)$ indicates that this point is included on the graph ($g(-1) = 3$), while the open circle at $(-1, 1)$ indicates that this point is excluded.

4.4.1 A Formal Perspective on Functions

The informal notion of a function as a "rule" is useful but can be ambiguous. The rules $f(x) = x^2 - 1$ and $g(x) = (x-1)(x+1)$ are described differently, but they produce the same output for every input and are the same function. This suggests the essential aspect of a function is the pairing of inputs to outputs.

This leads to a more rigorous, set-theoretic definition which relies on the concept of an ordered pair.

Ordered Pairs

For completeness, we prove that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

Remark. Recall $(a, b) := \{\{a\}, \{a, b\}\}$.

***Theorem* 4.4.2.** If $(a, b) = (c, d)$, then $a = c$ and $b = d$.

Proof. The hypothesis $(a, b) = (c, d)$ means $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$. The first component, a, is the unique element that belongs to every member of the set $\{\{a\}, \{a, b\}\}$. The same must be true for c. Thus, $a = c$.

With $a = c$, the equality becomes $\{\{a\}, \{a, b\}\} = \{\{a\}, \{a, d\}\}$. Case 1: $a = b$. The set is $\{\{a\}\}$. The set $\{\{a\}, \{a, d\}\}$ must also have only one member, which implies $\{a\} = \{a, d\}$. This is true only if $d = a$. Thus $b = a = d$. Case 2: $a \neq b$. The set $\{\{a\}, \{a, b\}\}$ has two distinct members. Thus, $\{a, b\} = \{a, d\}$. Since $b \neq a$, it follows that $b = d$. In both cases, we conclude $b = d$. ■

Formal Definition of a Function

This precise notion of an ordered pair allows for a rigorous definition of a function.

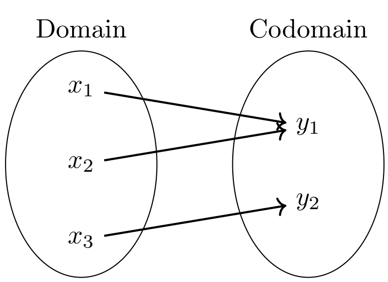
Definition 4.4.4. ***Function***. A function f is a set of ordered pairs (x, y) such that for every first element x, there is exactly one corresponding second element y.

For our example $f(x) = x^2 - 1$, the function f is the infinite set of all pairs $(x, x^2 - 1)$. A sample of these pairs is:

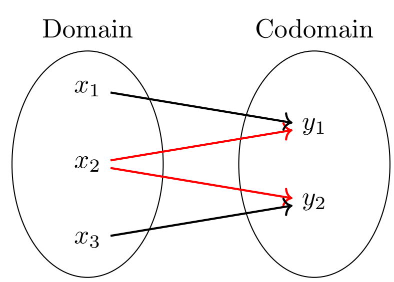
$$f = \{\dots, (-2, 3), (-1, 0), (0, -1), (1, 0), (2, 3), \dots\}$$

The condition that "for every first element x, there is exactly one corresponding second element y" is precisely what the Vertical Line Test verifies. If a vertical line at $x = a$ intersected the graph at two points, (a, b) and (a, c), the set would contain two pairs with the same first element, which is forbidden by the definition.

The domain is the set of all first elements, and the range is the set of all second elements. The notation $y = f(x)$ is a statement that the pair (x, y) is a member of the set that constitutes the function f.



(a) A valid function.



(b) Not a function.

Figure 4.11: Visualising the formal definition of a function.

Definition 4.4.5. ***Equality of Functions***. Two functions f and g are equal, written $f = g$, if they have the same domain and $f(x) = g(x)$ for all x in that domain. That is, the sets of ordered pairs that constitute the functions must be identical.

Example 4.4.6. Let $f(x) = x$ with domain $\mathbb{R}$ and $g(x) = x^2/x$ with its natural domain $\mathbb{R} \setminus \{0\}$. Although the rules agree for all $x \neq 0$, the functions are not equal because their domains differ.

We use the notation $f : A \to B$ to specify that f is a function with domain A and codomain B. The rule itself can be specified with the 'maps to' arrow, $\mapsto$. For an earlier example, we can write:

$$f : \mathbb{R} \to \mathbb{R} \qquad x \mapsto x^2 + 4$$

Here, $\mathbb{R}$ is the domain, and we have chosen $\mathbb{R}$ as a valid codomain, even though the range is the smaller set $[4, \infty)$.

4.5 Exercises

Part I: Basic Concepts and Representations

1. Let $f(x) = 2x^2 - 3x + 1$. Compute the following:

(a) $f(0)$

(b) $f(2)$

(c) $f(-1)$

(d) $f(a)$

(e) $f(x+1)$

(f) $\star$ $\frac{f(x+h)-f(x)}{h}$ (This is the difference quotient).

2. Translate the following verbal descriptions into algebraic functions.

(a) "Take a number, triple it, and then add five." Let the function be $g(x)$.

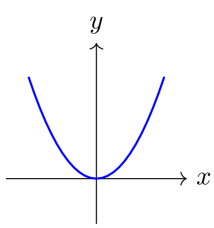
(b) "Find the distance of a point $(x, 3)$ from the origin." Let the function be $d(x)$.

(c) "Calculate the area of a circle given its diameter, d." Let the function be $A(d)$.

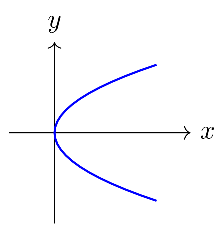
3. A function is defined by the formula $g(t) = t^3 - 4t$. Complete the following numerical table for g.

t	-2	-1	0	1	2	3
$g(t)$						

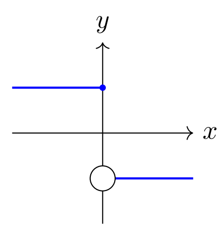
4. For each of the graphs below, determine if it represents y as a function of x. Justify your answer using the Vertical Line Test.



(a)



(b)



(c)

5. Determine whether the following equations define y as a function of x.

(a) $y - x^2 = 2$

(b) $x - y^2 = 2$

(c) $x^2 + y^2 = 9$

(d) $y = |x| + 1$

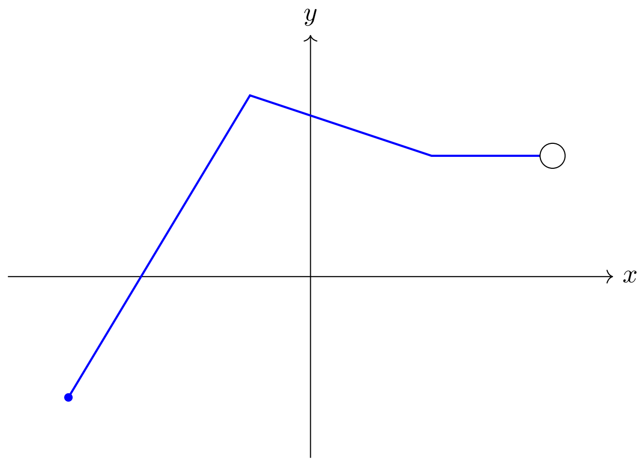
6. From the given graph of a function f, determine the following:

(a) The domain of f.

(b) The range of f.

(c) The value of $f(-1)$.

(d) The values of x for which $f(x) = 2$.



7. A function is described verbally as: "If the input x is a rational number, the output is 1. If the input x is an irrational number, the output is 0." This is the Dirichlet function.

(a) What is the domain of this function?

(b) What is the range of this function?

(c) What are the values of $f(2)$, $f(\pi)$, and $f(3/4)$?

8. Let $f(x) = |x|$. Express the difference quotient $\frac{f(x+h)-f(x)}{h}$ for $x > 0$ and for $x < 0$. What happens at $x = 0$?

Part II: Domain and Range

9. Find the natural domain of the following functions. Express your answer in interval notation.

(a) $f(x) = x^3 - 5x + 2$

(b) $g(t) = \frac{t+1}{t-4}$

(c) $h(y) = \frac{2y}{y^2-9}$

(d) $k(x) = \sqrt{x-7}$

(e) $m(x) = \frac{1}{\sqrt{x-7}}$

(f) $p(z) = \sqrt{16-z^2}$

10. Find the natural domain for the following more complex functions.

(a) $f(x) = \sqrt{x^2 - 5x + 6}$

(b) $g(x) = \frac{\sqrt{x+1}}{x^2-x-2}$

(c) $h(t) = \sqrt{\frac{t-2}{t+2}}$

(d) $\star$ $k(x) = \sqrt{x - \sqrt{x-4}}$

11. Find the domain and range for each of the following functions.

(a) $f(x) = 2x - 1$

(b) $g(x) = x^2 - 6x + 5$

Remark. Complete the square to find the vertex.

(c) $h(x) = |x-3| + 2$

(d) $k(x) = 5 - \sqrt{x}$

12. A rectangle has a perimeter of 20 metres.

(a) Express the area of the rectangle as a function of the length of one of its sides, l.

(b) What is the natural domain of this function in the context of the problem?

13. Consider the function $f(x) = \frac{x^2}{x^2+1}$.

(a) What is the domain of f?

(b) By analysing the expression, determine the range of f.

Remark. Is $f(x)$ ever negative? Can it be equal to 1?

14. Let $P(x, 0)$ be a point on the x-axis. Let Q be the fixed point (0, 4).

(a) Write a function $d(x)$ that gives the distance between P and Q.

(b) What is the domain of $d(x)$?

(c) What is the range of $d(x)$?

Part III: Piecewise-Defined Functions

15. Consider the function $f(x) = \begin{cases} 2x+5 & \text{if } x < -1 \\ x^2 - 1 & \text{if } x \geq -1 \end{cases}$. Evaluate $f(-3)$, $f(-1)$, and $f(2)$.

16. Sketch the graph of the function from the previous exercise. State its domain and range.

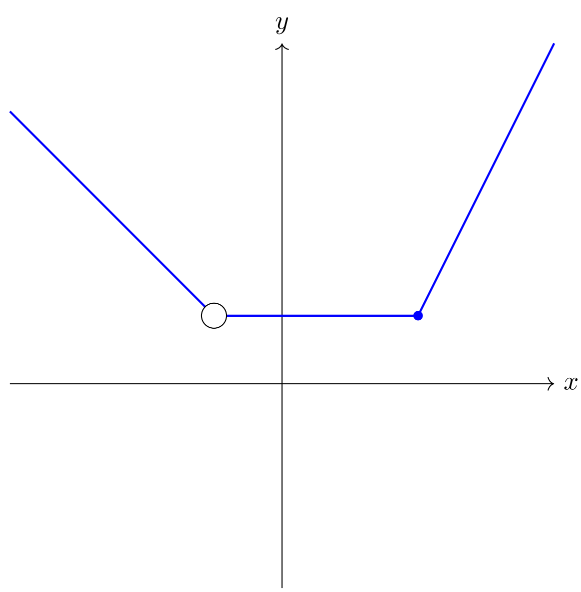
17. A mobile phone plan costs £20 per month. This includes 10 gigabytes (GB) of data. For each additional GB of data used, there is a charge of £2.

(a) Write a piecewise-defined function $C(d)$ for the monthly cost, where d is the number of GB of data used.

(b) What is the cost for using 8 GB? What about 15 GB?

18. Write a piecewise definition for the function $f(x) = |x - 2| - |x + 2|$. Simplify the expression in each interval.

19. Find an algebraic rule for the piecewise-defined function shown in the graph.



20. The Heaviside step function is defined as $H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$. Sketch the graph of $y = H(x) - H(x - 2)$. What does this new function represent?

Part IV: Functions in Geometry

21. Let s be the side length of an equilateral triangle.

(a) Express the height h of the triangle as a function of s, $h(s)$.
(b) Express the area A of the triangle as a function of s, $A(s)$.
(c) What are the domains of these functions in a geometric context?

22. A 10-foot ladder is leaning against a wall. Let x be the distance from the base of the wall to the bottom of the ladder. Let y be the height the ladder reaches on the wall. Express y as a function of x, $y(x)$. What is the domain of this function?

23. A point P moves along the line segment from A(0,0) to B(6,8). The x-coordinate of P is t.

(a) Express the y-coordinate of P as a function of t.
(b) Express the distance from P to the origin as a function of t, $d(t)$.
(c) What is the domain of these functions?

24. An open-topped box is to be made from a square piece of cardboard, 30 cm on a side, by cutting out equal squares of side length x from each of the four corners and folding up the sides.

(a) Express the volume V of the box as a function of x, $V(x)$.
(b) What is the domain of the function $V(x)$?

25. Consider a circle with equation $x^2 + y^2 = 25$.

(a) Express the length L of a horizontal chord as a function of its y-coordinate, $L(y)$.
(b) What is the domain of $L(y)$?

26. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a function that reflects a point across the x-axis. What is the algebraic rule for this function? That is, find an expression for $f(x, y)$.

27. Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be a function that rotates a point 90 degrees counter-clockwise about the origin.

(a) By testing points like (1,0), (0,1), and (2,3), find the rule for $g(x, y)$.
(b) Use the distance formula to prove that the distance from the origin to a point P is the same as the distance from the origin to its image g(P).

28. Let $h : \mathbb{R}^2 \to \mathbb{R}$ be a function that gives the perpendicular distance from a point (x, y) to the line $y = x$. Find the rule for $h(x, y)$.

Part V: Formal Definitions and Advanced Concepts

29. Which of the following sets of ordered pairs represents a function? If not, explain why.

(a) $\{(1, 2), (2, 4), (3, 6), (4, 8)\}$
(b) $\{(1, 2), (1, 3), (2, 4), (3, 5)\}$
(c) $\{(x, x^2) \mid x \in \mathbb{R}\}$
(d) $\{(x^2, x) \mid x \in \mathbb{R}\}$

30. Consider the functions $f(x) = \frac{x^2-1}{x-1}$ and $g(x) = x + 1$. Are these functions equal? Justify your answer based on the formal definition of function equality.

31. A function f is called even if $f(-x) = f(x)$ for all x in its domain. A function is called odd if $f(-x) = -f(x)$ for all x in its domain.

(a) What kind of geometric symmetry does the graph of an even function have? What about an odd function?

(b) Classify the following functions as even, odd, or neither: x^2, x^3, $x + 1$, $|x|$, $x^5 - x$.

32. A function f is one-to-one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$. A visual test for this is the Horizontal Line Test. Are the following functions one-to-one on their natural domains?

(a) $f(x) = 2x + 1$

(b) $g(x) = x^2$

(c) $h(x) = x^3$

(d) $k(x) = \sqrt{x}$

33. Let $f : A \to B$. The range of f is a subset of the codomain B. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ where the range is not equal to the codomain.

34. Let $f(x) = ax + b$. For what values of a and b is it true that $f(f(x)) = 4x + 3$?

35. Let A be a set with m elements and B be a set with n elements. How many different functions are there from A to B?

36. Prove that any function f with a domain symmetric about the origin (i.e., if x is in the domain, so is $-x$) can be written as the sum of an even function and an odd function.

Remark. Consider the functions $g(x) = \frac{f(x)+f(-x)}{2}$ and $h(x) = \frac{f(x)-f(-x)}{2}$.

Part VI: Challenge Problems

37. The "floor" function, $\lfloor x \rfloor$, gives the greatest integer less than or equal to x. The "ceiling" function, $\lceil x \rceil$, gives the smallest integer greater than or equal to x.

(a) Sketch the graphs of $f(x) = \lfloor x \rfloor$ and $g(x) = \lceil x \rceil$.

(b) What are the domain and range of each function?

(c) Sketch the graph of $h(x) = x - \lfloor x \rfloor$. What does this function represent?

38. Find the point on the line $y = x + 1$ that is closest to the point (4, 1).

Remark. Define a squared-distance function $D(x)$ from a point $(x, x + 1)$ on the line to (4,1). $D(x)$ will be a quadratic expression in the form $ax^2 + bx + c$. Complete the square to find the value of x that minimizes this distance.

39. A function f satisfies the property $f(xy) = f(x) + f(y)$ for all $x, y > 0$. If $f(2) = 1$, find the values of $f(4)$, $f(8)$, $f(1)$, and $f(1/2)$. (This is a property of the logarithm function).

40. Find the domain of the function $f(x) = \sqrt{1 - \sqrt{2 - \sqrt{3 - x}}}$.

4.6 Operations on Points and Functions

With the plane represented as the set of ordered pairs $\mathbb{R}^2$, we can define arithmetic operations on points. These operations form the algebraic basis for geometric transformations.

Scalar Multiplication: Dilations and Reflections

We can multiply a point by a real number (a scalar). This operation scales the point's coordinates.

Definition 4.6.1. ***Scalar Multiplication***. Let $A = (a_1, a_2)$ be a point in the plane and let c be a real number. The scalar product cA is the point defined by:

$$cA = (ca_1, ca_2)$$

Geometrically, this corresponds to a dilation (stretching or shrinking) centred at the origin, possibly combined with a reflection.

- If $|c| > 1$, the point A is stretched away from the origin.
- If $0 < |c| < 1$, the point A is shrunk towards the origin.
- If $c < 0$, the point is also reflected through the origin.

The mapping $A \mapsto rA$ for a positive number r is called a dilation by a factor of r. The mapping $A \mapsto -A$ is a reflection through the origin.

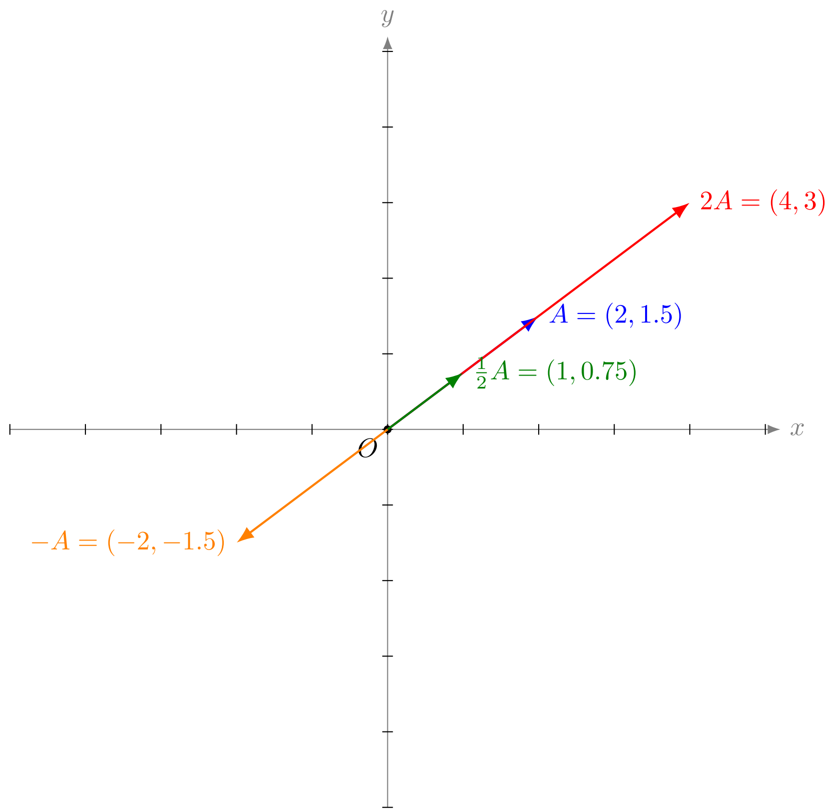


Figure 4.12: Scalar multiplication as dilation and reflection.

Scalar multiplication has a predictable effect on the distance between points.

***Theorem* 4.6.1.** Let A and B be points and let c be a scalar. Then $d(cA, cB) = |c| \cdot d(A, B)$.

Proof. Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$. Then $cA = (ca_1, ca_2)$ and $cB = (cb_1, cb_2)$.

$$\begin{aligned} d(cA, cB)^2 &= (cb_1 - ca_1)^2 + (cb_2 - ca_2)^2 \\ &= (c(b_1 - a_1))^2 + (c(b_2 - a_2))^2 \\ &= c^2(b_1 - a_1)^2 + c^2(b_2 - a_2)^2 \\ &= c^2\left((b_1 - a_1)^2 + (b_2 - a_2)^2\right) \\ &= c^2 \cdot d(A, B)^2 \end{aligned}$$

Taking the square root of both sides gives $d(cA, cB) = \sqrt{c^2} \cdot d(A, B) = |c| \cdot d(A, B)$. ■

Addition of Points: Translations

We can also define the addition and subtraction of points.

Definition 4.6.2. ***Addition and Subtraction of Points***. Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$ be points. Their sum $A + B$ and difference $A - B$ are defined as:

$$A + B = (a_1 + b_1, a_2 + b_2)$$

$$A - B = (a_1 - b_1, a_2 - b_2)$$

Geometrically, the sum $A + B$ can be found by constructing a parallelogram with vertices at the origin O, A, and B. The fourth vertex of this parallelogram is the point $A + B$. This is known as the parallelogram law.

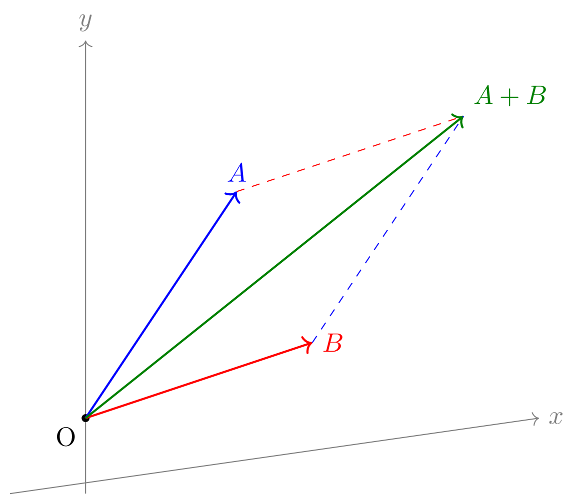


Figure 4.13: The parallelogram law for the addition of points.

The addition of a fixed point A to every point in the plane defines a fundamental transformation.

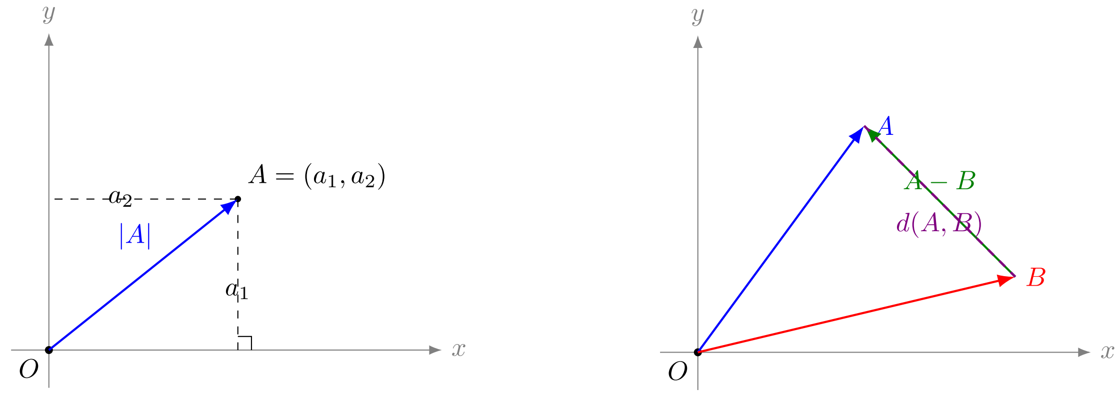
Definition 4.6.3. ***Translation***. The translation by a point A, denoted T_A, is the function that maps any point P to $P + A$.

$$T_A : \mathbb{R}^2 \to \mathbb{R}^2 \qquad P \mapsto P + A$$

We introduce a special notation for the distance of a point from the origin, called the norm.

Definition 4.6.4. ***Norm***. The norm of a point A, denoted $|A|$, is its distance from the origin: $|A| = d(A, O)$. If $A = (a_1, a_2)$, then $|A| = \sqrt{a_1^2 + a_2^2}$.

This notation allows us to express the distance between any two points concisely: $d(A, B) = |A - B|$.



(a) Norm of a point: $|A| = d(A, O) = \sqrt{a_1^2 + a_2^2}$.

(b) $d(A, B) = |A - B|$.

Figure 4.14: (Left) The norm as the length of $\overrightarrow{OA}$. (Right) Distance equals norm of a difference.

Remark. This notation is powerful as it recasts the distance between two points, A and B, as the norm of their difference. It follows directly from the symmetry of distance that $d(A, B) = d(B, A)$, which implies $|A - B| = |B - A|$. This algebraic framework is further strengthened by the property for scalar multiplication, $|cA| = |c||A|$, which relates dilation to the norm.

Combining Functions

Just as we can perform arithmetic on points, we can also combine real-valued functions (i.e., functions from $\mathbb{R} \to \mathbb{R}$) to create new ones.

Definition 4.6.5. ***Operations on Functions***. Let f and g be functions and c be a scalar. We define the sum $f + g$ and the scalar product cf as follows:

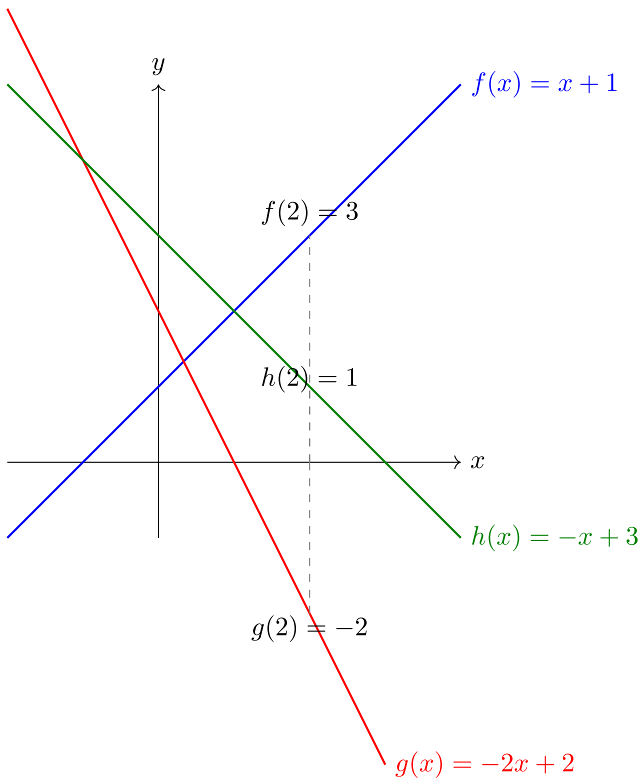
- $(f + g)(x) = f(x) + g(x)$
- $(cf)(x) = c \cdot f(x)$

The domain of the new function is the intersection of the domains of the original functions. That is, $\text{Domain}(f + g) = \text{Domain}(f) \cap \text{Domain}(g)$.

Example 4.6.1. (Combining Linear Functions). Let $f(x) = x + 1$ and $g(x) = -2x + 2$. Find the function $h = f + g$. The sum function $h(x)$ is given by:

$$h(x) = (f + g)(x) = f(x) + g(x) = (x + 1) + (-2x + 2) = -x + 3.$$

Since the domains of f and g are both $\mathbb{R}$, the domain of h is also $\mathbb{R}$. Visually, the graph of h is obtained by adding the y-coordinates of f and g at each x-value.



Formal Treatment of Function Operations Using the formal definition of a function as a set of ordered pairs, we can define these operations with greater rigour.

Definition 4.6.6. ***Formal Operations on Functions***. Let f and g be functions.

- The sum $f + g$ is the set of ordered pairs $\{(a, b + c) \mid (a, b) \in f \text{ and } (a, c) \in g\}$.
- The difference $f - g$ is the set of ordered pairs $\{(a, b - c) \mid (a, b) \in f \text{ and } (a, c) \in g\}$.

This construction defines a valid function. For any input a in the common domain of f and g, the output $f(a) = b$ is unique, and the output $g(a) = c$ is unique. Therefore, their sum $b + c$ (or difference $b - c$) is also unique, satisfying the condition for a function. These operations inherit properties from the addition of real numbers.

***Proposition* 4.6.1.** *Commutativity of Function Addition.* For any two real-valued functions f and g, $f + g = g + f$.

Proof. Let (a, d) be an ordered pair in $f + g$. By definition, there exist unique numbers b and c such that $(a, b) \in f$, $(a, c) \in g$, and $d = b + c$. Since the addition of real numbers is commutative, $b + c = c + b$. Therefore, we can write $d = c + b$, which means that the pair (a, d) is in $g + f$. Conversely, if a pair (a, d) is in $g + f$, the same argument shows it must be in $f + g$. As the two sets of ordered pairs contain the same elements, $f + g = g + f$. ∎

***Proposition* 4.6.2.** *Associativity of Function Addition.* For any three real-valued functions f, g, and h, $(f + g) + h = f + (g + h)$.

Proof. Let (a, d) be in $(f + g) + h$. By definition, there is a unique y such that $(a, y) \in f + g$ and a unique c such that $(a, c) \in h$, with $d = y + c$. Furthermore, for $(a, y) \in f + g$, there are unique numbers b_1 and b_2 such that $(a, b_1) \in f$, $(a, b_2) \in g$, and $y = b_1 + b_2$. Substituting this in, we get $d = (b_1 + b_2) + c$. Since addition of real numbers is associative, $(b_1 + b_2) + c = b_1 + (b_2 + c)$. Let $z = b_2 + c$. Since $(a, b_2) \in g$ and $(a, c) \in h$, the pair (a, z) is in $g + h$. Now we have $d = b_1 + z$, where $(a, b_1) \in f$ and $(a, z) \in g + h$. This means that (a, d) is in $f + (g + h)$. The argument is reversible, showing that the two sets are identical. ■

Multiplication and Division of Functions Following the same pattern, we can define multiplication and division of functions.

Definition 4.6.7. ***Formal Multiplication and Division***. Let f and g be functions.

- The product fg is the set of ordered pairs $\{(a, bc) \mid (a, b) \in f \text{ and } (a, c) \in g\}$.
- The quotient f/g is the set of ordered pairs $\{(a, b/c) \mid (a, b) \in f \text{ and } (a, c) \in g, c \neq 0\}$.

The domain of the product fg is the intersection of the domains of f and g. For the quotient f/g, we have an additional restriction: we must exclude any x from the domain for which $g(x) = 0$, as division by zero is undefined.

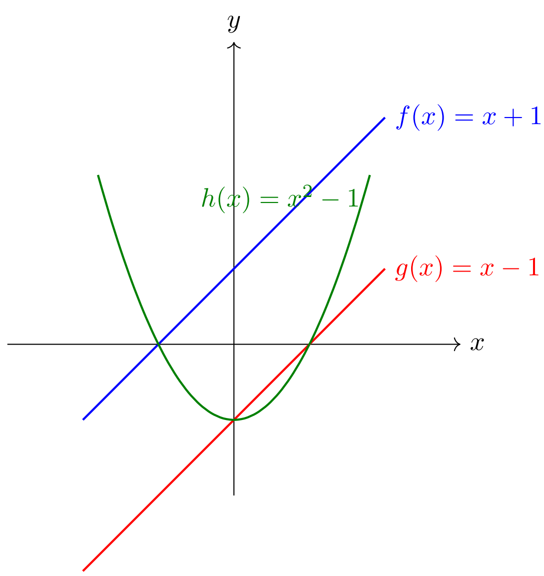
$$\text{Domain}(f/g) = \{x \in \text{Domain}(f) \cap \text{Domain}(g) \mid g(x) \neq 0\}.$$

Multiplication of functions is commutative and associative, inheriting these properties from the multiplication of real numbers.

Example 4.6.2. (Multiplication of Functions). Let $f(x) = x + 1$ and $g(x) = x - 1$. Find the function $h = fg$. The product function $h(x)$ is given by:

$$h(x) = (fg)(x) = f(x)g(x) = (x + 1)(x - 1) = x^2 - 1.$$

The domain is $\text{Domain}(f) \cap \text{Domain}(g) = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$. The product of two linear functions results in a quadratic function.



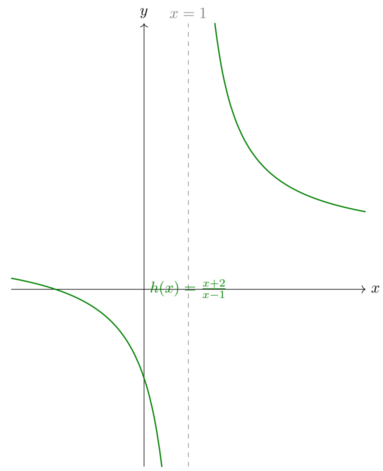
Example 4.6.3. (Division of Functions). Let $f(x) = x + 2$ and $g(x) = x - 1$. Find the function $h = f/g$. The quotient function $h(x)$ is given by:

$$h(x) = \frac{f(x)}{g(x)} = \frac{x + 2}{x - 1}.$$

The domains of f and g are both $\mathbb{R}$. However, for $h(x)$, we must exclude the value of x where the denominator $g(x)$ is zero.

$$g(x) = x - 1 = 0 \implies x = 1.$$

Therefore, the domain of h is $\{x \in \mathbb{R} \mid x \neq 1\}$.



4.6.1 Analytic Descriptions: Segments, Rays, and Lines

We recast 2.1.1 and 2.1.2 analytically, without redefining them. We continue developing the analytic language for straight figures in the plane using the point operations already introduced in 4.6.1 and 4.6.3 (cf. 2.1.1 and 2.1.2). Throughout, points are written as ordered pairs, and distances as in the distance formula.

Line Segments

***Proposition* 4.6.3.** *Analytic Description of a Segment.* Let P and Q be points in the plane. Then

$$\overline{PQ} = \{\, P + t\,(Q - P) : 0 \le t \le 1 \,\}.$$

Equivalently, every point on $\overline{PQ}$ can be written $(1-t)P + tQ$ with $0 \le t \le 1$. The length of $\overline{PQ}$ is $d(P, Q)$.

The description $(1-t)P + tQ$ shows that $\overline{PQ}$ and $\overline{QP}$ are the same set; the parameter simply runs in the reverse direction.

Example 4.6.4. (Midpoint and Translation). The midpoint of $\overline{PQ}$ is obtained by taking $t = \frac{1}{2}$:

$$M = P + \tfrac{1}{2}\,(Q - P) = \tfrac{1}{2}(P + Q).$$

If $Q = P + A$, then $\overline{PQ} = T_P(\,\overline{OA}\,)$, i.e. it is the translation by P of the segment from the origin to A (Figure 4.15).

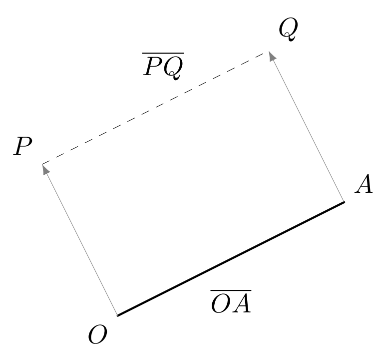


Figure 4.15: $\overline{PQ}$ is the translate of $\overline{OA}$ when $Q = P + A$.

Directed Segments (Located Vectors)

Definition 4.6.8. ***Directed Segment***. An ordered pair of points (P, Q) determines a directed segment (located vector), denoted $\overrightarrow{PQ}$. The point P is the initial point and Q is the terminal point. Its magnitude is $d(P, Q)$, and its direction is that of $Q - P$.

We locate $\overrightarrow{PQ}$ at P. Reversing the order changes the direction.

Rays

Proposition **4.6.4.** *Analytic Description of a Ray.* Let $A \neq O$. The ray with vertex P in the direction of A is the set

$$\{\, P + tA : t \geq 0 \,\}.$$

Equivalently, given distinct points P and Q, the ray from P through Q is $\{\, P + t(Q - P) : t \geq 0 \,\}$.

Example 4.6.5. (Computing Points on a Ray). Let $P = (-1, 3)$ and $A = (2, 1)$. Then $P + 5A = (9, 8)$ lies on the ray, as does $P + \frac{1}{2}A = (0, \frac{7}{2})$.

Remark. If $c > 0$, the rays $\{P + tA : t \geq 0\}$ and $\{P + s(cA) : s \geq 0\}$ coincide, since $t \mapsto s = \frac{t}{c}$ gives the same set of points. Thus multiplying a direction by a positive scalar does not change the ray.

Definition 4.6.9. ***Same Direction***. Non-zero points A and B have the same direction if there exists $c > 0$ with $B = cA$. Directed segments $\overrightarrow{PQ}$ and $\overrightarrow{MN}$ have the same direction if $Q - P = c\,(N - M)$ for some $c > 0$.

Example 4.6.6. (Matching Direction and Length). Let $P = (-3, 5)$, $Q = (1, 2)$, and $M = (4, 1)$. To find N so that $\overrightarrow{MN}$ has the same direction as $\overrightarrow{PQ}$ and the same length, set $N - M = Q - P$. Then

$$N = M + (Q - P) = (4, 1) + (1, 2) - (-3, 5) = (8, -2).$$

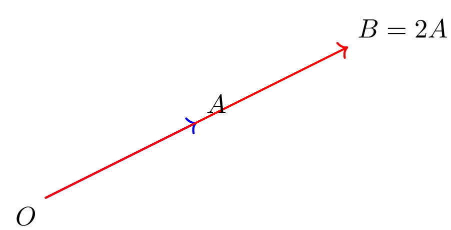


Figure 4.16: Two located vectors with the same direction.

Parallelism and Lines

Definition 4.6.10. ***Parallel Directions***. Non-zero points A and B are parallel if there exists $c \neq 0$ with $A = cB$. Directed segments (or ordinary segments) $\overrightarrow{PQ}$ and $\overrightarrow{MN}$ are parallel if $Q - P = c\,(N - M)$ for some $c \neq 0$.

Definition 4.6.11. ***Parametric Representation of a Line***. Given a point P and a non-zero direction A, the line through P parallel to A is the set

$$L = \{\, P + tA : t \in \mathbb{R} \,\}.$$

This is a parametric representation with parameter t.

Negative and positive values of t trace the entire line (Figure 4.17).

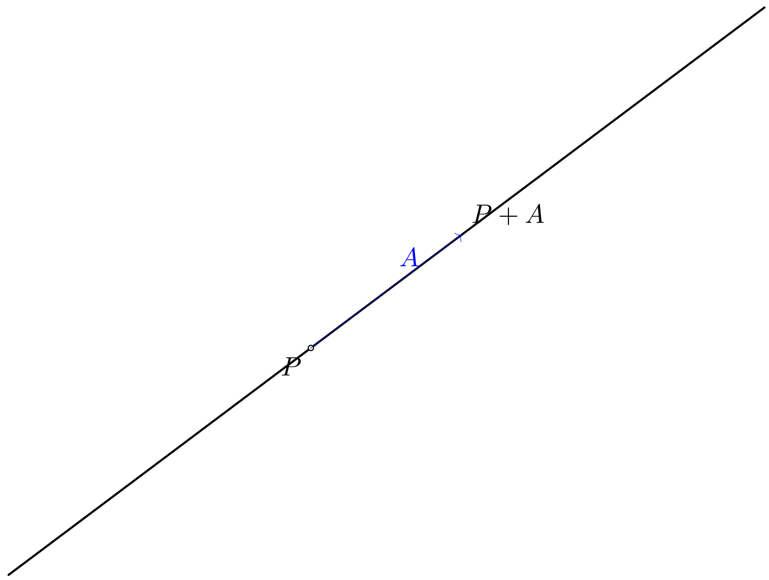


Figure 4.17: The line $\{P + tA\}$ with indicated direction vector A.

Example 4.6.7. (Line Through Two Points). Let $P = (1, 5)$ and $Q = (-2, 3)$. Then $A = Q - P = (-3, -2)$, and

$$\{P + tA\} = \{(1, 5) + t(-3, -2) : t \in \mathbb{R}\}.$$

In coordinates, $x(t) = 1 - 3t$, $y(t) = 5 - 2t$. At $t = 0$ the point is P, and at $t = 1$ the point is Q. The intersection with the x-axis occurs when $y(t) = 0$, i.e. when $5 - 2t = 0$, giving $t = \frac{5}{2}$ and point $(-\frac{13}{2}, 0)$.

Example 4.6.8. (Intersection of Two Lines). Consider

$$(1, 2) + t(3, 4) \quad \text{and} \quad (-1, 1) + s(2, -1).$$

Solve for t, s from $1 + 3t = -1 + 2s$ and $2 + 4t = 1 - s$. The solution is $t = -\frac{1}{5}$, $s = \frac{2}{5}$, so the common point is $(1, 2) + (-\frac{1}{5})(3, 4) = (\frac{2}{5}, \frac{6}{5})$.

Example 4.6.9. (Line–Circle Intersection). Let the line be $(-1, 2) + t(3, -4)$ and the circle be $x^2 + y^2 = 4$. Substituting $x(t) = -1 + 3t$, $y(t) = 2 - 4t$ gives

$$(-1 + 3t)^2 + (2 - 4t)^2 = 4 \iff 25t^2 - 22t + 1 = 0.$$

Solving yields $t = \dfrac{7 \pm \sqrt{96}}{25}$, giving two intersection points. If the discriminant were negative, there would be no real intersection.

Ordinary Equation of a Line

Eliminating the parameter from $x = p_1 + ta_1$, $y = p_2 + ta_2$ yields a linear equation in x and y. For example, with $P = (3, 5)$ and $A = (-2, 7)$, we have $x = 3 - 2t$, $y = 5 + 7t$. Multiplying by 7 and 2 respectively and adding cancels t:

$$7x + 2y = 7(3 - 2t) + 2(5 + 7t) = 31.$$

***Theorem* 4.6.2. Ordinary Line Equation**. Every non-vertical/non-horizontal line admits an equation of the form

$$ax + by = c,$$

with real a, b, c and not both a and b zero. Conversely, any such equation represents a straight line.

Proof. Given $L = \{P + tA\}$ with $A = (a_1, a_2) \neq (0, 0)$, one can choose $(a, b) = (a_2, -a_1)$, which is perpendicular to A. Then $ax + by$ is constant on L; setting $c = ax_P + by_P$ gives the required equation. Conversely, the solution set of $ax + by = c$ is an affine translate of the kernel of the linear form $(x, y) \mapsto ax + by$, hence a straight line. ∎

4.7 Exercises

Part I: Arithmetic of Points

1. Let $A = (3, -1)$, $B = (-2, 5)$, and $c = 3$. Compute the following:

(a) $A + B$
(b) $A - B$
(c) cA
(d) $2A + 3B$
(e) $|A|$ (the norm of A)
(f) $|B|$
(g) $|A + B|$
(h) $d(A, B)$ and verify that it equals $|A - B|$.

2. For each point P given below, plot the points P, $2P$, $\frac{1}{2}P$, and $-P$ on a single coordinate plane. Describe the geometric effect of each scalar multiplication.

(a) $P = (4, 2)$
(b) $P = (-3, -1)$

3. Let $A = (1, 4)$ and $B = (5, 2)$. Draw the parallelogram with vertices at O, A, B, and $A + B$. Find the coordinates of $A + B$.

4. Let $P = (6, 2)$. A translation T_A maps the origin $O = (0, 0)$ to the point P.

(a) What are the coordinates of the point A?
(b) What is the image of the point $Q = (-1, 3)$ under this translation?

5. Let $A = (a_1, a_2)$. Prove algebraically that $|-A| = |A|$. What does this mean geometrically?

6. Let $A = (6, 8)$ and $c = 5$.

(a) Calculate $|A|$ and $|cA|$.
(b) Verify that $|cA| = |c||A|$.

7. Find a point B such that $A + B = C$, where $A = (-1, 7)$ and $C = (4, 2)$.

8. Let $A = (1, 1)$ and $B = (5, 4)$. Find the point C such that OABC forms a parallelogram (where O is the origin). Is this point unique? What if the vertices were specified in the order OBAC?

Part II: Operations on Functions

9. Let $f(x) = x^2 - 4$ and $g(x) = x + 2$. For each of the following new functions, state its rule and its domain.

(a) $f + g$
(b) $f - g$
(c) fg

(d) f/g

(e) g/f

(f) $3f - 2g$

10. Let $f(x) = \sqrt{x}$ and $g(x) = \sqrt{4-x}$. Find the functions $f+g$, fg, and f/g, and state the domain for each.

11. Let $f(x) = \frac{1}{1+x}$.

(a) Find the rule for the composite function $(f \circ g)(x) = f(g(x))$. For which x does this make sense?

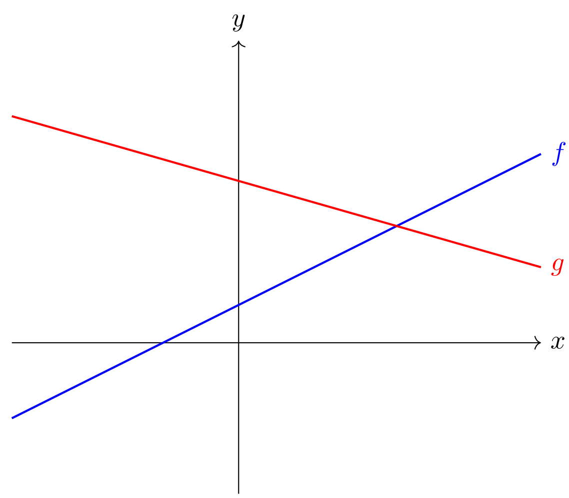
(b) Find the rule for the function $f(cx)$.

(c) Compare the functions $f(x+y)$ and $f(x) + f(y)$. Are they equal?

(d) For which numbers c is there a number x such that $f(cx) = f(x)$?

(e) $\star$ For which numbers c is it true that $f(cx) = f(x)$ for two different numbers x?

12. The graphs of two functions, f and g, are shown below. Sketch the graph of the function $f + g$.



13. Let $f(x) = x$, $g(x) = x^2$, and $h(x) = 1$. Sketch the graphs of f, g, and fg on the same axes. Then sketch the graphs of g, h, and $g + h$.

14. Prove that the product of two even functions is an even function.

15. Prove that the product of two odd functions is an even function.

16. Prove that the product of an even function and an odd function is an odd function.

17. If f and g are both linear functions, prove that $f+g$ and cf (for any scalar c) are also linear functions. What can you say about the function fg?

18. Let f be any function. Define $g(x) = \frac{f(x)+f(-x)}{2}$ and $h(x) = \frac{f(x)-f(-x)}{2}$.

(a) Show that g is an even function and h is an odd function.

(b) Show that $f = g + h$.

(c) For $f(x) = x^2 - 3x + 2$, find its even part $g(x)$ and its odd part $h(x)$.

19. Is function subtraction commutative? That is, is $f - g = g - f$? If not, provide a counterexample.

20. Let f be any function with domain $\mathbb{R}$. Prove that f can be written as $f = E + O$, where E is an even function and O is an odd function. Prove that this representation is unique.

Remark. Consider the functions $E(x) = \frac{f(x)+f(-x)}{2}$ and $O(x) = \frac{f(x)-f(-x)}{2}$.

21. For any function f, define $|f|$ by $|f|(x) = |f(x)|$. Also define $\max(f, g)(x) = \max(f(x), g(x))$ and $\min(f, g)(x) = \min(f(x), g(x))$. Find expressions for $\max(f, g)$ and $\min(f, g)$ using the absolute value function.

Remark. Recall that $\max(a, b) = \frac{a+b+|a-b|}{2}$.

22. The positive part of a function f is defined as $f_+ = \max(f, 0)$ and the **negative part** is $f_- = \max(-f, 0)$. Show that $f = f_+ - f_-$ and $|f| = f_+ + f_-$.

23. Let $f(x) = \frac{ax+b}{cx+d}$. Find the conditions on a, b, c, d for which $f(f(x)) = x$ for all x in the domain.

Part III: Analytic Descriptions of Segments, Rays, and Lines

24. Let $P = (1, 2)$ and $Q = (5, 5)$.

(a) Write the parametric representation of the line segment $\overline{PQ}$.

(b) Find the point on the segment that is $\frac{1}{3}$ of the way from P to Q.

(c) Find the midpoint of the segment.

25. For each pair of points P and Q, give a parametric equation for the line passing through them.

(a) P = (1, 4), Q = (3, 0)

(b) P = (-2, 5), Q = (-2, -1)

(c) P = (0, 0), Q = (a, b)

26. A point $X(t) = P + t(Q - P)$ lies on the line L_{PQ}.

(a) Where is $X(t)$ relative to P and Q if $0 < t < 1$?

(b) Where is $X(t)$ if $t > 1$?

(c) Where is $X(t)$ if $t < 0$?

27. Let $P = (3, 1)$. The ray from P passes through $Q = (5, 5)$.

(a) Write an analytic description of this ray.

(b) Does the point $(4, 3)$ lie on this ray?

(c) Does the point $(2, -1)$ lie on this ray?

28. Let $\overrightarrow{PQ}$ be a directed segment with P=(1,1) and Q=(4,5). Find a point N such that the directed segment $\overrightarrow{MN}$, with M=(-2,3), has the same direction and magnitude as $\overrightarrow{PQ}$.

29. Let L be the line defined by $P + tA$ where $P = (2, -1)$ and $A = (3, 2)$. Which of the following points are on L?

(a) $(5, 1)$

(b) $(-1, -3)$

(c) $(0, 0)$

30. Find the point of intersection of the following pairs of lines.

(a) $L_1 : (2, 1) + t(1, 2)$ and $L_2 : (5, 8) + s(-1, 1)$

(b) The line through (0,5) and (3,2), and the line through (-1,1) and (1,5).

31. Convert the following parametric equations into an ordinary line equation of the form $ax + by = c$.

(a) $x(t) = 2 + 3t, y(t) = 1 - t$

(b) The line through $P = (4, -2)$ and $Q = (6, 3)$.

32. Convert the ordinary equation $2x - 5y = 10$ into a parametric representation.

Remark. Find two points on the line, or find one point and the direction vector.

Is the parametric representation unique?

33. Find the intersection points of the line $P + tA$ with the circle $x^2 + y^2 = 25$, where $P = (1, 2)$ and $A = (1, 1)$.

34. A triangle has vertices $A = (0, 0)$, $B = (6, 0)$, and $C = (3, 4)$.

(a) Find the parametric equation for the median from vertex C to the side $\overline{AB}$.

(b) Find the parametric equation for the altitude from vertex C to the side $\overline{AB}$.

35. Show that the directed segments $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are parallel, where P=(1,1), Q=(3,5), R=(0,3), S=(-1,1). Do they have the same direction?

Part IV: Proofs Using Vector/Point Algebra

36. Let $M = \frac{1}{2}(A + B)$ be the midpoint of $\overline{AB}$. Prove that $d(A, M) = d(B, M) = \frac{1}{2}d(A, B)$ using the norm notation $d(P, Q) = |P - Q|$.

37. Use point algebra to prove that the diagonals of a parallelogram bisect each other.

Remark. Let the vertices be O, A, B, and $A + B$. Find the midpoint of each diagonal.

38. Let A, B, C, D be four points. Let M_{AB} be the midpoint of $\overline{AB}$, M_{BC} of $\overline{BC}$, etc. Prove that the quadrilateral $M_{AB}M_{BC}M_{CD}M_{DA}$ is a parallelogram (Varignon's theorem).

39. Centroid of a Triangle. The centroid of $\triangle ABC$ is the point $G = \frac{1}{3}(A + B + C)$. Show that the centroid lies on each of the three medians of the triangle.

Remark. Show that G lies on the segment connecting vertex A to the midpoint of $\overline{BC}$.

40. Prove that translations preserve distances. That is, for any points P, Q and a translation T_A, prove that $d(T_A(P), T_A(Q)) = d(P, Q)$.

41. Prove that dilations preserve collinearity. That is, if P, Q, R are collinear, then for any scalar c, the points cP, cQ, cR are also collinear.

42. The line segment $(1 - t)P + tQ$ is called a convex combination of P and Q. Show that for any function $f(x) = mx + b$, we have $f((1 - t)x_1 + tx_2) = (1 - t)f(x_1) + tf(x_2)$. What does this mean geometrically for the graph of a line?

43. Let A and B be non-zero points. Prove that $|A + B| = |A| + |B|$ if and only if A and B have the same direction. What is the geometric interpretation of this?

44. $\star$ Let f satisfy $f(x + y) = f(x) + f(y)$ for all x, y.

(a) Prove that $f(x_1 + \cdots + x_n) = f(x_1) + \cdots + f(x_n)$.

(b) Prove that there is a number c such that $f(x) = cx$ for all rational numbers x.

Remark. First prove it for integers, then for reciprocals of integers, then for all rational numbers.

45. For which functions f is there a function g such that $f = g^2$? For which functions f is there a function g such that $f = 1/g$?

Part V: Challenge Problems

46. A laser beam starts at point P(1,2), reflects off the y-axis, and then hits the point Q(5,4). Find the point of reflection on the y-axis.

Remark. The angle of incidence equals the angle of reflection. A simpler method is to reflect one of the points across the y-axis.

47. **Perpendicularity Condition.** Let $A = (2, 3)$ and $B = (3, -2)$ be two direction vectors.

(a) Calculate $|A|^2$, $|B|^2$, and $|A - B|^2$.

(b) Use the Converse of Pythagoras' Theorem to prove that the triangle with sides corresponding to A, B, and $A - B$ is right-angled at the origin. Conclude that the directions A and B are perpendicular.

(c) $\star$ Generalise this: Prove that two vectors $A = (a_1, a_2)$ and $B = (b_1, b_2)$ are perpendicular if and only if $a_1b_1 + a_2b_2 = 0$. (This quantity is called the dot product).

48. Find the parametric equation of the line that is the perpendicular bisector of the segment connecting $A = (-2, 3)$ and $B = (4, 7)$.

49. Let $f(x) = x^2$ and $g(x) = -x^2 + 8x - 12$.

(a) Find the x-coordinates of the intersection points of the two graphs.

(b) Define a function $d(x) = g(x) - f(x)$. What does this function represent geometrically?

50. Suppose that $f(y) - f(x) \leq (y - x)^2$ for all x, y. Prove that f must be a constant function.

Remark. Divide the interval from x to y into n equal pieces and apply the inequality to each subinterval.

51. Let $I(x) = x$. Suppose g is a function such that $g(x) \neq g(y)$ if $x \neq y$. Prove that there is a function f such that $f \circ g = I$.

52. Suppose f is a function such that for every number b, we can find a number a such that $b = f(a)$. Prove there is a function g such that $f \circ g = I$.

53. Suppose $f \circ g = I$ and $h \circ f = I$. Prove that $g = h$.

Remark. Use the associativity of function composition.

Chapter 5

Graphs

The algebraic definitions of functions and points in the plane are made more intuitive through their geometric representations. We begin by revisiting the fundamental connection between the set of real numbers and the points on a straight line.

5.1 Graphs of Functions

By 4.4.4, the graph is the set of ordered pairs $(x, f(x))$.

Definition 5.1.1. ***Graph of a Function***. The graph of a function f is the set of all points (x, y) in the coordinate plane such that $y = f(x)$. Formally, the graph is the set of ordered pairs that constitutes the function.

From the definition of a function (4.4.4), for any x in the domain, there is exactly one corresponding output $f(x)$. This leads to a simple geometric test for whether a set of points is the graph of a function. By Theorem 4.4.1, a curve represents a function iff no vertical line meets it more than once. An equation such as $x^2 + y^2 = 4$ does not define y as a function of x, because solving for y yields $y = \pm\sqrt{4 - x^2}$, providing two outputs for a single input x in the interval $(-2, 2)$.

Reading Domain and Range from a Graph

The domain and range of a function can be determined by inspecting its graph. The domain is the set of all x-coordinates covered by the graph, which can be visualised by projecting the graph onto the x-axis. The range is the set of all y-coordinates, found by projecting the graph onto the y-axis.

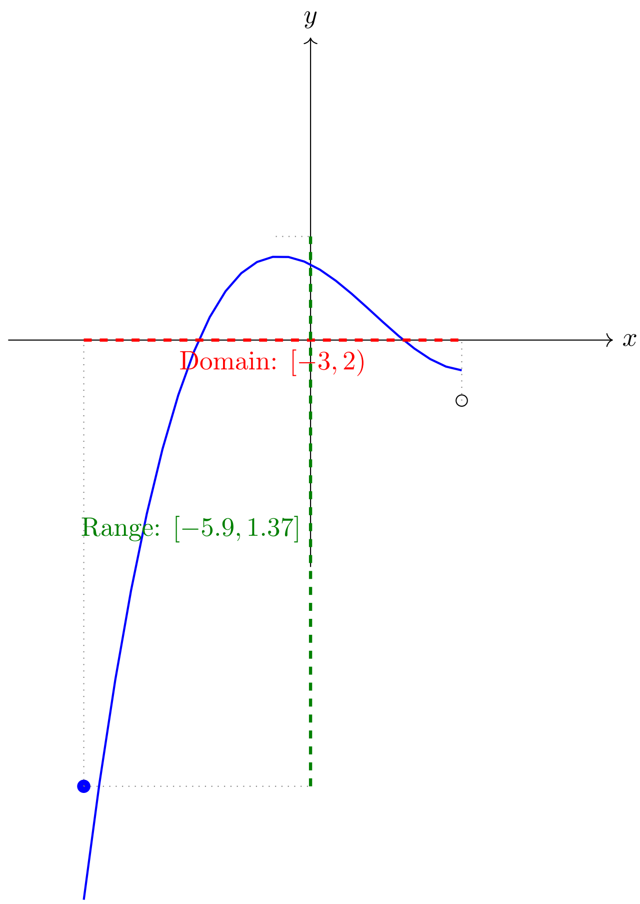


Figure 5.1: The domain and range projected onto the axes. A filled circle indicates inclusion of the endpoint, while an open circle indicates exclusion.

Solving Equations and Inequalities Graphically

The graph provides a visual method for solving equations and inequalities. The solutions to an equation $f(x) = g(x)$ are the x-coordinates of the points where the graphs of $y = f(x)$ and $y = g(x)$ intersect. Similarly, the solution to the inequality $f(x) < g(x)$ is the set of x-values for which the graph of f lies below the graph of g.

Example 5.1.1. (Graphical Solution). Solve the inequality $x^2 - 3 < x - 1$. Let $f(x) = x^2 - 3$ and $g(x) = x - 1$. We plot both functions, as shown in Figure 5.2. The intersections occur where $x^2 - 3 = x - 1$, which simplifies to $x^2 - x - 2 = 0$, or $(x-2)(x+1) = 0$. The solutions are $x = -1$ and $x = 2$. The inequality $f(x) < g(x)$ holds for x-values where the parabola is below the line. From the graph, this occurs between the intersection points. The solution is the interval $(-1, 2)$.

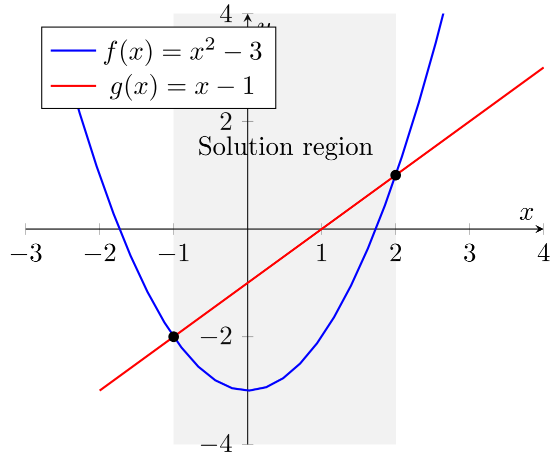


Figure 5.2: Solving $x^2 - 3 < x - 1$ graphically.

Monotonicity and Local Extrema

A function's graph reveals intervals over which its values are increasing or decreasing. This behaviour is called monotonicity.

Definition 5.1.2. ***Monotonic Functions***. Let f be a function defined on an interval I.

- f is increasing on I if for any $x_1, x_2 \in I$ with $x_1 < x_2$, we have $f(x_1) \leq f(x_2)$.
- f is strictly increasing on I if for any $x_1, x_2 \in I$ with $x_1 < x_2$, we have $f(x_1) < f(x_2)$.
- f is decreasing on I if for any $x_1, x_2 \in I$ with $x_1 < x_2$, we have $f(x_1) \geq f(x_2)$.
- f is strictly decreasing on I if for any $x_1, x_2 \in I$ with $x_1 < x_2$, we have $f(x_1) > f(x_2)$.

If a function is any of these, it is said to be monotonic on I.

Visually, a strictly increasing function's graph rises from left to right, while a strictly decreasing function's graph falls. The points where a function changes from increasing to decreasing are local maxima, and points where it changes from decreasing to increasing are local minima. These points are collectively known as local extrema.

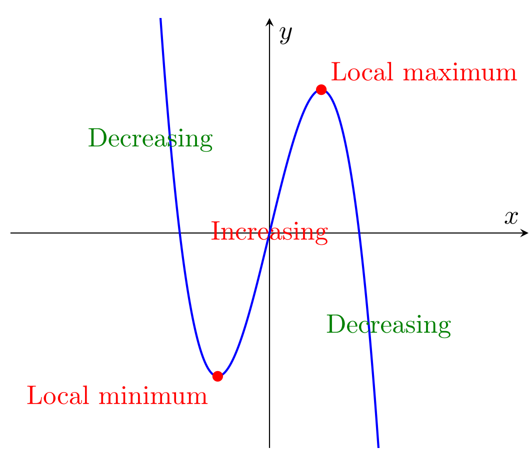


Figure 5.3: Intervals of increase/decrease and local extrema for $f(x) = -x^3 + 3x$.

A Gallery of Fundamental Graphs

Linear Functions A function of the form $f(x) = mx+d$, where m and d are constants, is a linear function. Its graph is a straight line. The constant d is the y-intercept, the value of the function when $x = 0$. The constant m is the slope of the line. It measures the rate of change of the function. For any two distinct points (x_1, y_1) and (x_2, y_2) on the line, the slope is given by the ratio:

$$m = \frac{\text{change in y}}{\text{change in x}} = \frac{y_2 - y_1}{x_2 - x_1}$$

This is illustrated in Figure 5.4. If $m > 0$, the line rises from left to right. If $m < 0$, it falls. If $m = 0$, the function is a constant function $f(x) = d$, and its graph is a horizontal line. The slope of a linear function is constant, meaning the function's rate of change is the same over any interval.

***Theorem* 5.1.1.** A function is linear if and only if its average rate of change is constant.

Proof. The average rate of change of a function f over an interval $[x_1, x_2]$ is $\frac{f(x_2)-f(x_1)}{x_2-x_1}$. If $f(x) = mx + d$, then

$$\frac{(mx_2 + d) - (mx_1 + d)}{x_2 - x_1} = \frac{m(x_2 - x_1)}{x_2 - x_1} = m.$$

The rate is the constant m. Conversely, if the average rate of change between any point $(x, f(x))$ and a fixed point (x_1, y_1) is a constant m, then $\frac{f(x)-y_1}{x-x_1} = m$, which implies $f(x) - y_1 = m(x - x_1)$. Rearranging gives $f(x) = mx - mx_1 + y_1$, which is a linear function with intercept $d = y_1 - mx_1$. ∎

Remark. The domain of a constant function ($f(x) = d$) is $\mathbb{R}$ and its range is $\{d\}$. For the identity function ($f(x) = x$), it maps every real number to itself, thus both its domain and range are $\mathbb{R}$.

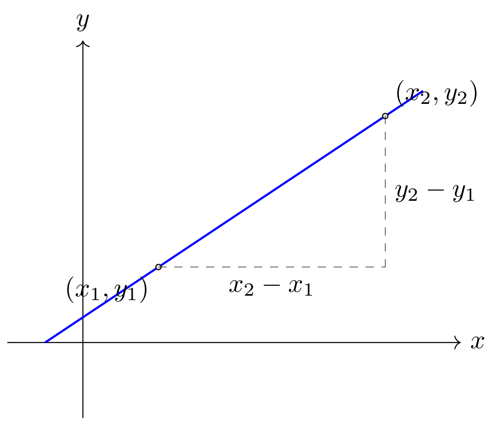


Figure 5.4: The slope of a line.

Polynomial Functions The general form of a quadratic function is $f(x) = ax^2 + bx + c$ with $a \neq 0$. The graph of any quadratic function is a parabola. The sign of the leading coefficient, a, determines the orientation of the parabola. If $a > 0$, the parabola opens upwards. If $a < 0$, it opens downwards.

Definition 5.1.3. *Even/Odd Function*. A function f is even if its domain is symmetric about 0 and $f(-x) = f(x)$ for all x in the domain. A function f is odd if its domain is symmetric about 0 and $f(-x) = -f(x)$ for all x in the domain.

Example 5.1.2. (Even and Odd Functions). The function $f(x) = x^2$ is even, as $f(-x) = (-x)^2 = x^2 = f(x)$. Its graph, a parabola shown in Figure 5.5(a), is symmetric with respect to the y-axis. The function $g(x) = x^3$ is odd, as $g(-x) = (-x)^3 = -x^3 = -g(x)$. Its graph, shown in Figure 5.5(b), is symmetric with respect to the origin. The function $h(x) = x + 1$ is neither even nor odd, as $h(-x) = -x + 1$, which is not equal to $h(x)$ or $-h(x) = -x - 1$.

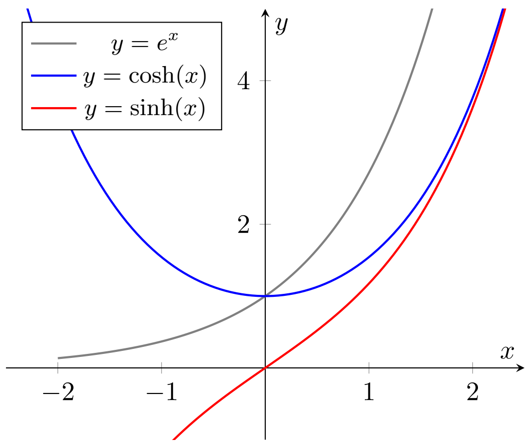
If a function f has a domain symmetric about the origin, it can be expressed as the sum of an even function and an odd function.

$$f(x) = \underbrace{\frac{1}{2}[f(x) + f(-x)]}_{\text{even part}} + \underbrace{\frac{1}{2}[f(x) - f(-x)]}_{\text{odd part}}$$

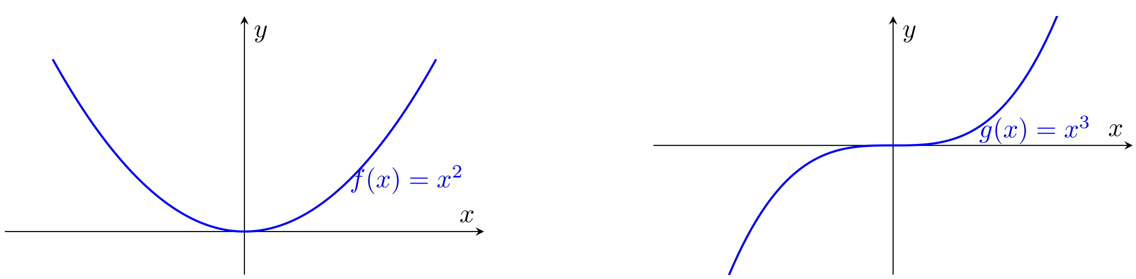
Example 5.1.3. (Even and Odd Parts). The even and odd parts of the exponential function e^x define the hyperbolic functions.

$$e^x = \cosh(x) + \sinh(x)$$

where the hyperbolic cosine is $\cosh(x) = \frac{e^x+e^{-x}}{2}$ (even) and the hyperbolic sine is $\sinh(x) = \frac{e^x-e^{-x}}{2}$ (odd).



The end behaviour of a general polynomial function $P(x) = a_n x^n + \cdots + a_0$ describes the direction of its graph as x approaches infinity ($x \to \infty$) or negative infinity ($x \to -\infty$). For very large values of $|x|$, the leading term $a_n x^n$ dominates the function. Thus, the end behaviour of $P(x)$ is the same as that of $y = a_n x^n$. There are four cases, depending on whether n is even or odd, and whether a_n is positive or negative, as shown in Figure 5.6.



(a) The parabola. (b) The cubic curve.

Figure 5.5: Graphs of basic polynomial functions.

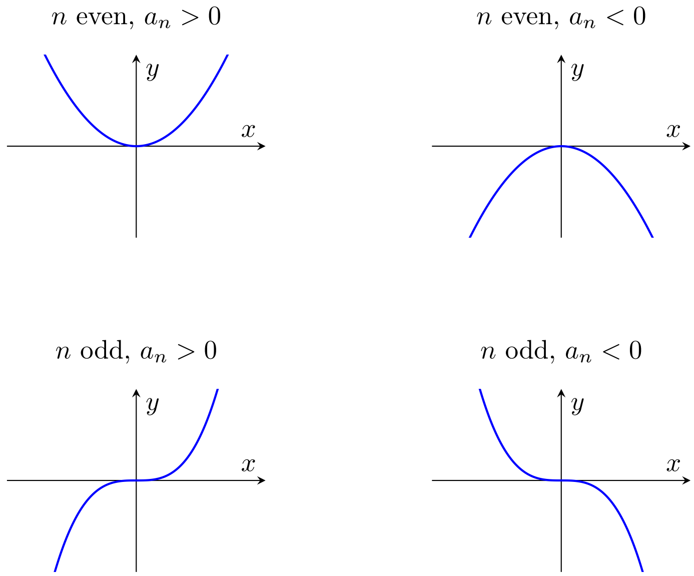
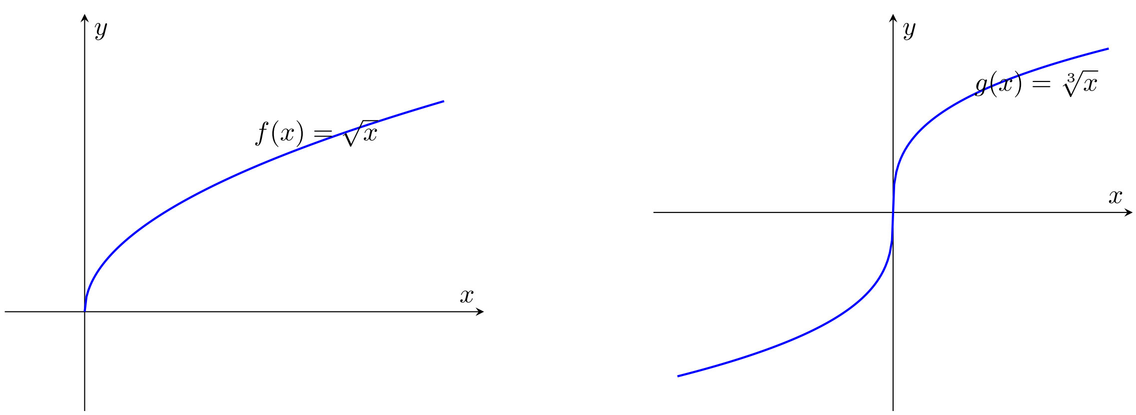


Figure 5.6: The four cases of polynomial end behaviour.

Power and Root Functions A function $f(x) = x^a$ where a is a constant is a power function. This family includes polynomials (a is a positive integer) and root functions. For $f(x) = \sqrt[n]{x}$, where n is a positive integer:

- If n is even, the domain of $f(x) = \sqrt[n]{x}$ is $[0, \infty)$.
- If n is odd, the domain is all real numbers, $\mathbb{R}$.

The graphs of $y = \sqrt{x}$ and $y = \sqrt[3]{x}$ are shown in Figure 5.7. The graph of $y = \sqrt{x}$ can be seen as the top half of a parabola $x = y^2$ that opens to the right.



(a) $f(x) = \sqrt{x}$ (b) $g(x) = \sqrt[3]{x}$

Figure 5.7: Graphs of root functions.

The Absolute Value Function The absolute value function $f(x) = |x|$ is a piecewise-defined function whose graph is a distinctive V-shape with its vertex at the origin. Its domain is $\mathbb{R}$ and its range is $[0, \infty)$. It is an even function, symmetric about the y-axis.

***Proposition* 5.1.1.** *Max–min via* $|\cdot|$. For any real numbers a, b,

$$\max(a,b) = \tfrac{1}{2}\big(a + b + |a - b|\big), \qquad \min(a,b) = \tfrac{1}{2}\big(a + b - |a - b|\big).$$

Proof. Let $max(a, b) = \frac{a+b+|a-b|}{2}$ and $min(a, b) = \frac{a+b-|a-b|}{2}$ for $a, b \in \mathbb{R}$.

- Case 1: $a \geq b$. Since $a \geq b$ then $a - b \geq 0$ therefore $|a - b| = a - b$. Thus maximum function is:

$$\frac{a + b + |a - b|}{2} = \frac{a + b + a - b}{2} = a$$

or $max(a, b) = a$ since $a \geq b$ while minimum function is:

$$\frac{a + b - |a - b|}{2} = \frac{a + b - (a - b)}{2} = b$$

or $min(a, b) = b$ since $b \leq a$

- Case 2: $a < b$. Since $a < b$ then $a - b < 0$ therefore $|a - b| = -(a - b) = b - a$. Thus maximum function is:

$$\frac{a + b + |a - b|}{2} = \frac{a + b + b - a}{2} = b$$

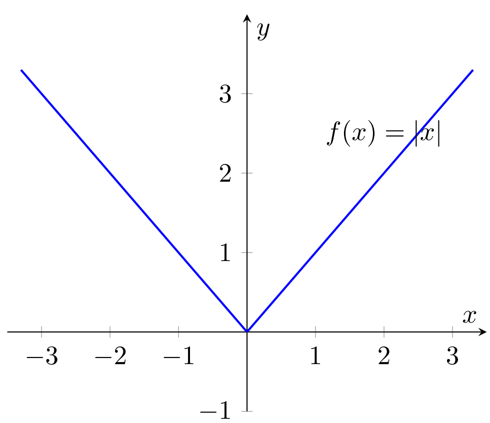
or $max(a, b) = b$ since $b > a$ while minimum function is:

$$\frac{a + b - |a - b|}{2} = \frac{a + b - (b - a)}{2} = a$$

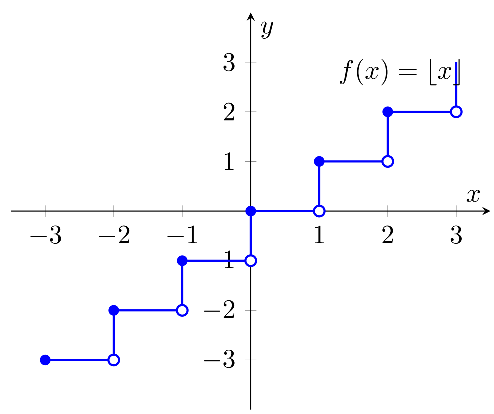
or $min(a, b) = a$ since $a < b$

■

Step Functions The greatest integer function, or floor function, $f(x) = \lfloor x \rfloor$, maps a real number x to the greatest integer less than or equal to x (e.g, $\lfloor 2.9 \rfloor = 2$). Its graph consists of a series of horizontal line segments, resembling a staircase. It is discontinuous at every integer value.



(a) The Absolute Value Function



(b) The Floor Function

Figure 5.8: Graphs of piecewise functions.

The Parabola as a Locus of Points

A parabola has a purely geometric definition independent of its role as the graph of a quadratic function.

Definition 5.1.4. ***Parabola***. A parabola is the set of all points in a plane that are equidistant from a fixed point (the focus) and a fixed line (the directrix).

Let us derive the equation for a parabola with its focus at $F = (0, p)$ and its directrix being the horizontal line L with equation $y = -p$, where $p \neq 0$. A point $P = (x, y)$ is on the parabola if its distance to the focus equals its distance to the directrix. The distance from P to the directrix is $|y - (-p)| = |y + p|$. The distance from P to the focus is $d(P, F) = \sqrt{(x-0)^2 + (y-p)^2}$. Equating these distances gives:

$$\sqrt{x^2 + (y-p)^2} = |y + p|$$

Squaring both sides yields:

$$x^2 + (y-p)^2 = (y+p)^2$$
$$x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2$$
$$x^2 = 4py \implies y = \frac{1}{4p}x^2$$

This confirms that the set of points defined geometrically is the graph of a quadratic function of the form $y = ax^2$, where $a = 1/(4p)$. Only parabolas with a horizontal directrix (and thus a vertical axis of symmetry) pass the vertical line test and are graphs of functions of x.

Rational Functions A rational function is a ratio of two polynomials of the form $R(x) = \frac{P(x)}{Q(x)}$. The domain of $R(x)$ is the set of all real numbers except for the values of x for which the denominator $Q(x)$ is zero. The simplest non-trivial example is $f(x) = 1/x$. Its domain is all real numbers except $x = 0$. Its graph is a hyperbola. As x approaches 0, $|f(x)|$ becomes very large. As $|x|$ becomes very large, $f(x)$ approaches 0. These limiting behaviours are described by asymptotes.

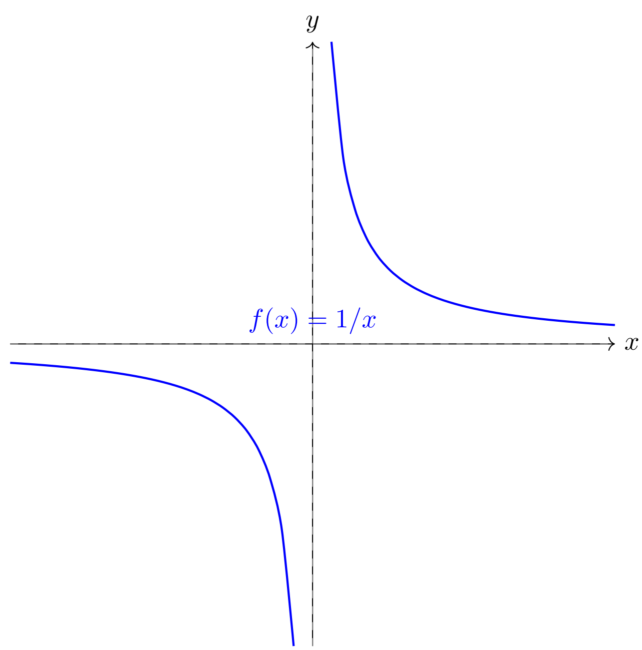


Figure 5.9: Graph of the rational function $f(x) = 1/x$.

Definition 5.1.5. ***Asymptotes***. Let $y = f(x)$.

- A vertical line $x = a$ is a *vertical asymptote* if $f(x) \to \pm\infty$ as $x \to a^{\pm}$.
- A horizontal line $y = L$ is a *horizontal asymptote* if $f(x) \to L$ as $x \to \pm\infty$.
- A non-horizontal line $y = mx + b$ is an *oblique (slant) asymptote* if $f(x) - (mx + b) \to 0$ as $x \to \pm\infty$.

Example 5.1.4. (Rational graph with vertical and oblique asymptotes). For $R(x) = \dfrac{2x^2+1}{x-1}$ we have a vertical asymptote at $x = 1$. Long division gives

$$\frac{2x^2+1}{x-1} = 2x + 2 + \frac{3}{x-1},$$

so $y = 2x + 2$ is an oblique asymptote. See Figure 5.10.

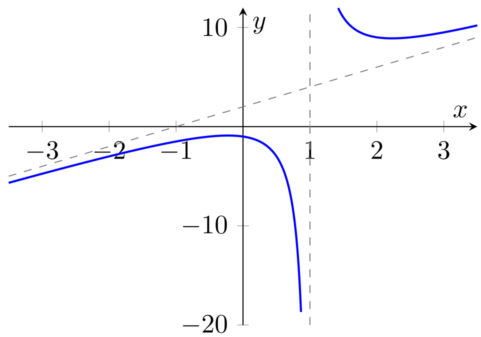


Figure 5.10: Vertical asymptote $x = 1$ and oblique asymptote $y = 2x + 2$ for $R(x)$.

Example 5.1.5. (Removable discontinuity). Consider

$$f(x) = \frac{x(x-1)(x-3)}{x(x^2-5x+6)} = \frac{x-1}{x-2} \quad \text{for } x \neq 0, 2, 3.$$

The simplified formula hides two removable discontinuities (holes) at $x = 0$ and $x = 3$, with hole points $(0, \frac{1}{2})$ and $(3, 2)$. There is a vertical asymptote at $x = 2$, and a horizontal asymptote $y = 1$.

Exponential and Logarithmic Functions The base–10 exponential function and its inverse, the common logarithm, are fundamental non-algebraic functions. Their graphs are inverse to each other and are symmetric about the line $y = x$.

Definition 5.1.6. ***Exponential and common logarithm***. For $x \in \mathbb{R}$ and $x > 0$ respectively,

$$y = 10^x \quad \text{and} \quad y = \log(x).$$

The exponential has domain $\mathbb{R}$ and range $(0, \infty)$; the logarithm has domain $(0, \infty)$ and range $\mathbb{R}$.

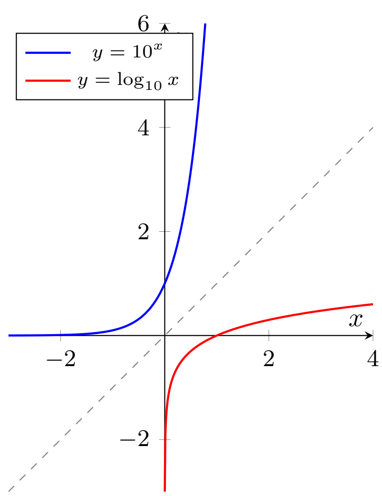


Figure 5.11: Inverse graphs: $y = 10^x$ and $y = \log(x)$ reflected across $y = x$.

Transformations of Graphs

The graph of a complicated function can often be understood as a transformation of the graph of a simpler, parent function.

Vertical and Horizontal Shifts Let c be a positive constant.

- The graph of $g(x) = f(x) + c$ is the graph of $f(x)$ shifted vertically upwards by c units.
- The graph of $g(x) = f(x) - c$ is the graph of $f(x)$ shifted vertically downwards by c units.
- The graph of $g(x) = f(x - c)$ is the graph of $f(x)$ shifted horizontally to the right by c units.
- The graph of $g(x) = f(x + c)$ is the graph of $f(x)$ shifted horizontally to the left by c units.

Vertical and Horizontal Scaling and Reflections Let $c > 1$.

- The graph of $g(x) = cf(x)$ is the graph of $f(x)$ stretched vertically by a factor of c.
- The graph of $g(x) = \frac{1}{c}f(x)$ is the graph of $f(x)$ compressed vertically by a factor of c.
- The graph of $g(x) = f(cx)$ is the graph of $f(x)$ compressed horizontally by a factor of c.
- The graph of $g(x) = f(x/c)$ is the graph of $f(x)$ stretched horizontally by a factor of c.

If the constant c is negative, the scaling is combined with a reflection.

- The graph of $g(x) = -f(x)$ is the graph of $f(x)$ reflected across the x-axis.
- The graph of $g(x) = f(-x)$ is the graph of $f(x)$ reflected across the y-axis.

Absolute-Value Transformations Two common absolute-value constructions alter a graph in distinct ways.

***Proposition* 5.1.2.** For any function f:

1. The graph of $y = |f(x)|$ is obtained from the graph of $y = f(x)$ by reflecting the portion below the x-axis across the x-axis.
2. The graph of $y = f(|x|)$ is obtained by taking the right-hand half of $y = f(x)$ ($x \geq 0$) and reflecting it across the y-axis.

Proof.

1. Since $|f(x)| = \begin{cases} f(x) & f(x) \geq 0 \\ -f(x) & f(x) < 0 \end{cases}$, negative ordinates are replaced by their positives.
2. Because $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$, for $x < 0$ we have $f(|x|) = f(-x)$, which mirrors the right-hand branch.

■

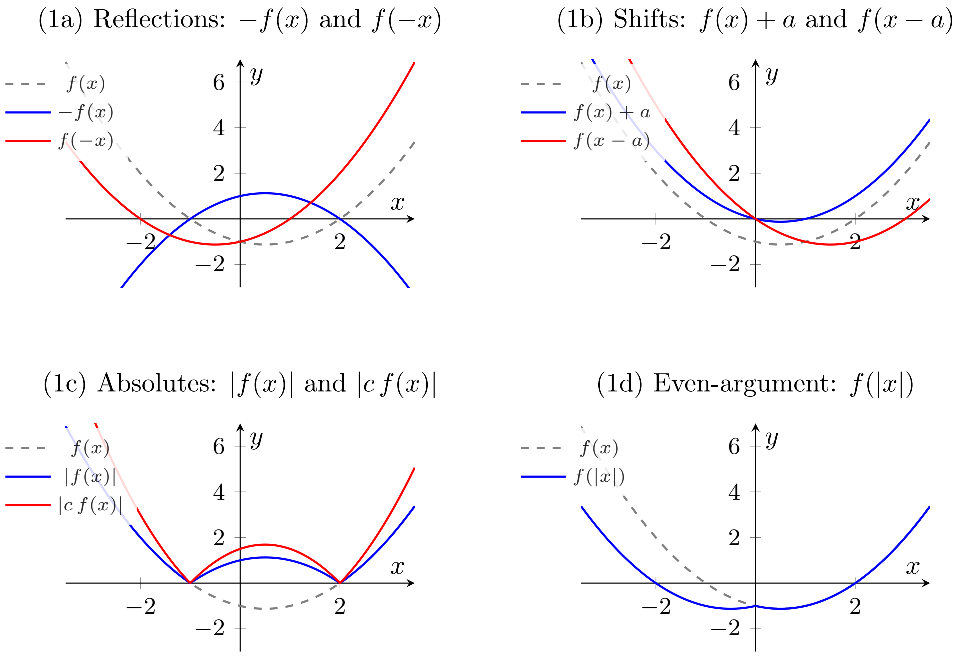
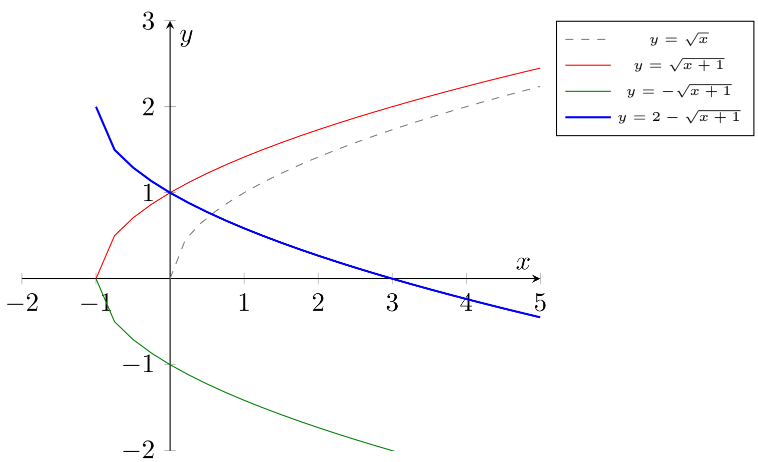


Figure 5.12: Common transformations for $f(x) = \frac{1}{2}x^2 - \frac{1}{2}x - 1$ with $a = 1$, $c = 1.5$.

Example 5.1.6. (Combining Transformations). Construct the graph of $g(x) = 2 - \sqrt{x+1}$ from the parent function $f(x) = \sqrt{x}$. We can rewrite $g(x)$ as $g(x) = -\sqrt{x+1} + 2$. The graph is obtained by applying a sequence of transformations to the graph of $f(x) = \sqrt{x}$:

1. Shift horizontally to the left by 1 unit to get $y = \sqrt{x+1}$.
2. Reflect across the x-axis to get $y = -\sqrt{x+1}$.
3. Shift vertically upwards by 2 units to get $y = -\sqrt{x+1} + 2$.

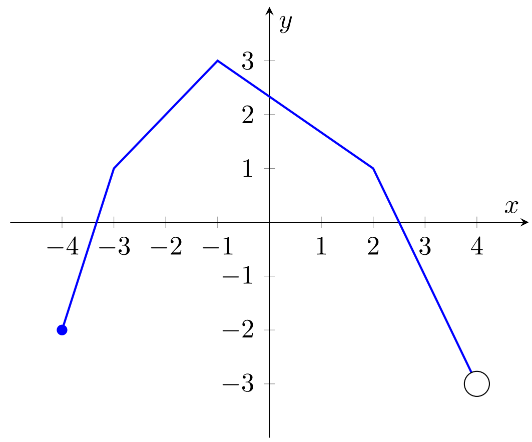


5.2 Exercises

Part I: Reading and Interpreting Graphs

1. For the function f whose graph is shown, find the following.

(a) The domain of f.
(b) The range of f.
(c) $f(0)$, $f(2)$, and $f(-3)$.
(d) The values of x for which $f(x) = 1$.
(e) The values of x for which $f(x) \leq 0$.
(f) The number of solutions to the equation $f(x) = 2.5$.
(g) The intervals on which f is strictly increasing.
(h) The intervals on which f is strictly decreasing.
(i) The coordinates of any local maxima and minima.

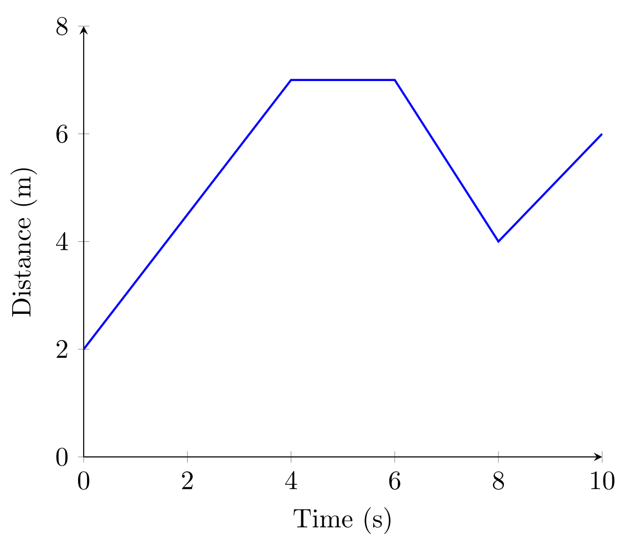


2. Use the graphs of $f(x) = x + 1$ and $g(x) = |x - 1|$ to solve the following equation and inequalities graphically.

(a) $f(x) = g(x)$
(b) $f(x) > g(x)$
(c) $f(x) \leq g(x)$

3. A man walks away from a streetlight. His distance, d (in metres), from a starting point is given as a function of time, t (in seconds), by the graph below.

(a) What is the man's initial distance from the starting point?
(b) On which time intervals is he moving away from the start? When is he moving towards it?
(c) On which interval is he moving fastest? How can you tell?
(d) Describe his motion between $t = 4$ and $t = 6$.



4. A function is defined by $f(x) = x^3 - 3x^2 + 2$. Plot this function (or use a graphing tool). From the graph, estimate:

(a) The local maximum and minimum values.

(b) The intervals where the function is increasing or decreasing.

(c) The solutions to $f(x) = 0$.

5. Explain why the equation of a circle, $x^2 + y^2 = r^2$, does not define y as a function of x. How could you modify it to get two separate functions whose graphs, when combined, form the circle?

6. Indicate on a straight line the set of all x satisfying the following conditions. Also name each set using interval notation.

(a) $|x - 3| < 1$

(b) $|x - a| < \epsilon$ (where a is a point and $\epsilon > 0$)

(c) $|x^2 - 1| < 1/2$

(d) $x^2 + 1 > 2$

(e) $(x + 1)(x - 1)(x - 2) > 0$

Part II: Graphing in the Plane

7. Draw the set of all points (x, y) satisfying the following conditions.

(a) $x > y$

(b) $y \leq x^2$

(c) $|x - y| < 1$

(d) $x + y$ is an integer.

(e) $x^2 + y^2 = 0$

(f) $xy = 0$

(g) $x^2 - 2x + y^2 = 3$

Remark. Complete the square.

(h) $x^2 = y^2$

8. Draw the set of all points (x, y) satisfying the following conditions.

(a) $|x| + |y| = 1$

(b) $|x| - |y| = 1$

(c) $|x - 1| = |y - 1|$

9. Sketch the graphs of the following, which are relations but not all functions of x.

(a) $x = y^2$

(b) $x = |y|$

(c) $x^2/9 + y^2/4 = 1$ (This is an ellipse).

10. Describe the general features of the graph of a function f if:

(a) f is even.

(b) f is odd.

(c) $f(x) = f(x + a)$ for all x and a fixed $a > 0$ (f is periodic).

Part III: A Library of Functions and Transformations

11. For a line passing through (a, b) and (c, d) with $a \neq c$, show that its equation can be written as $f(x) = \frac{d-b}{c-a}(x-a) + b$. Explain why this "point-slope" form is convenient.

12. When are the graphs of $f(x) = mx + d$ and $g(x) = m'x + d'$ parallel lines?

13. $\star$ Let a triangle be formed by the origin and the points of intersection of two lines $y = mx$ and $y = nx$. Use the Pythagorean theorem to prove that the lines are perpendicular if and only if $mn = -1$.

14. Sketch the graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4$ on the same axes. What features do the graphs for even n share? What about odd n?

15. Sketch the graphs of $f(x) = \sqrt[n]{x}$ for $n = 2, 3, 4, 5$. Note the difference in domain and shape for even and odd n.

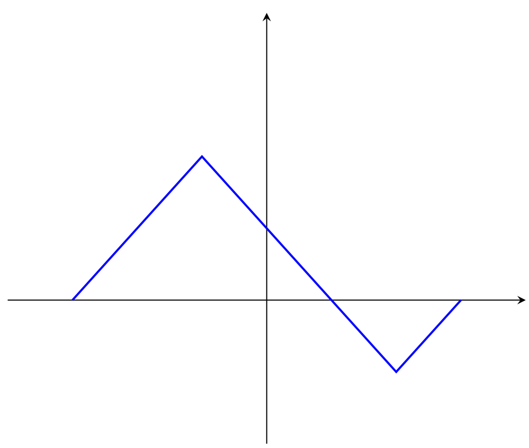
16. Sketch the graph of $f(x) = x + 1/x$.

Remark. What happens near $x = 0$? What happens for large $|x|$? How does the graph relate to the line $y = x$?

Do the same for $g(x) = x - 1/x$.

17. The graph of a function f is given. Sketch the graphs of the following transformations.

(a) $y = f(x) + c$
(b) $y = f(x + c)$
(c) $y = cf(x)$
(d) $y = f(cx)$
(e) $y = f(|x|)$
(f) $y = |f(x)|$
(g) $y = \max(f(x), 0)$



18. Start with the graph of $f(x) = x^2$ and use transformations to sketch $g(x) = ax^2 + bx + c$.

Remark. First complete the square to write $g(x)$ in the form $a(x-h)^2 + k$.

19. The function $f(x) = \lfloor x \rfloor$ denotes the greatest integer $\leq x$. Draw the graphs of the following functions.

(a) $g(x) = x - \lfloor x \rfloor$ (the "fractional part" function)
(b) $h(x) = \sqrt{x - \lfloor x \rfloor}$
(c) $k(x) = \lfloor x \rfloor + \sqrt{x - \lfloor x \rfloor}$

(d) $m(x) = \lfloor 1/x \rfloor$

20. Let $\{x\}$ be the function that gives the distance from x to the nearest integer. All this means is $\{x\} = \min\{r, 1-r\}$

(a) Sketch the graph of $f(x) = \{x\}$. Show that $\{x+1\} = \{x\}$ for all x.

(b) Sketch the graph of $g(x) = \{2x\}$.

(c) Sketch the graph of $h(x) = \{x\} + \frac{1}{2}\{2x\}$ and $z(x) = \left\{x + \dfrac{1}{2}\right\}$.

21. For the function $f(x) = x^2$, each point on its graph (x, x^2) is equidistant from a focus point F and a directrix line L. Find the coordinates of F and the equation of L.

Part IV: Proofs and Theoretic Explorations

22. Prove that the points on the interval $[a, b]$ can be described as the set $\{(1-t)a + tb \mid 0 \le t \le 1\}$. What is the geometric significance of t? What point corresponds to $t = 1/3$?

23. Use coordinate geometry to prove the triangle inequality: for any three points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, and $P_3(x_3, y_3)$, we have $d(P_1, P_3) \le d(P_1, P_2) + d(P_2, P_3)$. When does equality hold?

24. Prove that every line in the plane, including vertical lines, can be described by an equation of the form $Ax + By + C = 0$ where A and B are not both zero.

25. Prove that the lines given by $Ax + By + C = 0$ and $A'x + B'y + C' = 0$ are perpendicular if and only if $AA' + BB' = 0$.

26. Let f be a strictly increasing function on $\mathbb{R}$.

(a) Using the definition of monotonicity, prove that $g(x) = -f(x)$ must be strictly decreasing.

(b) Using the same definition, prove that $h(x) = f(x+c)$ for any constant c is also strictly increasing.

27. Prove that the graph of an even function is symmetric with respect to the y-axis.

28. Prove that the graph of an odd function is symmetric with respect to the origin.

29. If a polynomial contains only even powers of x (e.g., $P(x) = a_{2n}x^{2n} + \cdots + a_2x^2 + a_0$), prove it must be an even function. What is the corresponding statement for odd powers?

Part V: Challenge Problems

30. A function is defined as $f(x) = 0$ if the number of 7s in the decimal expansion of x is finite, and $f(x) = 1$ otherwise. Describe the graph of this function as best you can. Is it monotonic? Is it periodic?

31. Consider the function $f(x) = 0$ if x is irrational, and $f(x) = 1/q$ if $x = p/q$ is a rational number in lowest terms with $q > 0$.

(a) What is $f(1/2)$, $f(2/4)$, $f(1/3)$, $f(2/3)$, $f(\sqrt{2})$?

(b) Try to sketch the graph of this function on the interval $[0, 1]$. What does it look like near the x-axis?

32. Find the minimum distance from the point (c, d) to the line given by $f(x) = mx + b$.

Remark. Let $P(x, mx + b)$ be a point on the line. Write the squared distance $D^2(x)$ between P and (c, d). This will be a quadratic in x. Complete the square to find the minimum value.

33. Let f be a strictly increasing function. Does it follow that $f \circ f$ is also strictly increasing? Prove or give a counterexample. What if f is strictly decreasing?

34. Can you find functions f and g such that $f(x) + g(y) = xy$ for all x, y?

Remark. Try setting x or y to specific values, like 0 or 1, to see what constraints this places on f and g.

5.3 The Equation of a Circle

An equation involving x and y defines a set of points in the plane, known as a relation. A circle is the set of all points equidistant from a fixed centre (cf. 2.1.8). Let the centre be $C = (a, b)$ and the radius be $r > 0$. A point $P = (x, y)$ is on the circle if its distance from C is r. Using the distance formula, $d(P, C) = r$, we have:

$$\sqrt{(x-a)^2 + (y-b)^2} = r$$

Since both sides of this equation are positive, we can square them to obtain an equivalent equation which is often more convenient:

$$(x-a)^2 + (y-b)^2 = r^2$$

This is the standard equation of a circle with centre (a, b) and radius r.

Example 5.3.1. (Equations of Circles).

1. The equation for a circle with centre $(1, 4)$ and radius 3 is

$$(x-1)^2 + (y-4)^2 = 3^2 = 9.$$

2. The equation $(x-2)^2 + (y+5)^2 = 16$ represents a circle. The centre is $(2, -5)$ and the radius is $\sqrt{16} = 4$. Notice that $y + 5$ corresponds to $y - (-5)$.

3. The equation $x^2 + y^2 = 1$ represents a circle of radius 1 centred at the origin $(0, 0)$. This is the unit circle.

4. The equation $(x+2)^2 + (y+3)^2 = 7$ describes a circle with centre $(-2, -3)$ and radius $\sqrt{7}$.

A circle is not the graph of a function, as it fails the vertical line test (Figure 4.10). However, it is the union of the graphs of two functions: the upper semicircle $f(x) = b + \sqrt{r^2 - (x-a)^2}$ and the lower semicircle $g(x) = b - \sqrt{r^2 - (x-a)^2}$.

We can also understand the equation of a circle through the lens of transformations.

***Theorem* 5.3.1.** The circle of radius r and centre A is the image of the circle of radius r and centre O under the translation T_A.

Proof. Let X be a point on the circle of radius r centred at the origin O. By definition, this means its distance from the origin is r, so $|X| = r$. The image of X under the translation T_A is the point $Y = T_A(X) = X + A$. The distance of this image point from A is

$$d(Y, A) = |Y - A| = |(X + A) - A| = |X| = r.$$

This shows that any point Y which is the image of a point on the first circle lies on the circle of radius r centred at A. Conversely, let Y be any point on the circle centred at A, so $|Y - A| = r$. Let $X = Y - A$. Then $Y = X + A = T_A(X)$, and $|X| = |Y - A| = r$, so X is on the circle centred at O. Thus, the circle centred at A is precisely the set of images of the points of the circle centred at O. ∎

5.3.1 Rational Points on the Unit Circle

The problem of finding all integer solutions (a, b, c) to the Pythagorean equation $a^2 + b^2 = c^2$ is equivalent to finding all rational points on the unit circle. A rational point is a point (x, y) where both coordinates x and y are rational numbers. If (a, b, c) is an integer solution with $c \neq 0$, dividing by c^2 gives

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1.$$

This means that $x = a/c$ and $y = b/c$ is a rational point on the circle $x^2 + y^2 = 1$. Conversely, if (x, y) is a rational point on the circle, we can write x and y with a common denominator c, so $x = a/c$ and $y = b/c$ for integers a, b, c. Substituting these into the circle's equation and multiplying by c^2 recovers $a^2 + b^2 = c^2$.

All rational points on the unit circle, with one exception, can be generated by a single parameter.

***Theorem* 5.3.2. Parameterisation of Rational Points on a Circle**. Let (x, y) be a point on the unit circle $x^2 + y^2 = 1$.

1. If t is any number, the point given by

$$x = \frac{1 - t^2}{1 + t^2}, \quad y = \frac{2t}{1 + t^2}$$

   lies on the unit circle. If t is a rational number, then this point is a rational point.

2. Conversely, if (x, y) is a rational point on the unit circle and $x \neq -1$, then there exists a rational number t such that (x, y) is given by the formulas above. This rational number is $t = \frac{y}{x+1}$.

Proof. For the first part, we substitute the expressions for x and y into the circle equation:

$$x^2 + y^2 = \left(\frac{1 - t^2}{1 + t^2}\right)^2 + \left(\frac{2t}{1 + t^2}\right)^2 = \frac{(1 - 2t^2 + t^4) + 4t^2}{(1 + t^2)^2} = \frac{1 + 2t^2 + t^4}{(1 + t^2)^2} = \frac{(1 + t^2)^2}{(1 + t^2)^2} = 1.$$

The point lies on the circle. If t is rational, then t^2, $1 - t^2$, $1 + t^2$, and $2t$ are all rational, so x and y are rational.

For the converse, let (x, y) be a point on the circle such that $x \neq -1$. Define $t = \frac{y}{x+1}$. If x, y are rational, then t is also rational. We must show that this value of t generates x and y via the given formulas. From the definition of t, we have $y = t(x + 1)$. Squaring both sides gives $y^2 = t^2(x + 1)^2$. Since $x^2 + y^2 = 1$, we can substitute $y^2 = 1 - x^2 = (1 - x)(1 + x)$.

$$t^2(x + 1)^2 = (1 - x)(1 + x)$$

Since $x \neq -1$, we can divide both sides by $x + 1$:

$$t^2(x + 1) = 1 - x \implies t^2x + t^2 = 1 - x \implies x(t^2 + 1) = 1 - t^2 \implies x = \frac{1 - t^2}{1 + t^2}$$

To find y, we substitute this expression for x back into $y = t(x + 1)$:

$$y = t\left(\frac{1 - t^2}{1 + t^2} + 1\right) = t\left(\frac{1 - t^2 + 1 + t^2}{1 + t^2}\right) = t\left(\frac{2}{1 + t^2}\right) = \frac{2t}{1 + t^2}.$$

This completes the proof. The point $(-1, 0)$ is the only rational point not generated by this method, as it would require an infinite value for t. ■

Geometrically, the parameter t is the slope of the line connecting the point $(-1, 0)$ to the point (x, y) on the circle. Any line through $(-1, 0)$ with a rational slope intersects the unit circle at $(-1, 0)$ and one other point, which must be a rational point.

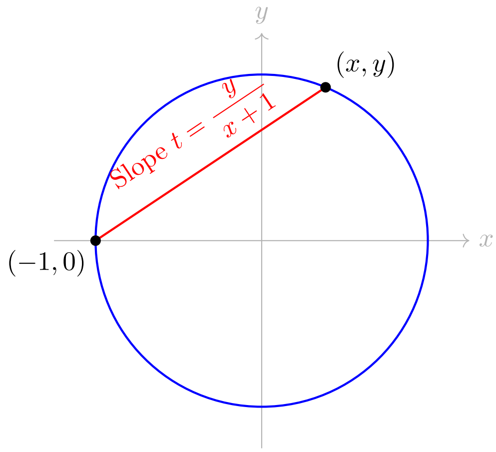


Figure 5.13: Geometric interpretation of the parameter t with $(x, y) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$.

Example 5.3.2. (Pythagorean Triples from Rational t).

1. Let $t = 2$. Then $x = \frac{1-4}{1+4} = -\frac{3}{5}$ and $y = \frac{4}{1+4} = \frac{4}{5}$. This corresponds to the triple $(-3, 4, 5)$ or $(3, 4, 5)$.
2. Let $t = 4$. Then $x = \frac{1-16}{1+16} = -\frac{15}{17}$ and $y = \frac{8}{1+16} = \frac{8}{17}$. This corresponds to the triple $(15, 8, 17)$.
3. Let $t = 1/2$. Then $x = \frac{1-1/4}{1+1/4} = \frac{3/4}{5/4} = \frac{3}{5}$ and $y = \frac{1}{1+1/4} = \frac{1}{5/4} = \frac{4}{5}$. This again gives the $(3, 4, 5)$ triple.

5.3.2 Graphs of Other Relations

Definition 5.3.1. ***Ellipse***. An ellipse is the set of points P such that the sum of the distances from P to two fixed points (the foci) is a constant.

Definition 5.3.2. ***Hyperbola***. A hyperbola is the set of points P such that the absolute difference of the distances from P to two fixed points (the foci) is a constant.

When the foci are placed symmetrically on the x-axis at $(\pm c, 0)$, their equations take standard forms.

- **Ellipse:** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
- **Hyperbola:** $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

These relations, like the circle, are not functions but can be decomposed into the graphs of functions representing their upper and lower branches.

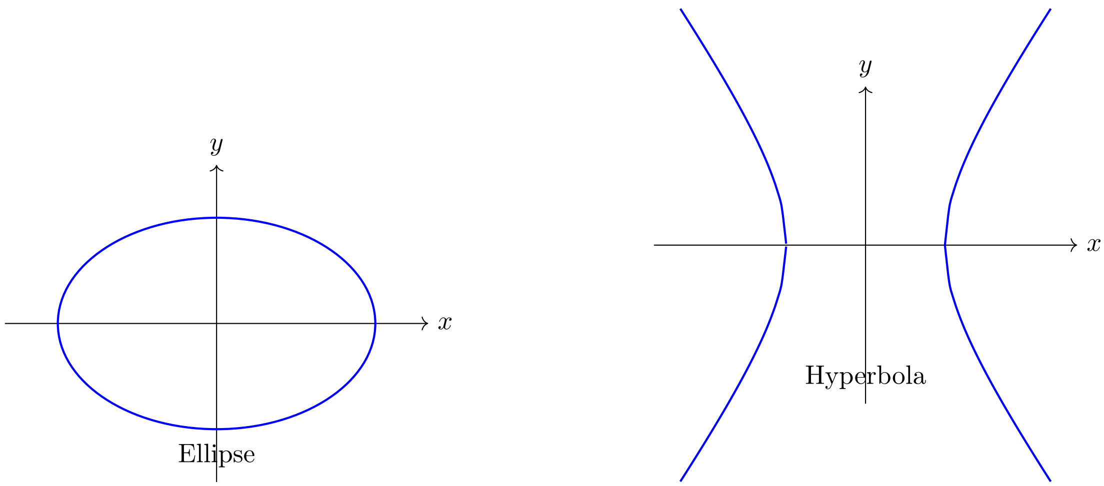


Figure 5.14: Standard graphs of an ellipse and a hyperbola.

5.4 Average Rate of Change

The slope of a line measures its constant rate of change. For a function whose graph is not a straight line, the rate of change is not constant. We can, however, speak of the average rate of change over an interval.

Definition 5.4.1. ***Average Rate of Change***. The average rate of change of a function f over the interval $[a, b]$ is the ratio of the change in the function's output to the change in its input:

$$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

Geometrically, this is the slope of the secant line passing through the points $(a, f(a))$ and $(b, f(b))$ on the graph of the function, as illustrated in Figure 5.15.

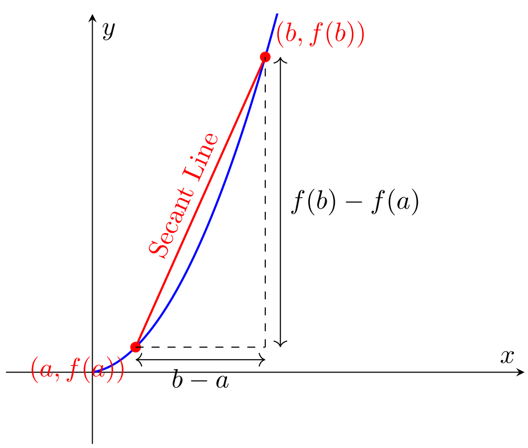


Figure 5.15: The average rate of change is the slope of the secant line.

Example 5.4.1. (Calculating Average Rate of Change). Find the average rate of change of $f(x) = x^2 + 1$ on the interval $[1, 3]$.

$$\frac{f(3) - f(1)}{3 - 1} = \frac{(3^2 + 1) - (1^2 + 1)}{2} = \frac{10 - 2}{2} = \frac{8}{2} = 4.$$

The slope of the line connecting $(1, 2)$ and $(3, 10)$ is 4.

If we consider the interval from a to $a+h$, the average rate of change is given by the difference quotient: $\frac{f(a+h)-f(a)}{h}$. This expression is fundamental to the study of calculus, where it is used to define the instantaneous rate of change.

5.5 Exercises

Part I: The Equation of a Circle

1. Write the standard equation for the circle with the given properties.

(a) Centre at $(0,0)$, radius 5.

(b) Centre at $(3,-2)$, radius 4.

(c) Centre at $(-1,0)$, passing through the point $(2,2)$.

(d) Diameter with endpoints $(-3,1)$ and $(5,7)$.

2. Find the centre and radius of the circle for each of the following equations.

(a) $x^2+y^2=10$

(b) $(x-4)^2+(y+1)^2=16$

(c) $x^2+(y-2)^2=8$

3. Find the centre and radius of the circle by first completing the square.

(a) $x^2+y^2-4x-6y-3=0$

(b) $x^2+y^2+8x-2y+1=0$

(c) $2x^2+2y^2-4x+12y+10=0$

4. A circle with centre (h,k) is tangent to the x-axis. What is its radius? Write the equation of a circle with centre $(4,3)$ that is tangent to the x-axis.

5. Describe the set of points (x,y) that satisfy the inequality $(x-1)^2+(y-2)^2<9$. What about the inequality $(x-1)^2+(y-2)^2\geq 9$? Sketch both sets.

6. Find the points of intersection of the line $y=2x-1$ and the circle $x^2+y^2=2$. Sketch a diagram of the situation.

7. Find the points of intersection of the circle $x^2+y^2=25$ and the circle $(x-8)^2+y^2=25$.

8. Show that the set of all points P such that the distance from P to A=(2,0) is twice the distance from P to B=(-1,0) forms a circle. Find its centre and radius. This is an example of a Circle of Apollonius.

9. Use coordinate geometry to prove Thales's Theorem: any angle inscribed in a semicircle is a right angle.

Remark. Place the circle with centre $(0,0)$ and radius r. Let the endpoints of the diameter be $A=(-r,0)$ and $B=(r,0)$. Let $P=(x,y)$ be any other point on the circle. Show that the product of the slopes of $\overline{PA}$ and $\overline{PB}$ is -1.

10. A point $P(x,y)$ moves so that the line segment joining it to $A(0,2)$ is always perpendicular to the line segment joining it to $B(0,-2)$. Find the equation of the locus of P and describe the shape.

11. Prove the theorem from the text that a circle with centre A and radius r is the image of a circle with centre O and radius r under the translation T_A. Now, consider the function $f(x)=\sqrt{r^2-x^2}$ (a semicircle). Show that the graph of $g(x)=a+f(x-b)$ is a semicircle with centre (b,a).

Part II: Rational Points and Pythagorean Triples

12. Use the parameterisation $x = \frac{1-t^2}{1+t^2}, y = \frac{2t}{1+t^2}$ to find the rational point on the unit circle and the corresponding primitive Pythagorean triple for the following values of t.

(a) $t = 3$
(b) $t = 1/3$
(c) $t = 4/3$

13. For each of the following Pythagorean triples, find the corresponding rational point (x, y) on the unit circle and the rational parameter t that generates it.

(a) $(5, 12, 13)$
(b) $(8, 15, 17)$
(c) $(7, 24, 25)$

14. The parameterisation in the text generates all rational points on the circle except for one. Which point is it, and what value of t would correspond to it if we could take limits? Explain the geometric reasoning.

15. Let $P(t) = (\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2})$. Show that $P(-t) = (x(t), -y(t))$. What does this mean geometrically about the points generated by t and $-t$?

16. Let $P(t) = (\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2})$. Show that the point $P(1/t)$ is a reflection of the point $P(t)$ across the y-axis.

17. A line is drawn through the point $(-1, 0)$ with a slope of $t = 5/12$. At what other point does this line intersect the unit circle?

Part III: Average Rate of Change

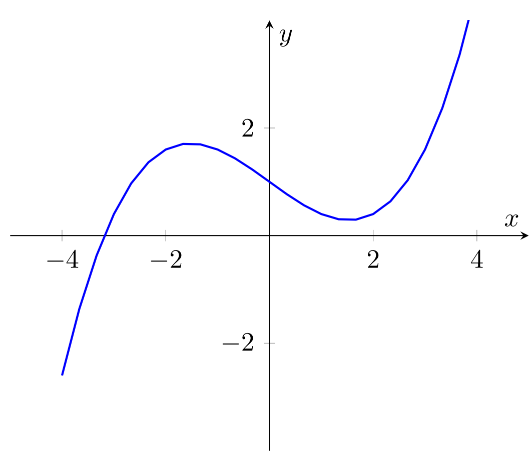
18. For each function and interval, calculate the average rate of change.

(a) $f(x) = x^2 - 2x$ on $[2, 5]$
(b) $g(x) = \frac{1}{x}$ on $[1, 4]$
(c) $h(t) = \sqrt{t+3}$ on $[1, 6]$

19. For the function $f(x) = x^2 - 2x$ from the previous exercise, find the equation of the secant line passing through the points at $x = 2$ and $x = 5$.

20. Let $f(x) = x^3$. Calculate the difference quotient $\frac{f(a+h)-f(a)}{h}$. Simplify your answer.

21. The graph of a function f is shown. For which of the following intervals is the average rate of change the largest? The smallest (most negative)? Closest to zero? (a) $[-4, -2]$ (b) $[-2, 0]$ (c) $[0, 2]$ (d) $[2, 4]$



22. The area of a circle is a function of its radius, $A(r) = \pi r^2$. What is the average rate of change of the area as the radius increases from $r = 2$ to $r = 3$? What does this value represent geometrically?

23. If the average rate of change of a function f over every interval is a constant m, prove that f must be a linear function.

Part IV: Synthesis and Challenge Problems

24. The tangent line to a circle at a point P is perpendicular to the radius at that point. Find the equation of the tangent line to the circle $(x-1)^2 + (y-2)^2 = 25$ at the point $(4, 6)$.

25. Find the maximum vertical distance between the line $y = x + 2$ and the parabola $y = x^2$ for x in the interval $[-1, 2]$.

Remark. Define a difference function $d(x)$ and find its maximum value.

26. A function is defined as $d(P)$, where $P = (x, y)$ is a point and $d(P)$ is the distance from P to the fixed point $C = (1, 1)$. Is this a function from $\mathbb{R}^2$ to $\mathbb{R}$? What is the graph of the equation $d(x, y) = 3$?

27. Let $f(x) = x^2$. Consider the difference quotient $\frac{f(a+h)-f(a)}{h}$. What value does this expression approach as h gets very close to 0? This value is the slope of the tangent line to the parabola at $x = a$.

28. Find the equation of a circle that is tangent to both the x-axis and the y-axis and has a radius of 4. How many such circles are there?

29. Find the equation of the set of all points P such that the line segment from P to A=(1,0) is perpendicular to the line segment from P to B=(5,0). Show that this set is a circle and find its centre and radius.

30. A circle passes through the vertices of the triangle with vertices $A(0, 0)$, $B(4, 0)$, and $C(2, 2)$. Find the equation of this circle (the circumcircle).

Remark. The centre of the circle is the intersection of the perpendicular bisectors of the sides.

Chapter 6

Isometries and Other Mappings

In our study of geometry, we are often concerned with the idea of congruence—when two figures have the same size and shape. Intuitively, this means one figure can be moved, rotated, or flipped to lie perfectly on top of the other without stretching or tearing. To make this concept rigorous, we first introduce the general idea of a mapping, and then identify the specific class of mappings that preserve distance, which we call isometries.

6.1 Mappings of the Plane

A mapping of the plane is a function that takes a point in the plane as an input and returns another point in the plane as an output.

Definition 6.1.1. ***Mapping***. A mapping (or transformation) of the plane is a function $F : \mathbb{R}^2 \to \mathbb{R}^2$. For each point P, the point $P' = F(P)$ is called the image of P under F.

We can describe several fundamental mappings using the algebraic operations on points that we have developed.

The Identity Mapping The simplest mapping is the identity, which leaves every point unchanged.

$$I(P) = P$$

Translations A translation shifts every point in the plane by the same amount and in the same direction; recall 4.6.3. We denote the translation by the point A as $T_A(P) = P + A$.

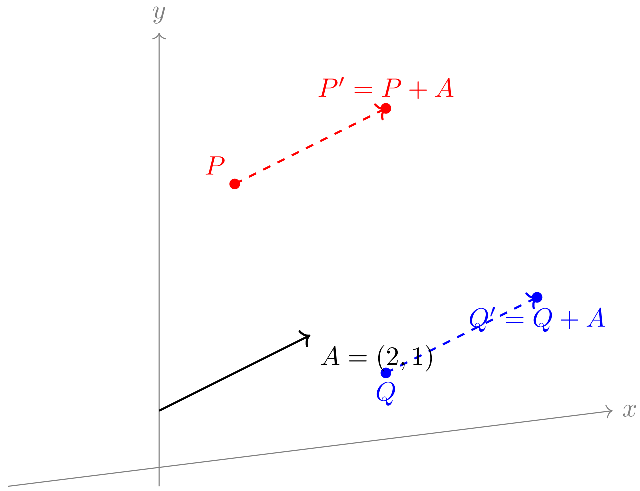
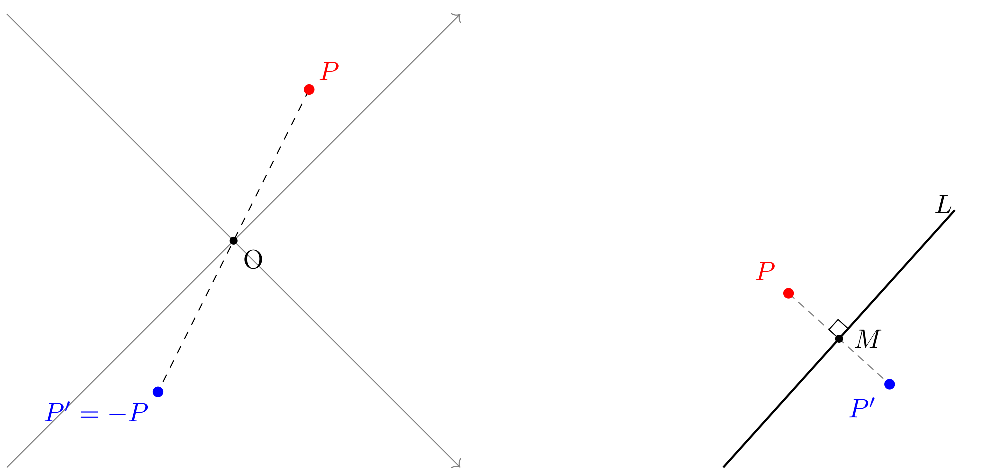


Figure 6.1: A translation by the point $A = (2, 1)$ shifts points P and Q.

Reflections A reflection 'flips' the plane across a point or a line. Reflection through the origin is defined simply using scalar multiplication.

Definition 6.1.2. ***Reflection Through the Origin***. The reflection through the origin, R_O, is the mapping defined by $R_O(P) = -P$.

Reflection across an arbitrary line L maps a point P to a point P' such that L is the perpendicular bisector of the segment $\overline{PP'}$.



(a) Reflection through the origin.

(b) Reflection through a line L.

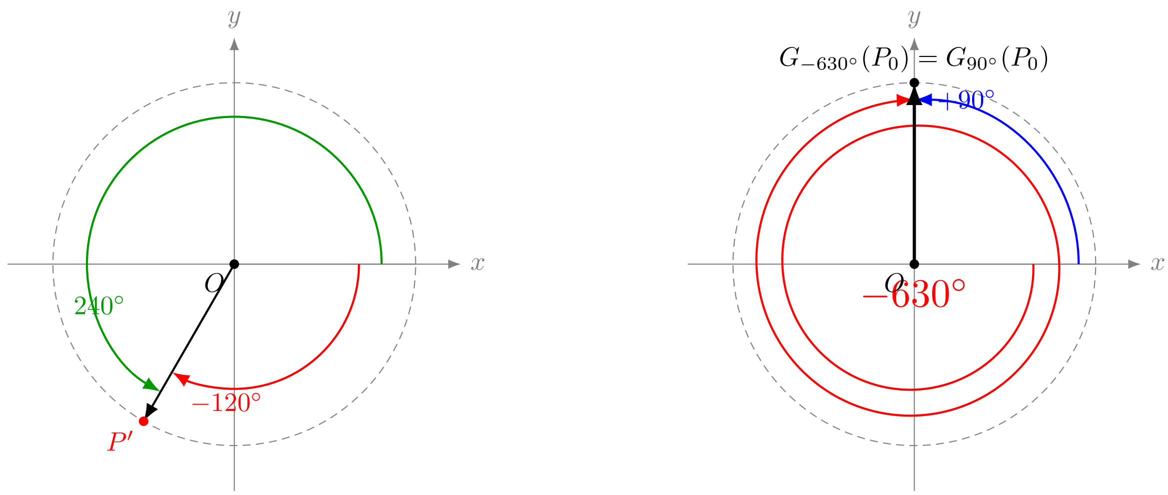
Figure 6.2: Two types of reflection.

Rotations A rotation pivots the plane around a fixed centre.

Definition 6.1.3. ***Rotation***. A rotation about the origin O by an angle θ, denoted G_θ, is a mapping that takes a point P to a point P' such that $d(O, P) = d(O, P')$ and the directed angle $\angle POP'$ has measure θ.

By convention, a positive angle corresponds to a counter-clockwise rotation, and a negative angle corresponds to a clockwise rotation. Since a full rotation corresponds to 360°, the mapping is periodic. A rotation by θ is identical to a rotation by $\theta + 360n°$ for any integer n. For example, a rotation of 450° is equivalent to a rotation of $450° - 360° = 90°$. Similarly, a rotation of $-120°$ (clockwise) is equivalent to a counter-clockwise rotation of $-120° + 360° = 240°$ (see Figure 6.3).

Certain rotations correspond to other mappings we have defined. A rotation by $180°$ is equivalent to a reflection through the origin, thus $G_{180°} = R_O$. A rotation by $0°$ or any integer multiple of $360°$ is the identity mapping, $G_{360k°} = I$ for $k \in \mathbb{Z}$.



(a) $-120° \equiv 240°$

(b) $-630° \equiv 90°$

Figure 6.3: Rotations about the origin. (a) Clockwise $-120°$ and counter-clockwise $240°$ land at the same point. (b) A continuous spiraling arc of $-630°$ shows $-630° \equiv 90°$.

Dilations As in 4.6.1, scalar multiplication by a non-zero factor c defines a dilation centred at the origin. We shall write this as the mapping $D_c(P) = cP$.

6.2 Exercises

Part I: Applying Basic Mappings

1. Let $P = (2, -4)$, $Q = (-1, 3)$, and $A = (3, 1)$. Compute the following points:

(a) The identity mapping on P, $I(P)$.

(b) The translation of P by A, $T_A(P)$.

(c) The reflection of Q through the origin, $R_O(Q)$.

(d) The dilation of P by a factor of 3, $D_3(P)$.

(e) The dilation of Q by a factor of -1/2, $D_{-1/2}(Q)$.

2. A triangle has vertices $V_1 = (1, 1)$, $V_2 = (4, 1)$, and $V_3 = (1, 5)$. Sketch the original triangle and its image under each of the following transformations.

(a) Translation by $A = (-2, -1)$.

(b) Reflection through the origin, R_O.

(c) Rotation by $180°$ about the origin, $G_{180°}$.

(d) Dilation by a factor of 2, D_2.

3. Find the image of the point $P = (5, 0)$ under the following rotations about the origin.

(a) $G_{90°}$

(b) G_{-90°

(c) G_{180°

(d) G_{270°

(e) G_{450°

4. A point P is mapped to $P' = (-6, 8)$. Find the original point P if the transformation was:

(a) A translation by $A = (4, -2)$.

(b) A reflection through the origin.

(c) A dilation by a factor of $c = -2$.

5. What is the image of the line given by the parametric equation $L = \{(1,1) + t(2,3) \mid t \in \mathbb{R}\}$ under the translation T_A where $A = (-1, 2)$? Give the parametric equation for the new line.

6. What is the image of the line $y = 2x + 1$ under a reflection through the origin? Find the equation of the new line.

7. A circle is given by the equation $(x-2)^2 + (y-3)^2 = 9$.

(a) What is the equation of the image of this circle under a translation by $A = (-2, 1)$?

(b) What is the equation of the image of this circle under a reflection through the origin?

(c) What is the equation of the image of this circle under a dilation D_3?

8. A mapping F is applied to a square with vertices at $(0,0), (1,0), (1,1), (0,1)$. The image is a square with vertices at $(0,0), (3,0), (3,3), (0,3)$. What mapping F was applied? Is the answer unique?

Part II: Properties of Mappings

9. A mapping that preserves distance is called an **isometry**. That is, F is an isometry if $d(F(P), F(Q)) = d(P, Q)$ for all points P, Q.

(a) Using the algebraic definitions, prove that any translation T_A is an isometry.

(b) Prove that reflection through the origin R_O is an isometry.

(c) From its geometric definition, argue that any rotation about the origin G_θ is an isometry.

(d) Show that a dilation D_c is an isometry if and only if $|c| = 1$.

10. A point P is a **fixed point** of a mapping F if $F(P) = P$.

(a) Does the identity mapping have any fixed points?

(b) For which point A does the translation T_A have fixed points?

(c) What is the only fixed point of a reflection through the origin, R_O?

(d) What is the only fixed point of a dilation D_c for $c \neq 1$?

(e) What is the set of all fixed points of a reflection across a line L?

11. A mapping F preserves collinearity if for any three collinear points P, Q, R, their images $F(P), F(Q), F(R)$ are also collinear.

(a) Prove that translations preserve collinearity.

(b) Prove that dilations preserve collinearity.

12. A mapping F preserves midpoints if the image of the midpoint of a segment $\overline{PQ}$ is the midpoint of the image segment $\overline{F(P)F(Q)}$. Prove that translations and dilations preserve midpoints.

13. If a mapping is an isometry, must it preserve midpoints? Explain your reasoning.

14. Let S be a square of area 5. What is the area of the image of S under the dilation D_3? What about under D_c? Justify your answer.

Part III: Compositions of Mappings

15. Let $A = (1, 2)$ and $B = (-3, 1)$. Let $F = T_A \circ T_B$ be the composition of a translation by B followed by a translation by A.

(a) Find the image of the point $P = (x, y)$ under F.
(b) Show that F is itself a translation, T_C, and find the point C.
(c) In general, prove that $T_A \circ T_B = T_{A+B}$.

16. Let R_O be reflection through the origin and T_A be translation by $A = (2, -1)$.

(a) Find a formula for the mapping $F = T_A \circ R_O$.
(b) Find a formula for the mapping $G = R_O \circ T_A$.
(c) Is the composition of mappings commutative? That is, is $F = G$?

17. Describe the mapping $F = R_O \circ R_O$. What is its name?

18. A **glide reflection** is the composition of a reflection across a line L and a translation by a point A that is parallel to L. Let L be the x-axis and $A = (3, 0)$. Find the image of the point $P = (1, 2)$ under this glide reflection.

19. Consider the composition of two dilations, $F = D_{c_1} \circ D_{c_2}$. Show that F is also a dilation and find its factor.

20. Let G_{90° be a rotation by 90° about the origin. Let $F = G_{90^\circ} \circ G_{90^\circ}$. What single mapping is equivalent to F?

21. A mapping F has an **inverse**, denoted F^{-1}, if $(F^{-1} \circ F)(P) = P$ for all points P.

(a) What is the inverse of the translation T_A?
(b) What is the inverse of the reflection R_O?
(c) What is the inverse of the rotation G_θ?
(d) What is the inverse of the dilation D_c (for $c \neq 0$)?

22. Show that the composition of two isometries is also an isometry.

Part IV: Proofs and Broader Concepts

23. Prove that any translation maps a line to a parallel line.

Remark. Use the parametric representation of a line, $L = \{P + tA \mid t \in \mathbb{R}\}$, and apply the translation mapping.

24. Let G_θ be a rotation about the origin. Let L be a line through the origin. Does G_θ always map L to another line? If so, what is the angle between L and its image $G_\theta(L)$?

25. Consider reflection across the y-axis, $R_y(x, y) = (-x, y)$. Show that this is an isometry.

26. The mapping $F(x, y) = (x, ky)$ for $k > 0$ is a vertical stretch (if $k > 1$) or compression (if $k < 1$). Is this mapping an isometry? Under what condition?

27. Prove that if a mapping F is an isometry, then it must map a triangle to a congruent triangle.

28. Can an isometry map a square to a line segment? Explain why or why not.

29. Consider the mapping $F(P) = P + A$, where P is a point on the circle $|P| = r$ and A is a fixed point. We proved this maps the circle to another circle. What happens if A is not a fixed point, but is instead defined by $A = P$? Describe the image of the circle $|P| = r$ under the mapping $F(P) = P + P = 2P$.

30. Let R_x be the reflection across the x-axis, R_y be the reflection across the y-axis, and R_O be the reflection through the origin. Prove that $R_x \circ R_y = R_y \circ R_x = R_O$.

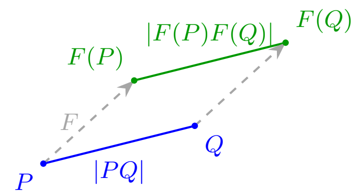
6.3 Isometries

We now define the class of mappings that formally capture the idea of a "rigid motion".

Definition 6.3.1. ***Isometry***. An isometry is a mapping of the plane that preserves distance. A mapping F is an isometry if for any two points P and Q,

$$d(F(P), F(Q)) = d(P, Q).$$

Using norm notation, this is equivalent to $|F(P) - F(Q)| = |P - Q|$.



We can now test which of our fundamental mappings are isometries.

***Theorem* 6.3.1.** Translations are isometries.

Proof. Let T_A be a translation by point A. For any points P and Q, the images are $P + A$ and $Q + A$. The distance between the images is:

$$d(P + A, Q + A) = |(P + A) - (Q + A)| = |P + A - Q - A| = |P - Q| = d(P, Q).$$

Since the distance is preserved, every translation is an isometry. ■

***Theorem* 6.3.2.** Reflections and rotations are isometries.

Proof. For a reflection through the origin, $R_O(P) = -P$. The distance between the images of P and Q is:

$$d(-P, -Q) = | - P - (-Q)| = | - P + Q| = |Q - P| = |P - Q| = d(P, Q).$$

The reflection is an isometry. We accept as axioms that reflections across any line and rotations about any point are also isometries, as rigorous proofs require a more advanced framework. The geometric intuition is that these transformations move figures without altering their size or shape. ■

***Theorem* 6.3.3.** Dilations are not generally isometries.

Proof. Let D_c be a dilation by a factor c. From the theorem on scalar multiplication and distance, we know:

$$d(D_c(P), D_c(Q)) = d(cP, cQ) = |c| \cdot d(P, Q).$$

The distance is only preserved if $|c| = 1$. If $c = 1$, the dilation is the identity map. If $c = -1$, the dilation is a reflection through the origin. For any other factor c, the distance is scaled and the mapping is not an isometry. ■

6.4 Exercises

Part I: Verifying Isometries

1. Let $P = (1, 2)$ and $Q = (4, 6)$. Calculate the distance $d(P, Q)$. Then, for each mapping F below, find the images $P' = F(P)$ and $Q' = F(Q)$, calculate $d(P', Q')$, and determine if F acted as an isometry on this specific pair of points.

(a) $F(P) = T_A(P)$, where $A = (-2, 1)$. (Translation)

(b) $F(P) = R_O(P)$. (Reflection through origin)

(c) $F(P) = D_3(P)$. (Dilation)

(d) $F(P) = (x, -y)$. (Reflection across the x-axis)

2. A mapping is defined by the rule $F(x, y) = (x + 2, y - 3)$.

(a) What is the name of this type of mapping?

(b) Prove from the definition that this mapping is an isometry for any pair of points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$.

3. A mapping is defined by $F(x, y) = (2x, 2y)$. Show with a specific counterexample that this mapping is not an isometry.

4. Consider the mapping $F(x, y) = (y, x)$, which is a reflection across the line $y = x$. Prove that this mapping is an isometry.

5. Consider the mapping $F(x, y) = (x, 0)$, which is a projection onto the x-axis. Is this mapping an isometry? Justify your answer with a counterexample.

6. Consider the mapping $F(x, y) = (x + y, y)$, known as a shear. Test this mapping with the points $P = (0, 0)$ and $Q = (1, 1)$, and then with $P = (0, 0)$ and $Q = (0, 1)$. Is it an isometry?

7. A rotation about the origin by 90° maps a point (x, y) to $(-y, x)$. Use this rule to prove that this rotation is an isometry.

Part II: Properties of Isometries

8. Let F be an isometry. Prove that if F has a fixed point at the origin (i.e., $F(O) = O$), then $|F(P)| = |P|$ for all points P.

9. Let F be an isometry.

(a) Prove that F preserves collinearity. That is, if P, Q, R are collinear, then $F(P), F(Q), F(R)$ are also collinear.

Remark. Use the property that Q is between P and R if and only if $d(P, Q) + d(Q, R) = d(P, R)$.

(b) Does an isometry also preserve the "betweenness" of points? Explain.

10. Let F be an isometry. Prove that F maps a line segment to a line segment of the same length.

11. Let F be an isometry. Prove that F maps any triangle $\triangle PQR$ to a congruent triangle $\triangle F(P)F(Q)F(R)$.

Remark. Use the SSS congruence postulate.

12. Using the result from the previous exercise, prove that isometries preserve angles.

13. Prove that an isometry maps a circle of radius r and centre C to another circle of radius r. What is the centre of the new circle?

14. If a mapping preserves the shape of every figure, must it be an isometry? If a mapping preserves the area of every figure, must it be an isometry? Provide reasoning for your answers.

15. Prove that an isometry must map a pair of parallel lines to another pair of parallel lines.

Part III: Compositions and Inverses

16. Prove that the composition of two isometries is also an isometry.

17. Let F be a mapping. An inverse mapping F^{-1} is such that $F^{-1}(F(P)) = P$ for all P.

(a) Find the inverse of the translation $T_A(P) = P + A$.

(b) Find the inverse of the reflection $R_O(P) = -P$.

(c) Find the inverse of the dilation $D_c(P) = cP$ for $c \neq 0$.

18. If a mapping F is an isometry, prove that its inverse F^{-1}, if it exists, must also be an isometry.

19. What single isometry is equivalent to the composition of a reflection through the origin followed by another reflection through the origin? $F = R_O \circ R_O$.

20. Consider the reflection across the x-axis, $R_x(x, y) = (x, -y)$, and the reflection across the y-axis, $R_y(x, y) = (-x, y)$.

(a) Prove that both R_x and R_y are isometries.

(b) Show that $R_x \circ R_y = R_O$.

(c) Does $R_x \circ R_y = R_y \circ R_x$?

21. A mapping F is applied to a triangle $\triangle ABC$, resulting in a congruent triangle $\triangle A'B'C'$. Does this guarantee that F is an isometry?

Remark. Consider a mapping that acts as an isometry on the three vertices but does not preserve distances for other points.

Part IV: Challenge and Synthesis

22. Can an isometry have exactly one fixed point? Provide an example.

23. Can an isometry have a line of fixed points? Provide an example.

24. Can a non-identity isometry have every point as a fixed point? Explain.

25. A mapping F is called a **conformal mapping** if it preserves angles. We have seen that all isometries are conformal. Is the converse true? Is every conformal mapping an isometry? Provide a counterexample.

26. Let F be an isometry. Show that F can be written in the form $F(P) = U(P) + A$, where U is an isometry that fixes the origin ($U(O) = O$) and A is a point.

Remark. Let $A = F(O)$ and define $U(P) = F(P) - A$.

27. Let F be an isometry that fixes the origin. Prove that F preserves norms, i.e., $|F(P)| = |P|$.

28. Let $P = (x, y)$. Show that the reflection of P across the line $y = mx$ is not a simple algebraic operation like translation or dilation. However, if $m = 1$, what is the formula for the reflection?

29. It is a theorem that any isometry of the plane can be described as a composition of at most three reflections.

(a) Describe a translation as a composition of two reflections.

(b) Describe a rotation about the origin as a composition of two reflections.

30. Can an isometry map a right-handed glove to a left-handed glove? Which type of isometry does this? This property is related to preserving "orientation".

31. Let S be the set of all isometries of the plane. Consider the operation of composition, $\circ$.

(a) Show that S is closed under composition (if $F, G \in S$, then $F \circ G \in S$).

(b) Show that there is an identity element in S.

(c) Show that every element in S has an inverse in S.

(These properties mean that the set of isometries forms a mathematical structure called a group.)

6.5 Composition of Mappings

Mappings can be applied in succession to create a new mapping. This process of combining mappings is called composition.

Definition 6.5.1. ***Composition***. Let F and G be two mappings of the plane. The composition of G followed by F, denoted $F \circ G$, is the mapping defined by:

$$(F \circ G)(P) = F(G(P))$$

for every point P. The operation is performed from right to left: first apply G to P, then apply F to the result.

When we compose two isometries, the resulting mapping is also an isometry.

***Theorem* 6.5.1.** The composition of two isometries is an isometry.

Proof. Let F and G be isometries. Consider the distance between the images of two points P and Q under the composition $F \circ G$.

$$d((F \circ G)(P), (F \circ G)(Q)) = d(F(G(P)), F(G(Q)))$$

Because F is an isometry, it preserves the distance between the points $G(P)$ and $G(Q)$:

$$d(F(G(P)), F(G(Q))) = d(G(P), G(Q))$$

And because G is an isometry, it preserves the distance between P and Q:

$$d(G(P), G(Q)) = d(P, Q)$$

Chaining these equalities together, we have $d((F \circ G)(P), (F \circ G)(Q)) = d(P, Q)$, which proves that $F \circ G$ is an isometry. ■

***Proposition* 6.5.1.** Composition of mappings is associative. That is, for any mappings F, G, H, we have $(F \circ G) \circ H = F \circ (G \circ H)$.

Proof. We test the action of both composite mappings on an arbitrary point P.

$$((F \circ G) \circ H)(P) = (F \circ G)(H(P)) = F(G(H(P)))$$

$$(F \circ (G \circ H))(P) = F((G \circ H)(P)) = F(G(H(P)))$$

Since both mappings produce the same image for any point P, they are the same mapping. ■

Composition of Translations

***Proposition* 6.5.2.** The composition of two translations, T_A followed by T_B, is another translation, given by T_{A+B}. That is, $T_B \circ T_A = T_{A+B}$.

Proof. For any point P, we have:

$$(T_B \circ T_A)(P) = T_B(T_A(P)) = T_B(P + A) = (P + A) + B = P + (A + B) = T_{A+B}(P).$$

The result follows from the associativity of point addition. ■

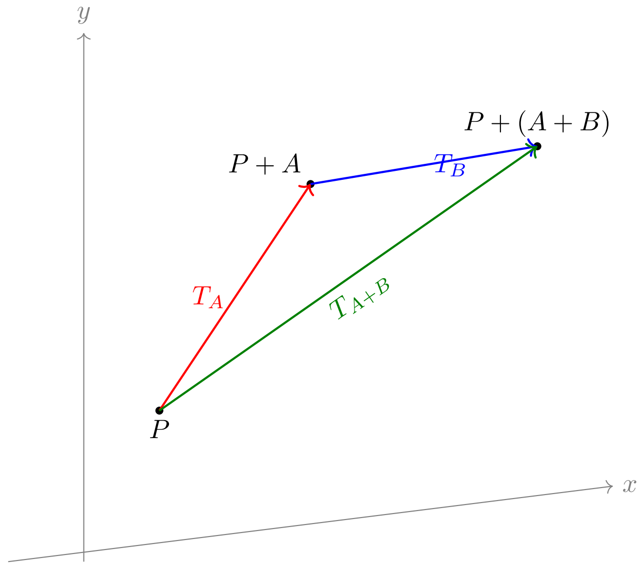
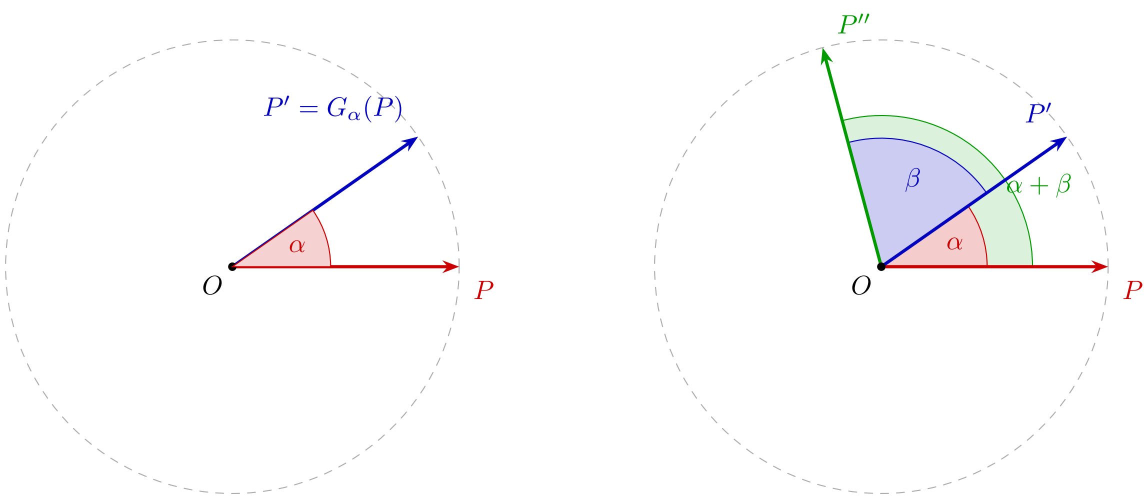


Figure 6.4: Composition of translations: $T_B \circ T_A = T_{A+B}$.

Composition of Rotations

***Proposition* 6.5.3.** The composition of two rotations about the same centre O, G_α followed by G_β, is a rotation about O by the sum of the angles, $G_{\alpha+\beta}$. That is, $G_\beta \circ G_\alpha = G_{\alpha+\beta}$.

Proof. Let P be any point. The mapping G_α rotates P to P' such that $\angle POP' = \alpha$. The mapping G_β then rotates P' to P'' such that $\angle P'OP'' = \beta$. The total rotation from P to P'' is the sum of the individual rotations, so the angle $\angle POP'' = \alpha + \beta$. This corresponds to the definition of the single rotation $G_{\alpha+\beta}$. ■



(a) First rotation by α: $P \mapsto P'$.

(b) Then rotate by β: $P' \mapsto P''$ (total $\alpha + \beta$).

Figure 6.5: Composition of rotations about the same centre: $G_\beta \circ G_\alpha = G_{\alpha+\beta}$. The blue β wedge sits *after* the red α wedge, with the faint green showing the total.

Powers of Mappings We use exponential notation for repeated composition of a mapping with itself.

$$F^2 = F \circ F, \qquad F^3 = F \circ F \circ F, \qquad \text{and so on.}$$

By definition, $F^0 = I$, the identity mapping. This notation follows the familiar rule $F^{m+n} = F^m \circ F^n$.

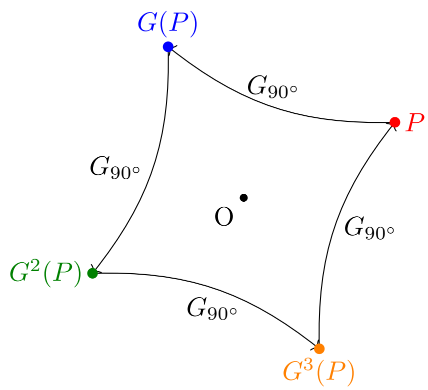
Example 6.5.1. (Powers of Isometries). (a) Let R_O be reflection through the origin. Applying this transformation twice returns every point to its original position.

$$R_O^2(P) = R_O(R_O(P)) = R_O(-P) = -(-P) = P = I(P).$$

Thus, $R_O^2 = I$.

(b) Let G_{90° be a rotation by 90° about the origin.

- $G_{90^\circ}^2 = G_{90^\circ+90^\circ} = G_{180^\circ}$, which is a reflection through the origin.
- $G_{90^\circ}^3 = G_{180^\circ+90^\circ} = G_{270^\circ}$.
- $G_{90^\circ}^4 = G_{270^\circ+90^\circ} = G_{360^\circ}$, which is the identity mapping I.
- $G_{90^\circ}^5 = G_{360^\circ+90^\circ} = G_{90^\circ}$. The sequence of transformations is cyclical.



Composition of Functions

The final way to combine functions is composition, which involves applying one function to the output of another.

Definition 6.5.2. ***Composition***. Let f and g be two functions. The composition of g followed by f, denoted $f \circ g$, is the function defined by:

$$(f \circ g)(x) = f(g(x))$$

The operation is performed from right to left: first, apply g to the input x, and then apply f to the resulting output, $g(x)$.

This process can be visualised as a chain of function machines (cf. Figure 4.8), where the output of the first becomes the input for the second.

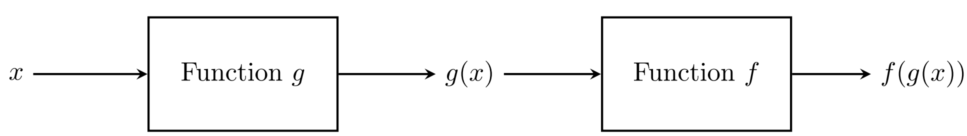


Figure 6.6: Composition as a chain of functions.

Unlike the addition and multiplication of functions, composition is not commutative. The order in which the functions are applied generally matters.

Example 6.5.2. (Composition is Not Commutative). Let $f(x) = x + 1$ and $g(x) = x^2$.

- $(f \circ g)(x) = f(g(x)) = f(x^2) = x^2 + 1$.
- $(g \circ f)(x) = g(f(x)) = g(x + 1) = (x + 1)^2 = x^2 + 2x + 1$.

Clearly, $f \circ g \neq g \circ f$.

The domain of a composite function $f \circ g$ requires special attention. An input x is in the domain of $f \circ g$ only if it is in the domain of g, and its output, $g(x)$, is in the domain of f.

Example 6.5.3. (Domain of a Composition). Let $f(x) = \sqrt{x}$ and $g(x) = x-3$. Find the function $h = f \circ g$ and its domain. The composite function is $h(x) = f(g(x)) = f(x - 3) = \sqrt{x - 3}$. The domain of g is $\mathbb{R}$, but the domain of f is $[0, \infty)$. For an input x to be in the domain of h, the output $g(x)$ must be in the domain of f.

$$g(x) \geq 0 \implies x - 3 \geq 0 \implies x \geq 3.$$

Thus, the domain of $h(x) = \sqrt{x - 3}$ is $[3, \infty)$.

Formally, composition is defined on the sets of ordered pairs.

Definition 6.5.3. ***Formal Composition***. Let f and g be functions. The composition $f \circ g$ is the set of ordered pairs

$$\{(a, c) \mid \text{there exists some } b \text{ such that } (a, b) \in g \text{ and } (b, c) \in f\}.$$

While not commutative, composition is associative.

***Proposition* 6.5.4.** *Composition is associative.* For any functions f, g, h, we have $(f \circ g) \circ h = f \circ (g \circ h)$.

Proof. We show that both functions have the same effect on any input x.

$$((f \circ g) \circ h)(x) = (f \circ g)(h(x)) = f(g(h(x))).$$

$$(f \circ (g \circ h))(x) = f((g \circ h)(x)) = f(g(h(x))).$$

Since the outputs are identical for every valid input x, the functions are equal. ■

6.6 Exercises

Part I: Composition of Mappings

1. Let $A = (2, -1)$, $B = (-3, 4)$, and $P = (1, 1)$. Compute the following:

(a) $(T_A \circ T_B)(P)$

(b) $(T_B \circ T_A)(P)$

(c) $(T_{A+B})(P)$

2. Let G_{90° be a counter-clockwise rotation by 90° about the origin and let R_O be a reflection through the origin. Let $P = (3, 2)$.

(a) Find $(G_{90^\circ} \circ R_O)(P)$.

(b) Find $(R_O \circ G_{90^\circ})(P)$.

(c) What single rotation G_θ is equivalent to R_O?

3. Let R_x be the reflection across the x-axis, so $R_x(x, y) = (x, -y)$. Let G_{90° be the rotation by 90° about the origin.

(a) Find a single algebraic rule for the composition $F = G_{90^\circ} \circ R_x$.

(b) Find a single algebraic rule for the composition $G = R_x \circ G_{90^\circ}$.

(c) Are F and G the same mapping?

4. A mapping F reflects a point across the line $y = x$, so $F(x, y) = (y, x)$. A mapping G translates a point by $A = (2, 0)$. Find the image of the point $P = (1, 3)$ under $F \circ G$ and $G \circ F$.

5. Prove that the composition of two reflections through the origin is the identity mapping, $R_O \circ R_O = I$, using the algebraic definition $R_O(P) = -P$.

6. Let F be the dilation D_2 and G be the translation T_A with $A = (3, 0)$.

(a) Find the image of $P = (1, 1)$ under $F \circ G$ and $G \circ F$.

(b) Does composition of a dilation and a translation commute?

7. A mapping is called an involution if $F^2 = I$. Which of the following are involutions?

(a) The identity I.

(b) A translation T_A for $A \neq (0, 0)$.

(c) A reflection R_O.

(d) A rotation G_{180°.

(e) A rotation G_{90°.

(f) A reflection across a line L.

Part II: Composition of Functions

8. Let $f(x) = 2x - 3$ and $g(x) = x^2 + 1$. Find the rule for each of the following composite functions.

(a) $(f \circ g)(x)$

(b) $(g \circ f)(x)$

(c) $(f \circ f)(x)$

(d) $(g \circ g)(x)$

9. Let $h(x) = \sqrt{x^2 + 4}$. Find two functions f and g such that $h = f \circ g$. Is the choice of f and g unique?

10. Find the domain of the composite function $f \circ g$ for each pair of functions.

(a) $f(x) = \frac{1}{x}$, $g(x) = x + 2$

(b) $f(x) = \sqrt{x}$, $g(x) = x^2 - 9$

(c) $f(x) = \frac{1}{x-1}$, $g(x) = \frac{1}{x-2}$

11. Let $f(x) = \frac{x+1}{x-1}$. Find a formula for $(f \circ f)(x)$. What is the domain of this new function? Simplify your result.

12. Using the functions $S(x) = x^2$, $P(x) = x + 1$, and $R(x) = 1/x$, write each of the following functions as a composition of S, P, and R.

(a) $f(x) = (x + 1)^2$

(b) $g(x) = \frac{1}{x^2+1}$

(c) $h(x) = (x^2 + 1)^2$

13. If f and g are both odd functions, is $f \circ g$ even, odd, or neither? Prove your result.

14. If f is even and g is odd, what can you say about $f \circ g$? What about $g \circ f$?

Part III: Properties and Proofs

15. Provide a specific example of two mappings F and G to show that composition is not commutative.

16. Prove that the composition of two dilations centred at the origin, D_{c_1} and D_{c_2}, is commutative. That is, show $D_{c_1} \circ D_{c_2} = D_{c_2} \circ D_{c_1}$.

17. We proved that $T_B \circ T_A = T_{A+B}$. Use this result to argue that the composition of translations is commutative.

18. A function f has an inverse g if $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$.

(a) Let $f(x) = 2x + 1$. Find its inverse function $g(x)$.
(b) Verify that $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$.

19. Let F and G be two mappings. If F and G both have inverses, F^{-1} and G^{-1}, prove that the inverse of their composition is given by $(F \circ G)^{-1} = G^{-1} \circ F^{-1}$.

20. Let f be a linear function, $f(x) = mx + d$, and g be another linear function, $g(x) = nx + c$. Prove that their composition $f \circ g$ is also a linear function. What is the slope of the new function?

21. If a function f is strictly increasing and g is strictly increasing, prove that their composition $f \circ g$ is also strictly increasing.

22. Let $f(x) = ax$. For what value of a is it true that $f \circ f = f$? For what value is it true that $f \circ f = I$, where $I(x) = x$?

23. A function is defined by the set of ordered pairs $f = \{(1, 2), (2, 3), (3, 1)\}$. Let $g = \{(1, 3), (2, 1), (3, 2)\}$. Find the set of ordered pairs representing $f \circ g$ and $g \circ f$.

Part IV: Synthesis and Challenge

24. Let $f(x)$ be the function that gives the distance from a point $(x, 0)$ on the x-axis to the fixed point $(0, 1)$. Let $g(x) = x^2$. Find the rule for the function $h = g \circ f$. What does this new function represent geometrically?

25. It is a theorem that any isometry of the plane is either a translation, a rotation, a reflection, or a glide reflection.

(a) Let H_A be the rotation by 180° about a point A. This mapping takes a point P to P' such that A is the midpoint of $\overline{PP'}$. Show that $H_A(P)$ can be written as $T_{2A}(R_O(P))$, proving it is a composition of a reflection (rotation about O) and a translation.
(b) Show that the composition of a reflection across the x-axis followed by a reflection across the y-axis is a rotation about the origin. What is the angle of rotation?

26. Let $f(x) = 1/(1-x)$. Find $(f \circ f)(x)$ and $(f \circ f \circ f)(x)$. What is $(f \circ f \circ \cdots \circ f)(x)$ where f is composed 100 times?

27. Consider the set of four mappings $\{I, G_{90°}, G_{180°}, G_{270°}\}$. Show that this set is closed under the operation of composition (i.e., the composition of any two mappings in the set results in a mapping that is also in the set).

28. Can you find a non-identity function f such that $f \circ f = f$?

Remark. Consider functions that project points onto a line or constant functions.

29. Let $f(x) = ax + b$ and $g(x) = cx + d$. Find the condition on the constants a, b, c, d such that $f \circ g = g \circ f$.

30. Let f be a function and I be the identity mapping. Suppose we can find a function g such that $f \circ g = I$. Does this imply that $g \circ f = I$? Find a counterexample.

Remark. Consider functions with different domains and ranges, for example $f(x) = \sqrt{x}$ from $[0, \infty)$ to $\mathbb{R}$ and $g(x) = x^2$ from $\mathbb{R}$ to $\mathbb{R}$.

6.7 Inverse Mappings

For many mappings, it is possible to define an inverse mapping that reverses the transformation, returning every image to its original point.

Definition 6.7.1. ***Inverse Mapping***. Let F be a mapping. An inverse for F, denoted F^{-1}, is a mapping such that

$$F \circ F^{-1} = I \quad \text{and} \quad F^{-1} \circ F = I,$$

where I is the identity mapping.

***Proposition* 6.7.1.** *Uniqueness of the Inverse.* If a mapping F has an inverse, it is unique.

Proof. Suppose G and H are both inverses for F. By definition, $F \circ G = I$ and $H \circ F = I$. Consider the composition $H \circ F \circ G$. Using the associative property of composition:

$$(H \circ F) \circ G = I \circ G = G.$$

$$H \circ (F \circ G) = H \circ I = H.$$

Since composition is associative, the results must be equal, so $G = H$. ■

Remark. The relations $P' = F(P)$ and $P = F^{-1}(P')$ are equivalent. If P' is the image of P under F, we say that P is the inverse image of P' under F.

We can find the inverses for the fundamental isometries.

Example 6.7.1. (Inverses of Isometries). (a) **Reflection:** Let R_L be the reflection across a line L. Applying the reflection twice maps every point back to its original position. Thus, $R_L \circ R_L = R_L^2 = I$. This means a reflection is its own inverse: $R_L^{-1} = R_L$. The same is true for reflection through the origin, $R_O^{-1} = R_O$.

(b) **Rotation:** Let G_θ be a rotation by an angle θ about the origin. The inverse mapping must rotate every point back by the same angle in the opposite direction. This is a rotation by $-\theta$.

$$G_\theta^{-1} = G_{-\theta}$$

This is verified by composition: $G_{-\theta} \circ G_\theta = G_{-\theta+\theta} = G_{0^\circ} = I$.

(c) **Translation:** Let T_A be the translation by a point A. The inverse must be a translation that shifts every point back to its starting place. This is a translation by the point $-A$.

$$T_A^{-1} = T_{-A}$$

We can verify this by composition: $T_{-A} \circ T_A = T_{-A+A} = T_O = I$.

The inverse of a composition of mappings follows a "reverse order" rule.

***Theorem* 6.7.1. Inverse of a Composition**. If mappings F and G have inverses F^{-1} and G^{-1}, then the composition $F \circ G$ has an inverse, given by:

$$(F \circ G)^{-1} = G^{-1} \circ F^{-1}.$$

Proof. We verify that composing this expression with $F \circ G$ yields the identity.

$$(G^{-1} \circ F^{-1}) \circ (F \circ G) = G^{-1} \circ (F^{-1} \circ F) \circ G = G^{-1} \circ I \circ G = G^{-1} \circ G = I.$$

A similar check on the other side, $(F \circ G) \circ (G^{-1} \circ F^{-1})$, also yields the identity. The property is proven. ■

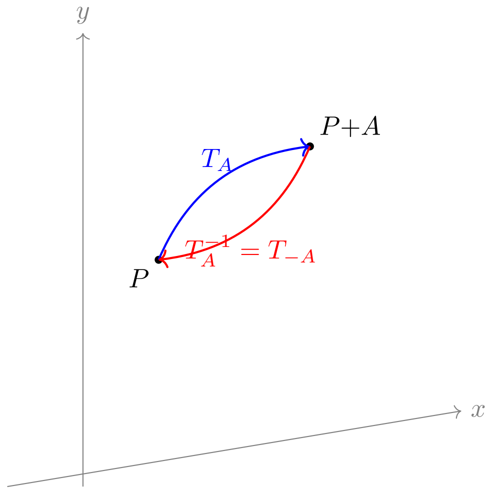


Figure 6.7: A translation and its inverse.

Powers of Mappings We can extend the notation for powers of mappings to include negative integers. For a positive integer k, we define $F^{-k} = (F^{-1})^k$, which represents applying the inverse mapping k times. With the definition $F^0 = I$, the rule of exponents $F^{m+n} = F^m \circ F^n$ holds for all integers m, n.

Injective, Surjective, and Bijective Functions

To classify functions further, we consider the uniqueness of their outputs and the extent to which they cover their codomain.

Definition 6.7.2. ***Injective and Surjective Functions***. Let $f : A \to B$ be a function.

- f is injective (or one-to-one) if for every $x_1, x_2 \in A$, the condition $f(x_1) = f(x_2)$ implies $x_1 = x_2$. This means distinct inputs always produce distinct outputs.
- f is surjective (or onto) if its range is equal to its codomain, i.e., $f(A) = B$. This means every element in the codomain B is the image of at least one element from the domain A.

A function that is both injective and surjective is called bijective. A function has a well-defined inverse function if and only if it is bijective.

Inverse Functions

Just as isometries can have inverses, so too can general functions. An inverse function, if it exists, is a function that "undoes" the action of the original function.

Definition 6.7.3. ***Inverse Function***. Let $f : A \to B$ be a function. An inverse for f is a function $f^{-1} : B \to A$ such that

$$f^{-1} \circ f = I_A \quad \text{and} \quad f \circ f^{-1} = I_B,$$

where I_A and I_B are the identity functions on the sets A and B, respectively ($I_A(x) = x$ for all $x \in A$).

If $y = f(x)$, then applying the inverse function gives $f^{-1}(y) = f^{-1}(f(x)) = x$. This means that if (x, y) is a pair in the function f, then (y, x) is a pair in the function f^{-1}. This implies that the domain of f^{-1} is the range of f, and the range of f^{-1} is the domain of f.

Theorem **6.7.2. Domain and Range of an Inverse Function**. Let f be an invertible function with inverse f^{-1}. Then:

- The domain of f^{-1} is the range of f.
- The range of f^{-1} is the domain of f.

Proof. From the formal definition, a function f is a set of ordered pairs (x, y). Its inverse, f^{-1}, is the set of all ordered pairs (y, x) for every pair (x, y) in f. The domain of f^{-1} is the set of all first elements of its pairs. The pairs in f^{-1} are of the form (y, x), so its domain is the set of all possible y-values. These y-values are precisely the second elements of the pairs in f, which is, by definition, the range of f. Similarly, the range of f^{-1} is the set of all second elements of its pairs, which are the x-values. This set of x-values is, by definition, the domain of f. ∎

Example 6.7.2. (Finding an Inverse Function). Let $f(x) = \frac{3x+1}{2x-7}$. Find its inverse. First, we find the domain of f by setting the denominator to zero: $2x - 7 = 0 \implies x = 7/2$. So $\text{dom}(f) = \mathbb{R} \setminus \{7/2\}$. To find the range, we set $y = f(x)$ and solve for x:

$$\begin{aligned} y(2x-7) &= 3x+1 \\ 2xy - 7y &= 3x+1 \\ 2xy - 3x &= 7y+1 \\ x(2y-3) &= 7y+1 \\ x &= \frac{7y+1}{2y-3} \end{aligned}$$

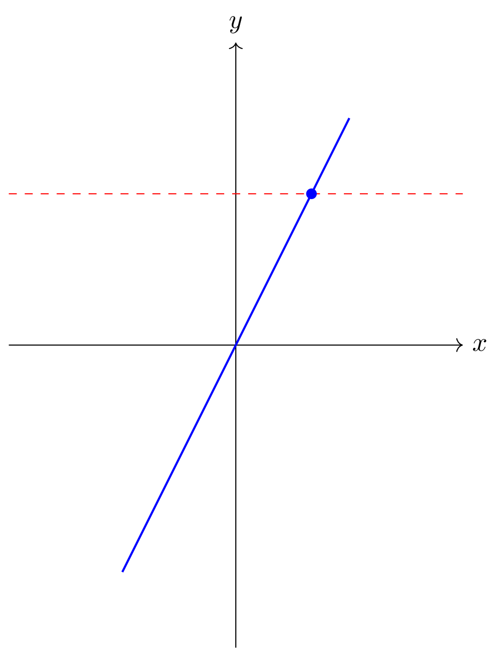
The expression for x is defined for all y except where $2y-3 = 0 \implies y = 3/2$. So the range of f is $\mathbb{R} \setminus \{3/2\}$. This formula for x defines the inverse function.

$$f^{-1}(y) = \frac{7y+1}{2y-3}$$

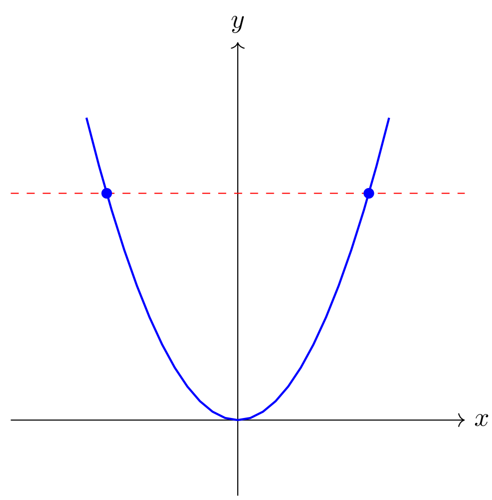
The domain of f^{-1} is the range of f, and the range of f^{-1} is the domain of f.

For an inverse to exist as a function, each output y must be traceable back to exactly one original input x. A function with this property is called one-to-one.

***Theorem* 6.7.3. Horizontal Line Test**. A function is one-to-one, and thus has an inverse function, if and only if no horizontal line intersects its graph more than once.



(a) Invertible: Passes the test.



(b) Not invertible: Fails the test.

Figure 6.8: The Horizontal Line Test for Invertibility.

Example 6.7.3. (Finding an Inverse Function). To find the inverse of a function $f(x)$ algebraically:

1. Write the equation as $y = f(x)$.
2. Solve this equation for x in terms of y.
3. The resulting expression is $f^{-1}(y)$. We can then replace y with x to get the conventional formula for $f^{-1}(x)$.

Let $f(x) = 2x - 3$.

$$\begin{aligned} y &= 2x - 3 \\ y + 3 &= 2x \\ x &= \frac{y+3}{2} \end{aligned}$$

So, $f^{-1}(y) = \frac{y+3}{2}$. The inverse function is $f^{-1}(x) = \frac{x+3}{2}$.

Many functions that are not one-to-one can be made so by restricting their domain.

Example 6.7.4. (Restricting a Domain to Find an Inverse). The function $f(x) = x^2$ is not one-to-one on its natural domain $\mathbb{R}$. However, if we restrict its domain to $[0, \infty)$, the new function is one-to-one. Let $f(x) = x^2$ with domain $[0, \infty)$ and range $[0, \infty)$. To find the inverse, we solve $y = x^2$ for x. This gives $x = \pm\sqrt{y}$. Since we restricted the domain of f to be non-negative, the range of the inverse must also be non-negative. We must therefore choose the positive root: $x = \sqrt{y}$. The inverse function is $f^{-1}(x) = \sqrt{x}$. Geometrically, the graph of f^{-1} is a reflection of the graph of f across the line $y = x$.

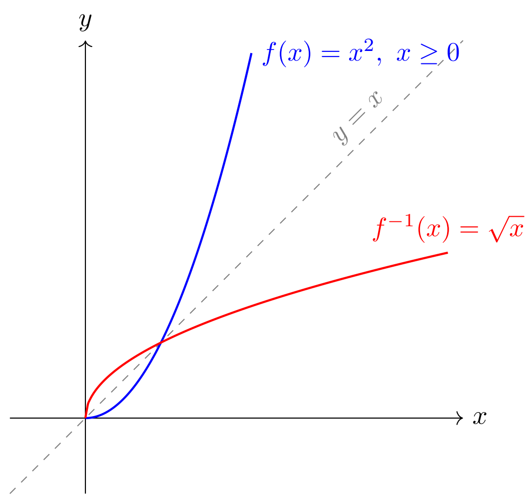


Figure 6.9: A restricted function and its inverse.

Restriction and Local Inverses

A function that is not injective on its natural domain can be made so by considering a smaller domain.

Definition 6.7.4. ***Restriction of a Function***. Let $f : A \to B$ be a function and let $S \subseteq A$. The restriction of f to S, denoted $f|_S$, is the function $g : S \to B$ defined by $g(x) = f(x)$ for all $x \in S$.

If the restriction $f|_S$ is injective, it has an inverse function.

Definition 6.7.5. ***Local Inverse***. Let $f : A \to B$ be a function and $S \subseteq A$ be a subset such that the restriction $f|_S$ is injective. Let $T = \{f(x) \mid x \in S\}$ be the set of outputs from S. The function $g : T \to S$ defined by $g(y) = x$ if and only if $f(x) = y$ is the local inverse of f with respect to S.

Strict Monotonicity and Invertibility on an Interval

Suppose f is strictly monotonic on an interval $I \subseteq \mathbb{R}$. Then f passes the horizontal line test on I (cf. Figure 6.8) and is one-to-one there, so the inverse $f^{-1} : f(I) \to I$ exists.

***Proposition* 6.7.2.** If f is strictly increasing on I, then f^{-1} is strictly increasing on $f(I)$. If f is strictly decreasing on I, then f^{-1} is strictly decreasing on $f(I)$.

Proof. Assume f is strictly increasing on I. Take $y_1 < y_2$ in $f(I)$ with $y_i = f(x_i)$ and $x_i \in I$. Since $y_1 < y_2$ and f is strictly increasing, we must have $x_1 < x_2$. Thus $f^{-1}(y_1) = x_1 < x_2 = f^{-1}(y_2)$, so f^{-1} is strictly increasing. The decreasing case is analogous: if f is strictly decreasing, then $y_1 < y_2$ forces $x_1 > x_2$, hence $f^{-1}(y_1) > f^{-1}(y_2)$. ■

***Corollary* 6.7.1.** If f is strictly monotonic on a closed interval $[a, b]$, then f is invertible on $[a, b]$ and

$$f([a,b]) = \begin{cases} [f(a), f(b)] & \text{if } f \text{ is strictly increasing,} \\ [f(b), f(a)] & \text{if } f \text{ is strictly decreasing.} \end{cases}$$

6.8 Exercises

Part I: Inverses of Mappings

1. For each mapping F, find its inverse F^{-1} and verify that $F^{-1} \circ F = I$.

(a) The translation T_A with $A = (4, -3)$.

(b) The rotation G_{270° about the origin.

(c) The reflection across the x-axis, $R_x(x, y) = (x, -y)$.

(d) The dilation $D_{1/3}$ centred at the origin.

2. A mapping is defined by $F(x, y) = (x + 2, -y)$.

(a) Find the rule for $F^{-1}(x, y)$.

(b) Describe F as a composition of a translation and a reflection.

(c) Use the theorem on the inverse of a composition to re-derive your answer for F^{-1}.

3. Let G_{90° be a rotation by 90° and T_A be a translation by $A = (1, 0)$. Let $F = G_{90^\circ} \circ T_A$. Find the inverse mapping F^{-1}.

4. If a mapping F is an involution (meaning $F^2 = I$), what is its inverse?

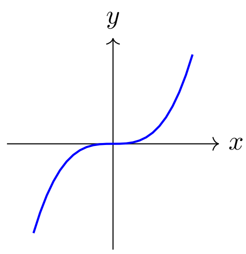
5. Consider the shear transformation $F(x, y) = (x + y, y)$.

(a) Find a formula for the inverse mapping F^{-1}.

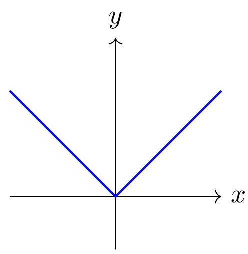
(b) Is a shear an isometry?

Part II: Inverse Functions

6. For each graph, determine if the function is one-to-one by applying the Horizontal Line Test.



(a)



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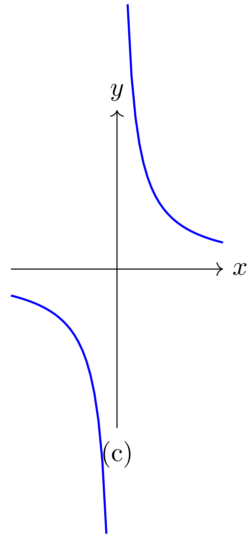


(c)

7. For each of the following one-to-one functions, find the rule for its inverse f^{-1}.

(a) $f(x) = 5x + 3$

(b) $f(x) = \frac{1}{x-1}$

(c) $f(x) = \sqrt[3]{x+2}$

(d) $f(x) = \frac{2x-1}{x+3}$

8. Let $f(x) = \frac{1}{x-1}$ from the previous problem.

(a) State the domain and range of f.

(b) State the domain and range of f^{-1}.

(c) Verify that the domain of f^{-1} is the range of f, and vice versa.

9. Let $f(x) = \sqrt{x-2}$.

(a) Find the domain and range of f.

(b) Find the rule for f^{-1}.

(c) Sketch the graphs of f, f^{-1}, and the line $y = x$ on the same axes.

10. The function $f(x) = x^2 + 4x + 5$ is not one-to-one on $\mathbb{R}$.

(a) Find the vertex of the parabola by completing the square.

(b) Restrict the domain of f to an interval on which it is one-to-one.

(c) Find the inverse of this restricted function.

11. If a function is odd and invertible, is its inverse function also odd? Prove your answer.

12. If a function is even, can it be invertible? Explain.

13. Let $f(x) = x$. Is this function its own inverse? Can you find another function $g(x)$ that is its own inverse?

Remark. Consider the graph of such a function. What symmetry must it have?

14. A function is defined by the set of ordered pairs $f = \{(-1, 3), (0, 1), (1, 4), (2, 2)\}$.

(a) Is f invertible?

(b) If so, write the set of ordered pairs that constitutes f^{-1}.

Part III: Injective, Surjective, Bijective

15. For each function, state whether it is injective, surjective, both (bijective), or neither.

(a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 1$
(b) $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$
(c) $h : [0, \infty) \to \mathbb{R}$, $h(x) = x^2$
(d) $k : [0, \infty) \to [0, \infty)$, $k(x) = x^2$
(e) $m : \mathbb{R} \to \mathbb{R}$, $m(x) = x^3$

16. Give an example of a function that is:

(a) Injective but not surjective.
(b) Surjective but not injective.
(c) Neither injective nor surjective.
(d) Bijective.

17. If f and g are both injective functions, is their composition $f \circ g$ also injective? Prove your claim.

18. If f and g are both surjective functions, is their composition $f \circ g$ also surjective? Prove your claim.

19. If a function $f : A \to B$ is bijective, we know it has an inverse $f^{-1} : B \to A$. Is the inverse function always bijective as well? Explain your reasoning.

20. Consider the floor function $f : \mathbb{R} \to \mathbb{Z}$ where $f(x) = \lfloor x \rfloor$. Is this function injective? Is it surjective?

Part IV: Synthesis and Challenge Problems

21. Let $f(x) = \frac{ax+b}{cx+d}$. Under what condition is f its own inverse?

22. Let $f(x) = 2 - \sqrt{x+1}$. We found its graph by transformations in the previous chapter. Now, find the domain and range of f, find its inverse f^{-1}, and state the domain and range of the inverse.

23. A function f is strictly increasing on an interval I. We know from the text that its inverse f^{-1} is also strictly increasing. Use this fact to argue that the graphs of f and f^{-1} can only intersect on the line $y = x$. Is the same true for a strictly decreasing function?

24. Let $f(x) = mx + b$. Prove that f has an inverse if and only if $m \neq 0$. Find the inverse.

25. Consider the mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x, y) = (x + y, x - y)$. Show that this mapping is bijective by finding its inverse.

26. A function f is defined by $f(x) = x^3 + x$.

(a) Prove that f is strictly increasing by showing that it is the sum of two strictly increasing functions. Why does this guarantee that f has an inverse?
(b) Find $f^{-1}(2)$ and $f^{-1}(10)$.
Remark. You do not need to find a formula for f^{-1}. Think about what $f^{-1}(2) = a$ means.

27. Let $f(x) = (1 - x^{1/3})^3$. Find $f^{-1}(x)$ and show that $f = f^{-1}$.

28. If a function has a domain that is a finite set of points, can it be surjective onto $\mathbb{R}$? Can it be injective?

29. Prove that any linear function $f(x) = mx + b$ with $m \neq -1$ is "conjugate" to a pure scaling function. That is, show there exists an invertible linear function g such that $g \circ f \circ g^{-1}$ is of the form $h(x) = kx$.

30. Let f be a function that is strictly monotonic on a closed interval $[a, b]$. We know it has an inverse on the interval $f([a, b])$. If we are given that f is also continuous (its graph can be drawn without lifting the pen), must its inverse also be continuous? Argue based on the graphical idea of reflecting across the line $y = x$.

6.9 The Structure of Isometries and Congruence

We have defined several types of isometries: translations, rotations, and reflections. The main result of this section is that any isometry of the plane can be expressed as a composition of these fundamental types. This powerful result allows us to provide a rigorous foundation for the concept of geometric congruence.

6.9.1 The Characterisation of Isometries

We build our argument by first considering isometries with fixed points.

***Theorem* 6.9.1. Isometries with Two Fixed Points**. Let F be an isometry that leaves two distinct points P and Q fixed. Then F is either the identity mapping I or the reflection R_L through the line L passing through P and Q.

Proof. Let M be any point not on the line L. Let $M' = F(M)$. Because F is an isometry and leaves P and Q fixed, we have:

$$d(P, M') = d(F(P), F(M)) = d(P, M)$$

$$d(Q, M') = d(F(Q), F(M)) = d(Q, M)$$

This means both P and Q are equidistant from M and M', and therefore must lie on the perpendicular bisector of the segment $\overline{MM'}$. Since P and Q are distinct, the line L passing through them must be the perpendicular bisector of $\overline{MM'}$. This implies that M' is the reflection of M across L, so $F(M) = R_L(M)$.

Now, consider the composite isometry $R_L \circ F$. This mapping has the following properties:

- $(R_L \circ F)(P) = R_L(F(P)) = R_L(P) = P$ (since P is on the line of reflection).
- $(R_L \circ F)(Q) = R_L(F(Q)) = R_L(Q) = Q$ (similarly).
- $(R_L \circ F)(M) = R_L(F(M)) = R_L(M') = M$.

An isometry that fixes three non-collinear points must be the identity mapping. Therefore, $R_L \circ F = I$. Composing both sides with R_L on the left:

$$R_L \circ (R_L \circ F) = R_L \circ I \implies (R_L \circ R_L) \circ F = R_L \implies I \circ F = R_L \implies F = R_L.$$

The only other possibility is that for all points M not on L, $F(M) = M$. In that case, F fixes all points, so $F = I$. ■

***Theorem* 6.9.2. Isometries with One Fixed Point**. An isometry F that leaves a point O fixed is either a rotation about O or a composition of a rotation about O and a reflection through a line passing through O.

Proof. Let P be any point other than O, and let $P' = F(P)$. Since F is an isometry that fixes O, $d(O, P) = d(F(O), F(P)) = d(O, P')$. Thus, P and P' lie on the same circle centred at O. There exists a unique rotation G about O that maps P to P'. Consider the composite isometry $G^{-1} \circ F$. This mapping fixes O, and it also fixes P, since $(G^{-1} \circ F)(P) = G^{-1}(F(P)) = G^{-1}(P') = P$. Since $G^{-1} \circ F$ is an isometry with two fixed points (O and P), by the previous theorem it must be either the identity I or the reflection R_L through the line L passing through O and P. **Case 1:** $G^{-1} \circ F = I$. Composing with G on the left gives $F = G$. Thus, F is a rotation. **Case 2:** $G^{-1} \circ F = R_L$. Composing with G on the left gives $F = G \circ R_L$. Thus, F is a rotation composed with a reflection. ■

***Theorem* 6.9.3. Fundamental Theorem of Isometries**. Any isometry F of the plane is a composition of a translation, a rotation, and at most one reflection.

Proof. Let O be the origin. Let $A = F(O)$. Let T_A be the translation that maps O to A. Consider the composite isometry $H = T_A^{-1} \circ F$. This mapping fixes the origin:

$$H(O) = (T_A^{-1} \circ F)(O) = T_A^{-1}(F(O)) = T_A^{-1}(A) = O.$$

By the previous theorem, H must be either a rotation G or a rotation composed with a reflection, $G \circ R_L$. **Case 1:** $H = G$. Then $T_A^{-1} \circ F = G$, which implies $F = T_A \circ G$. **Case 2:** $H = G \circ R_L$. Then $T_A^{-1} \circ F = G \circ R_L$, which implies $F = T_A \circ G \circ R_L$. In either case, F is a composition of the fundamental types of isometries. ■

6.9.2 Congruence

The characterisation of isometries provides the foundation for a rigorous definition of congruence.

Definition 6.9.1. ***Congruence***. Two sets of points S and S' are congruent, written $S \cong S'$, if there exists an isometry F such that $F(S) = S'$.

We can now prove the classical congruence conditions using this definition.

***Theorem* 6.9.4.** Any two segments of the same length are congruent.

Proof. Let $\overline{PQ}$ and $\overline{MN}$ be two segments of the same length. Let T be the translation that maps M to P, so $T(M) = P$. Let $N' = T(N)$. Since T is an isometry, $d(P, N') = d(T(M), T(N)) = d(M, N)$. We are given $d(P, Q) = d(M, N)$, so we have $d(P, N') = d(P, Q)$. This means N' and Q lie on the same circle centred at P. Let G be the rotation about P that maps N' to Q. The composite isometry $F = G \circ T$ maps M to P and N to Q. Since isometries map line segments to line segments, F maps the segment $\overline{MN}$ onto the segment $\overline{PQ}$. Therefore, $\overline{PQ} \cong \overline{MN}$. ■

***Theorem* 6.9.5. Side-Side-Side Congruence**. Two triangles $\triangle PQM$ and $\triangle P'Q'M'$ are congruent if their corresponding sides have equal lengths.

Proof. There exists an isometry F_1 (a composition of a translation and a rotation, as in the previous proof) that maps the segment $\overline{P'Q'}$ onto the segment $\overline{PQ}$. Let $M'' = F_1(M')$. Then $\triangle PQM''$ is the image of $\triangle P'Q'M'$ under F_1. Since F_1 is an isometry, the side lengths are preserved:

$$d(P, M'') = d(P', M') \quad \text{and} \quad d(Q, M'') = d(Q', M').$$

We are given $d(P, M) = d(P', M')$ and $d(Q, M) = d(Q', M')$. Thus,

$$d(P, M'') = d(P, M) \quad \text{and} \quad d(Q, M'') = d(Q, M).$$

This means that both P and Q lie on the perpendicular bisector of the segment $\overline{MM''}$. Therefore, the line L through P and Q is the perpendicular bisector of $\overline{MM''}$. This implies that M is the reflection of M'' across the line L. Let this reflection be R_L. Then $R_L(M'') = M$, $R_L(P) = P$, and $R_L(Q) = Q$. The composite isometry $F = R_L \circ F_1$ maps $P' \to P$, $Q' \to Q$, and $M' \to M$. This isometry maps $\triangle P'Q'M'$ onto $\triangle PQM$. Thus, the triangles are congruent. ■

6.10 Exercises

Part I: Understanding the Characterisation of Isometries

1. Let F be an isometry. If F fixes three non-collinear points P, Q, M, prove that F must be the identity mapping I.

   Remark. Let X be any other point. Show that its image $F(X)$ must be X by considering its distances to P, Q, M.

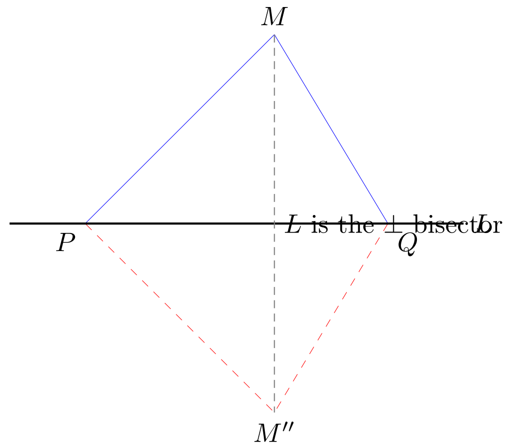


Figure 6.10: The final reflection step in the SSS congruence proof.

2. Let F be an isometry that fixes two distinct points P and Q. The proof in the text states that F is either the identity or the reflection across the line L_{PQ}.

(a) If F maps a third point M (not on the line L_{PQ}) to itself, which of the two possibilities must F be?

(b) If F maps a third point M to a different point M', where must M' be located with respect to M and the line L_{PQ}?

3. Let F be an isometry that fixes exactly one point, the origin O. According to the theorem, what two types of mappings could F be?

4. Let F be an isometry of the plane. Let $P' = F(P)$ and $Q' = F(Q)$. If we apply a translation $T_{-P'}$ to the whole plane, where do the points P' and Q' get mapped?

5. Let F be an arbitrary isometry. Explain the three main steps used in the proof of the Fundamental Theorem of Isometries to decompose F into simpler mappings.

6. An isometry F maps $O = (0, 0)$ to $A = (3, 1)$, and it maps $P = (1, 0)$ to $B = (4, 1)$.

(a) Define an isometry $H = T_{-A} \circ F$. Where does H map the origin?

(b) As H fixes the origin, it must be a rotation or a rotation composed with a reflection. Is H a rotation in this case? Why?

(c) Express F as a composition of simpler isometries.

7. An isometry is called orientation-preserving if it can be written as a composition of a translation and a rotation (an even number of reflections). It is orientation-reversing if it requires one reflection. Classify the following:

(a) A translation.

(b) A rotation.

(c) A reflection across a line.

(d) The identity mapping.

Part II: Congruence

8. Let S be the square with vertices at $(0, 0), (1, 0), (1, 1), (0, 1)$ and let S' be the square with vertices at $(2, 2), (3, 2), (3, 3), (2, 3)$. Find an isometry F such that $F(S) = S'$. Is this isometry unique?

9. Let $\overline{AB}$ be the segment from $(1, 1)$ to $(3, 2)$. Let $\overline{CD}$ be the segment from $(-1, 4)$ to $(1, 5)$.

(a) Show that the two segments have the same length.

(b) Find a translation T that maps A to C.

(c) Find a rotation G about C that maps $T(B)$ to D.

(d) State the composite isometry $F = G \circ T$ that maps $\overline{AB}$ to $\overline{CD}$.

10. Prove that any two circles with the same radius are congruent.

11. Let $\triangle ABC$ have vertices $A = (0, 0), B = (3, 0), C = (0, 4)$. Let $\triangle A'B'C'$ have vertices $A' = (1, 1), B' = (1, -2), C' = (5, 1)$.

(a) Show that the triangles are congruent by calculating their side lengths.

(b) Find an isometry that maps $\triangle ABC$ to $\triangle A'B'C'$.

12. The relation of congruence is an equivalence relation. This means it satisfies three properties. For any sets S_1, S_2, S_3:

(a) **Reflexive:** $S_1 \cong S_1$. Prove this.

(b) **Symmetric:** If $S_1 \cong S_2$, then $S_2 \cong S_1$. Prove this.

(c) **Transitive:** If $S_1 \cong S_2$ and $S_2 \cong S_3$, then $S_1 \cong S_3$. Prove this.

13. Is the set of all points on the graph of $y = x^2$ congruent to the set of all points on the graph of $y = x^2 + 1$? If so, find the isometry.

14. Is the graph of $y = x^2$ congruent to the graph of $y = 2x^2$? Justify your answer.

15. Let L_1 be the line $y = x$ and L_2 be the line $y = x + 2$. Are these two lines congruent sets? Find the isometry.

Part III: Proofs of Congruence Conditions

16. Side-Angle-Side (SAS) Congruence. Prove that if two triangles have two corresponding sides of equal length and the included angles are equal, then the triangles are congruent.

Remark. Use an isometry to map one side $\overline{P'Q'}$ onto $\overline{PQ}$. Then, argue that the third vertex M' must map to M or its reflection across the line L_{PQ}. Use the angle information to rule out the reflection.

17. Angle-Side-Angle (ASA) Congruence. Prove that if two triangles have two corresponding angles of equal measure and the included sides are equal, then the triangles are congruent.

18. A mapping F is a similarity transformation with factor $k > 0$ if $d(F(P), F(Q)) = k \cdot d(P, Q)$ for all points P, Q.

(a) Which of the basic mappings (translation, rotation, reflection, dilation) are similarity transformations?

(b) Two figures S and S' are similar if there exists a similarity transformation F such that $F(S) = S'$. Prove that any two squares are similar.

19. Use the definition of congruence to prove that in an isosceles triangle, the angles opposite the equal sides are equal.

Remark. Consider the reflection of the triangle across the line containing the altitude to the base.

20. Use the definition of congruence to prove that the diagonals of a rectangle are equal in length.

Remark. Find an isometry that maps the rectangle to itself but swaps the endpoints of the two diagonals.

Part IV: The Group of Isometries and Further Structures

21. An isometry F fixes a set S if $F(S) = S$. This is called a symmetry of the set S.

(a) Describe all the symmetries of an equilateral triangle. How many are there?

(b) Describe all the symmetries of a square. How many are there?

(c) Describe all the symmetries of a circle.

22. A glide reflection is the composition of a reflection across a line L and a translation by a vector parallel to L. Let F be a reflection across the x-axis and T be a translation by $A = (3, 0)$. Show that $F \circ T = T \circ F$.

23. It can be shown that any isometry without a fixed point is either a translation or a glide reflection. If F is an isometry that maps $(0, 0) \to (2, 0)$ and $(0, 1) \to (2, 1)$, what kind of isometry must it be?

24. Let R_L be the reflection across a line L and R_M be the reflection across a line M.

(a) If L and M are parallel and separated by a distance d, show that the composition $R_M \circ R_L$ is a translation. Show that the magnitude of the translation is $2d$.

(b) If L and M intersect at a point P with an angle θ between them, show that the composition $R_M \circ R_L$ is a rotation about P.

Remark. Test this with L as the x-axis and M as the line $y = x$ (angle $45°$).

Show that the angle of rotation is 2θ.

25. Using the result from the previous problem, prove that any translation or rotation can be expressed as the composition of two reflections.

26. It is a theorem that any isometry of the plane can be written as the composition of at most three reflections. Combine this with the Fundamental Theorem of Isometries to conclude that translations and rotations can be written as the composition of reflections.

27. Does the set of all translations form a "closed" system under composition? Does the set of all rotations about the origin? Does the set of all reflections?

28. A mapping F is an affine transformation if it is of the form $F(P) = M(P)+A$, where A is a fixed point (a translation) and M is an invertible linear mapping. Show that all isometries are affine transformations.

29. Prove that the composition of two orientation-preserving isometries is also orientation-preserving. What happens when you compose an orientation-preserving isometry with an orientation-reversing one?

30. Is the set of congruent triangles finite or infinite? What about the set of congruent squares? Justify your answers.

Chapter 7

Trigonometry

This chapter introduces the principles of trigonometry, beginning with a system for angle measurement that is intrinsic to the geometry of the circle. This section may be understood following the establishment of coordinates and the distance formula, and does not depend on the analytic descriptions of lines and segments.

7.1 Radian Measure

The measurement of angles by degrees is a historical convention. A more natural system of measurement, founded on the properties of the circle itself, is that of radians.

Recall that the measure of an angle A in degrees is defined by the ratio of the area of the sector it cuts from a disc to the area of the entire disc, scaled by 360. We adapt this principle to define radian measure, scaling instead by the circumference of a unit circle, 2π.

Definition 7.1.1. ***Radian Measure***. Let A be an angle with vertex P, and let D be a disc of any radius centred at P. Let S be the sector of D determined by the angle A. The measure of A in radians, denoted $m_{\text{rad}}(A)$, is defined as

$$m_{\text{rad}}(A) = 2\pi \times \frac{\text{area}(S)}{\text{area}(D)}.$$

Unless specified otherwise, all angle measures will henceforth be in radians.

Remark. Here π is the area of the unit disc from 2.3.1.

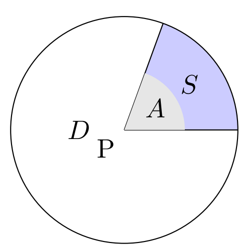


Figure 7.1: The ratio of the area of sector S to the area of disc D defines the radian measure of angle A.

The constant π is the area of a disc of radius 1. For a unit disc, $\text{area}(D) = \pi$. The definition of radian measure simplifies in this case. If an angle has measure x radians,

$$x = 2\pi \times \frac{\text{area}(S)}{\pi} = 2 \cdot \text{area}(S).$$

***Corollary* 7.1.1.** In a unit disc, the area of a sector determined by an angle of x radians is $x/2$.

Relationship with Arc Length Radian measure is directly related to the length of the arc an angle subtends on a unit circle. The ratio of the length of an arc to the circumference is equal to the ratio of the area of its sector to the area of the disc. For a unit circle, the circumference is 2π.

$$\frac{\text{length of arc}}{\text{circumference}} = \frac{\text{area}(S)}{\text{area}(D)}$$

Letting the angle measure be x radians, we have $\frac{\text{area}(S)}{\text{area}(D)} = \frac{x}{2\pi}$.

$$\frac{\text{length of arc}}{2\pi} = \frac{x}{2\pi} \implies \text{length of arc} = x.$$

***Proposition* 7.1.1.** An angle of x radians subtends an arc of length x on a circle of radius 1.

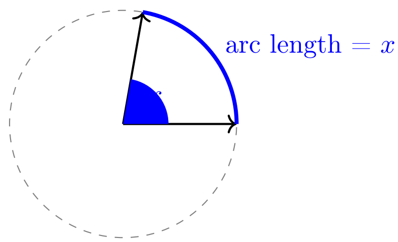


Figure 7.2: For a unit circle, an angle of x radians subtends an arc of length x.

***Proposition* 7.1.2.** Arc length on a circle of radius r Let $x \in \mathbb{R}$. On a circle of radius $r > 0$, an angle of x radians subtends an arc of length $r\,x$.

Proof. Dilate the unit circle by factor r. Lengths scale by r, so the unit-circle case ("arc length $=$ angle") becomes "arc length $= r\times$ angle". ■

***Corollary* 7.1.2.** *Circumference.* The circumference of a circle of radius r is $2\pi r$. In particular, the unit circle has circumference 2π.

Proof. Apply the previous proposition with $x = 2\pi$. ■

Note. Hence the ratio circumference/diameter of the unit circle is $\dfrac{2\pi}{2} = \pi$, consistent with 2.3.1 where π is the area of the unit disc.

Remark. The choice of π as the ratio of a circle's circumference to its diameter, rather than its radius, is a historical convention. Had the fundamental circle constant been defined as $\tau = 2\pi$ (the ratio of circumference to radius), many formulae, including the measure of a full rotation in radians, would appear simpler.

Conversion between Radians and Degrees A full rotation of $360°$ corresponds to the ratio area$(S)/$area$(D) = 1$, which gives an angle of 2π radians. This fundamental equivalence, $360° = 2\pi$ rad, allows for conversion between the two systems.

- $180° = \pi$ radians
- $90° = \pi/2$ radians
- $60° = \pi/3$ radians
- $45° = \pi/4$ radians
- $30° = \pi/6$ radians

To convert from degrees to radians, multiply by $\frac{\pi}{180}$. To convert from radians to degrees, multiply by $\frac{180}{\pi}$.

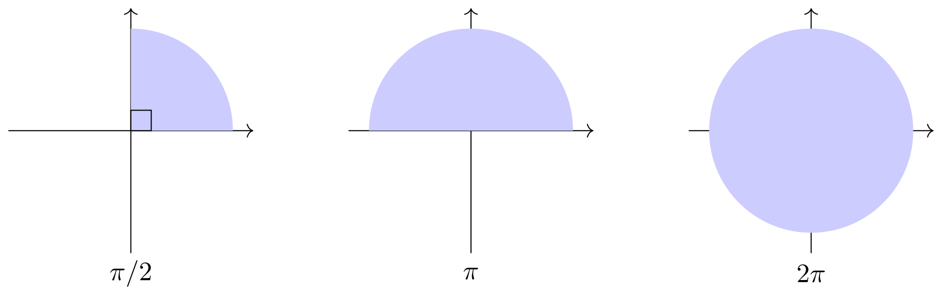


Figure 7.3: Common angles shown in radian measure.

Generalised Angles We extend the concept of angle measure to any real number. An angle of x radians, for any $x \in \mathbb{R}$, is defined to be coterminal with an angle of w radians, where $x = 2\pi k + w$ for some integer k and $0 \leq w < 2\pi$. The integer k represents the number of full counter-clockwise rotations.

A negative angle, such as $-x$ where $x > 0$, represents a rotation in the clockwise direction. For example, an angle of $-\pi/2$ radians is a clockwise rotation by $\pi/2$, which is coterminal with an angle of $3\pi/2$ radians.

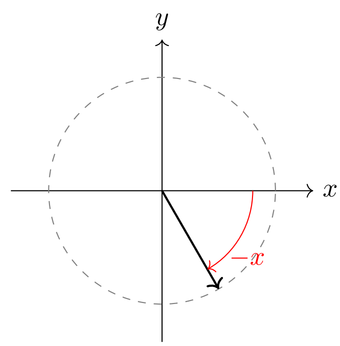


Figure 7.4: A negative angle corresponds to a clockwise rotation.

7.2 Exercises

Part I: Understanding Radian Measure

1. Convert the following degree measures to radians. Express your answers as a multiple of π.

(a) 135°
(b) 270°
(c) -60°
(d) 720°
(e) 150°
(f) 1°

2. Convert the following radian measures to degrees.

(a) $\frac{3\pi}{4}$
(b) $\frac{11\pi}{6}$
(c) $-\frac{\pi}{3}$
(d) 4π
(e) 1 radian (approximate to the nearest degree).

(f) $\frac{\pi}{12}$

3. A wheel rotates through an angle of 480°. How many radians has it rotated through? How many full rotations has it completed?

4. For each of the following angles, find a coterminal angle θ such that $0 \leq \theta < 2\pi$.

(a) $\frac{9\pi}{4}$
(b) $-\frac{\pi}{6}$
(c) 7π
(d) $-\frac{11\pi}{3}$

5. Explain in your own words why the radian measure of an angle is independent of the size of the circle used to measure it. Refer to the definition based on area ratios.

Part II: Arc Length and Sector Area

6. For a circle of radius r and a central angle θ (in radians), the area of the sector is given by $A = \frac{1}{2}r^2\theta$.

(a) Justify this formula using the definition of radian measure.
(b) Find the area of the sector of a circle with radius 10 cm and a central angle of $\pi/4$.
(c) Find the area of the sector of a circle with radius 6 m and a central angle of 120°.

7. Find the length of the arc subtended by the given central angle θ in a circle of radius r.

(a) $r = 5$ cm, $\theta = 2$ radians.
(b) $r = 8$ m, $\theta = 3\pi/4$.
(c) $r = 4$ cm, $\theta = 90°$.

8. A circle has a radius of 6 cm.

(a) Find the central angle (in radians and degrees) that subtends an arc of length 15 cm.
(b) Find the central angle of a sector that has an area of 12 cm^2.

9. A fan blade is 20 cm long. If the fan rotates at a speed of 100 revolutions per minute, how far does the tip of the blade travel in one second?

10. The Earth has an approximate radius of 6400 km.

(a) What is the distance on the surface between two points on the equator whose longitudes differ by 1°?
(b) What is the distance between two points on the same line of longitude whose latitudes differ by 1°? (Assume the Earth is a perfect sphere).

11. A circular pizza of radius 20 cm is cut into 8 equal slices. What is the area of the crust of one slice, if the crust has a uniform width of 3 cm?

Part III: Geometric Problems and Proofs

12. What is the angle (in radians) between the hands of a clock at 4:00?

13. A rope is wrapped around a circular drum of radius 50 cm. How much rope is unwound if the drum is rotated by an angle of 90°?

14. Consider a regular n-gon inscribed in a unit circle.

(a) What is the measure, in radians, of the central angle subtended by one side of the n-gon?

(b) Use this to find the perimeter of an inscribed square.

(c) Use this to find the perimeter of an inscribed hexagon.

15. Two circles have radii r_1 and r_2. An angle θ subtends an arc of length s_1 on the first circle and s_2 on the second. Prove that $\frac{s_1}{s_2} = \frac{r_1}{r_2}$.

16. The area of a segment of a circle (the region between a chord and its arc) can be found by taking the area of the sector and subtracting the area of the triangle formed by the radii and the chord. Find the area of the segment formed by a chord of length 8 in a circle of radius 5.

17. A satellite in a circular orbit 1000 km above the Earth's surface (radius 6400 km) completes one full revolution in 105 minutes. How far does the satellite travel in one minute?

18. A pendulum of length 80 cm swings through an angle of $\pi/6$. What is the length of the arc that the tip of the pendulum traces out?

19. In a unit circle, a central angle θ subtends a chord of length L. Show that $L = 2\sin(\theta/2)$.

Remark. This problem can be solved with basic geometry by bisecting the angle and forming two right triangles.

20. A car has tires of radius 30 cm. If the car is travelling at 60 km/h, what is the angular speed of the tires in radians per second?

7.3 Sine and Cosine

The radian measure of an angle allows for the definition of the fundamental trigonometric functions, sine and cosine, through the geometry of the coordinate plane.

Consider an angle θ with its vertex at the origin O, and its initial side along the positive x-axis. Let $P = (x, y)$ be any point on the terminal side of the angle, other than the origin itself. The distance of this point from the origin is given by $r = d(O, P) = \sqrt{x^2 + y^2}$.

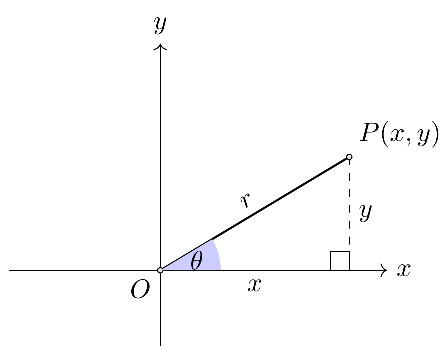


Figure 7.5: An angle θ in standard position.

Definition 7.3.1. ***Sine and Cosine***. The sine and cosine of the angle θ are defined as the ratios:

$$\sin\theta = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}} \qquad \text{and} \qquad \cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}$$

These definitions are independent of the specific point (x, y) chosen on the terminal ray. If another point (x', y') is chosen on the same ray, there exists a positive scalar c such that $x' = cx$ and $y' = cy$. The distance to the origin is $r' = \sqrt{(cx)^2 + (cy)^2} = \sqrt{c^2(x^2 + y^2)} = c\sqrt{x^2 + y^2} = cr$. The ratios remain unchanged:

$$\frac{y'}{r'} = \frac{cy}{cr} = \frac{y}{r} \quad \text{and} \quad \frac{x'}{r'} = \frac{cx}{cr} = \frac{x}{r}$$

Geometrically, this independence is a consequence of the properties of similar right-angled triangles, as shown in Figure 7.6.

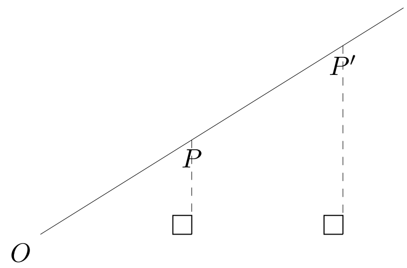


Figure 7.6: Similar triangles formed by different points on the same terminal ray.

The Unit Circle It is often convenient to select the point (x, y) on the unit circle, the circle of radius 1 centred at the origin (cf. 2.1.8). In this case, $r = 1$, and the definitions simplify to:

$$\sin\theta = y \qquad \text{and} \qquad \cos\theta = x$$

The coordinates of any point on the unit circle can thus be expressed as $(\cos\theta, \sin\theta)$, where θ is the angle its corresponding radius vector makes with the positive x-axis.

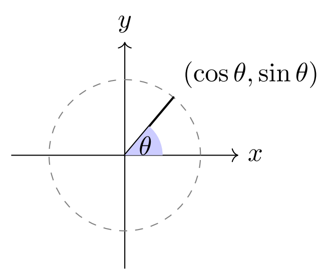


Figure 7.7: The unit circle, where coordinates are defined by cosine and sine.

The signs of the sine and cosine functions are determined by the quadrant in which the angle's terminal side lies.

- Quadrant I: $x > 0, y > 0 \implies \cos\theta > 0, \sin\theta > 0$
- Quadrant II: $x < 0, y > 0 \implies \cos\theta < 0, \sin\theta > 0$
- Quadrant III: $x < 0, y < 0 \implies \cos\theta < 0, \sin\theta < 0$
- Quadrant IV: $x > 0, y < 0 \implies \cos\theta > 0, \sin\theta < 0$

Right-Triangle Trigonometry For an acute angle θ in a right-angled triangle, the coordinate definitions align with the classical ratios. If we place the vertex of θ at the origin and the adjacent side along the x-axis, the coordinates of the third vertex are (adjacent, opposite). The hypotenuse corresponds to r.

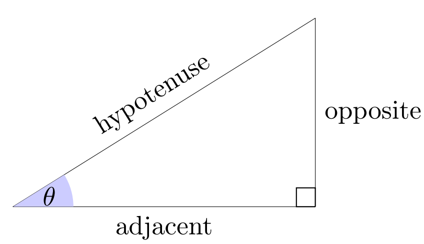


Figure 7.8: Trigonometric ratios in a right-angled triangle.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

Functions of a Real Number We extend the definitions to create functions whose domain is the set of all real numbers. For any number x, we write $x = 2\pi k + w$, where k is an integer and $0 \leq w < 2\pi$. We then define:

$$\sin x = \sin w \qquad \cos x = \cos w$$

This definition makes sine and cosine periodic functions. For any integer k:

$$\sin(x + 2\pi k) = \sin x \qquad \cos(x + 2\pi k) = \cos x$$

Common Values and Fundamental Identities

Certain values of sine and cosine can be derived from elementary geometry.

θ	$\sin\theta$	$\cos\theta$
0	0	1
$\pi/6$	$1/2$	$\sqrt{3}/2$
$\pi/4$	$1/\sqrt{2}$	$1/\sqrt{2}$
$\pi/3$	$\sqrt{3}/2$	$1/2$
$\pi/2$	1	0
π	0	-1

Example 7.3.1. (Derivations).

- For $\theta = \pi/4$ (45°), we consider an isosceles right-angled triangle with equal legs of length 1. By Pythagoras' theorem, the hypotenuse is $\sqrt{1^2 + 1^2} = \sqrt{2}$. Thus, $\sin(\pi/4) = 1/\sqrt{2}$ and $\cos(\pi/4) = 1/\sqrt{2}$.
- For $\theta = \pi/3$ (60°) and $\theta = \pi/6$ (30°), we consider an equilateral triangle with side length 2. The altitude to one base bisects the angle and the base, creating two right-angled triangles with angles $\pi/6, \pi/3, \pi/2$. The sides are 1, 2, and $\sqrt{2^2 - 1^2} = \sqrt{3}$. From this, we deduce:

$$\sin(\pi/3) = \frac{\sqrt{3}}{2}, \quad \cos(\pi/3) = \frac{1}{2}$$

$$\sin(\pi/6) = \frac{1}{2}, \quad \cos(\pi/6) = \frac{\sqrt{3}}{2}$$

***Theorem* 7.3.1. Pythagorean Identity**. For any number x,

$$\cos^2 x + \sin^2 x = 1.$$

Proof. Let $P = (a, b)$ be a point on the unit circle corresponding to the angle x. Then $a = \cos x$ and $b = \sin x$. Since P is on the unit circle, its coordinates must satisfy the equation $x^2+y^2 = 1$, so $(\cos x)^2+(\sin x)^2 = 1$. ∎

***Theorem* 7.3.2. Even-Odd Identities**. For any number x,

$$\cos(-x) = \cos x \quad \text{(even)} \qquad \text{and} \qquad \sin(-x) = -\sin x \quad \text{(odd)}.$$

Proof. Let (a, b) be the point on the unit circle corresponding to the angle x. The angle $-x$ corresponds to a reflection across the x-axis, so its point on the unit circle is $(a, -b)$. We have $\cos x = a$ and $\sin x = b$. Also, $\cos(-x) = a$ and $\sin(-x) = -b$. The identities follow directly. ∎

***Theorem* 7.3.3. Cofunction Identities**. For any number x,

$$\cos x = \sin(x + \pi/2) \qquad \text{and} \qquad \sin x = \cos(x - \pi/2).$$

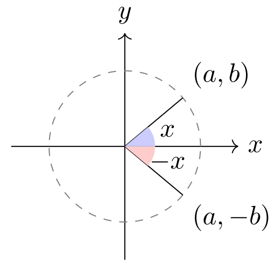


Figure 7.9: Geometric basis for the even-odd identities.

Proof. Let $P = (\cos x, \sin x)$ be a point on the unit circle. A rotation by $\pi/2$ radians, $G_{\pi/2}$, maps a point (a, b) to $(-b, a)$. Applying this rotation to P gives:

$$G_{\pi/2}(P) = (-\sin x, \cos x).$$

The point $G_{\pi/2}(P)$ is the point on the unit circle corresponding to the angle $x + \pi/2$. Therefore, its coordinates must be $(\cos(x + \pi/2), \sin(x + \pi/2))$. Equating the coordinates gives:

$$\cos(x + \pi/2) = -\sin x \quad \text{and} \quad \sin(x + \pi/2) = \cos x.$$

The second identity is proven. The first follows by replacing x with $x - \pi/2$ in the second identity. ■

Polar Coordinates

A point (x, y) in the plane can be described by an alternative coordinate system.

Definition 7.3.2. ***Polar Coordinates***. The polar coordinates of a point $P = (x, y)$ are an ordered pair (r, θ), where $r = \sqrt{x^2 + y^2}$ is the distance from the origin to P, and θ is an angle such that

$$x = r \cos \theta \qquad \text{and} \qquad y = r \sin \theta.$$

When $r = 0$, the point is the origin, and θ is undefined. Otherwise, for a given (x, y), θ is not unique; if θ is a valid angle, then so is $\theta + 2\pi k$ for any integer k. We typically choose the value of θ such that $0 \leq \theta < 2\pi$.

Example 7.3.2. (Coordinate Conversion). (a) Find polar coordinates for the point with rectangular coordinates $(1, \sqrt{3})$.

$$r = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1 + 3} = 2.$$

We require $\cos \theta = 1/2$ and $\sin \theta = \sqrt{3}/2$. The angle in $[0, 2\pi)$ that satisfies these conditions is $\theta = \pi/3$. The polar coordinates are $(2, \pi/3)$.

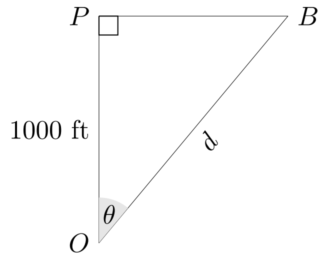
(b) Find the rectangular coordinates of the point with polar coordinates $(4, 2\pi/3)$.

$$x = 4 \cos(2\pi/3) = 4(-1/2) = -2$$

$$y = 4 \sin(2\pi/3) = 4(\sqrt{3}/2) = 2\sqrt{3}$$

The rectangular coordinates are $(-2, 2\sqrt{3})$.

Example 7.3.3. (Application). A boat starts at point P and moves down a straight river. An observer at point O, located 1000 ft from P on a line perpendicular to the river, measures the angle θ between the line $\overline{OP}$ and the line to the boat, $\overline{OB}$. At the exact moment the observer measures the angle such that its cosine is 0.7. Find the distance between the observer and the boat.



From the right-angled triangle $\triangle OPB$, we have $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1000}{d}$. Given $\cos\theta = 0.7$, we have $0.7 = \frac{1000}{d}$. Solving for d gives:

$$d = \frac{1000}{0.7} = \frac{10000}{7} \text{ ft.}$$

7.4 Exercises

Part I: The Unit Circle and Definitions

1. A point $P(x, y)$ on the terminal side of an angle θ is given. Find $\sin\theta$ and $\cos\theta$.

(a) $P(3, 4)$
(b) $P(-5, 12)$
(c) $P(-1, -1)$
(d) $P(8, -15)$

2. Determine the quadrant in which the angle θ lies for each of the following conditions.

(a) $\sin\theta > 0$ and $\cos\theta < 0$
(b) $\sin\theta < 0$ and $\cos\theta < 0$
(c) $\sin\theta < 0$ and $\cos\theta > 0$

3. Find the coordinates of the point on the unit circle corresponding to the following angles.

(a) $\theta = 0$
(b) $\theta = \pi$
(c) $\theta = 3\pi/2$
(d) $\theta = -\pi/2$

4. If $\cos\theta = 4/5$ and θ is in Quadrant IV, find the value of $\sin\theta$.

5. If $\sin\theta = -1/3$ and $\pi < \theta < 3\pi/2$, find the value of $\cos\theta$.

6. Let $P(t)$ be the point on the unit circle corresponding to the angle t. If $P(t)$ has coordinates $(-2/3, \sqrt{5}/3)$, find the coordinates of:

(a) $P(t + \pi)$
(b) $P(-t)$
(c) $P(t - \pi)$
(d) $P(t + \pi/2)$

7. Using only the unit circle definition and geometry, not a calculator, determine which is larger.

(a) $\sin(\pi/5)$ or $\sin(\pi/4)$
(b) $\cos(1)$ or $\cos(2)$ (angles in radians)
(c) $\sin(3)$ or 0

Part II: Fundamental Identities

8. Simplify the following expressions.

(a) $\sin^2(3\theta) + \cos^2(3\theta)$

(b) $1 - \cos^2(x)$

(c) $\cos(\theta)\cos(-\theta) - \sin(\theta)\sin(-\theta)$

9. Prove the identity $1 + \tan^2\theta = \sec^2\theta$, where $\tan\theta = \sin\theta/\cos\theta$ and $\sec\theta = 1/\cos\theta$, by starting with the Pythagorean identity.

10. Let $P(x, y)$ be a point on the terminal ray of an angle θ. We defined $\sin\theta = y/r$ and $\cos\theta = x/r$. Show that the point $Q(x/r, y/r)$ lies on the unit circle and corresponds to the same angle θ.

11. Use the cofunction identities to prove the following:

(a) $\cos(x) = \cos(2\pi - x)$

(b) $\sin(x) = \sin(\pi - x)$

(c) $\cos(x) = -\cos(\pi - x)$

12. The graph of $y = \sin(x)$ is shown. Use transformations and identities to sketch the graphs of:

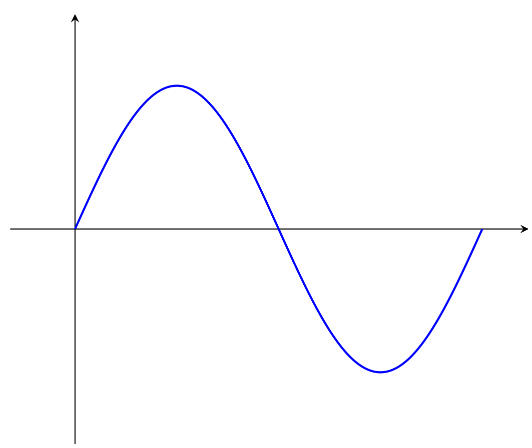
(a) $y = \sin(x) + 1$

(b) $y = \sin(x - \pi/2)$

(c) $y = -\sin(x)$

(d) $y = \cos(x)$

Remark. Use a cofunction identity.



13. A function is periodic with period P if $f(x + P) = f(x)$ for all x.

(a) What is the period of the sine and cosine functions?

(b) What is the period of the function $f(x) = \sin(2x)$?

(c) What is the period of the function $g(x) = \cos(x/3)$?

Part III: Polar Coordinates

14. Find the rectangular coordinates (x, y) for the point with the given polar coordinates (r, θ).

(a) $(3, \pi/2)$

(b) $(2, \pi/4)$

(c) $(5, 4\pi/3)$

(d) $(-2, \pi/6)$

Remark. A negative radius means move in the opposite direction.

15. Find the polar coordinates (r, θ) for the point with the given rectangular coordinates (x, y), with $r \geq 0$ and $0 \leq \theta < 2\pi$.

(a) $(-1, \sqrt{3})$

(b) $(0, -4)$

(c) $(-2, -2)$

(d) $(5, 0)$

16. Convert the following rectangular equations to polar equations.

(a) $x^2 + y^2 = 9$

(b) $x = 4$

(c) $y = x$

17. Convert the following polar equations to rectangular equations.

(a) $r = 3$

(b) $\theta = \pi/4$

(c) $r = \frac{2}{\cos\theta}$

18. Sketch the region in the plane defined by the polar inequalities $1 \leq r \leq 2$ and $0 \leq \theta \leq \pi/2$.

19. $\star$ The distance formula in polar coordinates for two points $P_1(r_1, \theta_1)$ and $P_2(r_2, \theta_2)$ is $d^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)$.

(a) How does this formula relate to the Law of Cosines?

(b) Use this formula to find the distance between $(2, \pi/6)$ and $(3, \pi/2)$.

Part IV: Applications and Challenge Problems

20. From the top of a 100-metre vertical cliff, the angle of depression to a boat at sea is $\pi/6$. How far is the boat from the base of the cliff?

21. A regular pentagon is inscribed in a unit circle. What is the length of one of its sides?

Remark. Use the polar distance formula or the Law of Cosines.

22. Let P_A have rectangular coordinates $(1, \sqrt{3})$ and P_B have rectangular coordinates $(-2, 2\sqrt{3})$.

(a) Convert P_A and P_B to polar coordinates (r_1, θ_1) and (r_2, θ_2).

(b) Define a new point P_C with polar coordinates $(r_1r_2, \theta_1 + \theta_2)$. Find the rectangular coordinates of P_C.

23. $\star$ **Rotation of Coordinates.** A rotation G_α maps a point (x, y) to a new point (x', y').

(a) By writing (x, y) in polar coordinates as $(r\cos\theta, r\sin\theta)$, find the polar coordinates of the rotated point.

(b) Use the sum formulas for sine and cosine (which we will prove later, but you may assume them here: $\cos(A + B) = \cos A\cos B - \sin A\sin B$ and $\sin(A + B) = \sin A\cos B + \cos A\sin B$) to find the formulas for x' and y' in terms of $x, y, \cos\alpha, \sin\alpha$.

24. Show that for any integer n, $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$, where $i = \sqrt{-1}$ (De Moivre's Formula). You can prove this by induction, using the sum formulas from the previous problem.

25. ⋆ A function is defined by $f(t) = \cos(t) + i\sin(t)$, where $i = \sqrt{-1}$. Show that this function satisfies the property $f(a+b) = f(a)f(b)$, which is a characteristic property of exponential functions. This motivates Euler's formula, $e^{it} = \cos t + i \sin t$.

26. A satellite orbits the Earth at an altitude of $h = 500$ km. The radius of the Earth is $R = 6400$ km. An observer on the equator sees the satellite directly overhead at time $t = 0$. At time $t = 1$, the angle between the observer's position vector (from Earth's center) and the satellite's position vector is $\pi/36$ radians (5 degrees).

(a) How far has the observer moved due to Earth's rotation? (Assume Earth is stationary for simplicity if preferred, or calculate arc length on surface).

(b) How far has the satellite travelled along its circular orbit?

27. Find the area of a regular octagon inscribed in a circle of radius R.

28. A spider is at one corner of a cube with side length 1. It wants to reach the opposite corner by walking along the surface. We have seen one path by unfolding the cube. Is there another, shorter path? Compare the lengths of the two possible "unfolded" straight-line paths.

29. Prove that the area of a triangle with vertices at the origin, $A(r_1, \theta_1)$, and $B(r_2, \theta_2)$ is given by $A = \frac{1}{2} r_1 r_2 |\sin(\theta_2 - \theta_1)|$.

30. ⋆ The lemniscate of Bernoulli is given by the polar equation $r^2 = \cos(2\theta)$. Convert this equation to rectangular coordinates.

7.5 The Graphs of Sine and Cosine

The behaviour of the sine and cosine functions can be visualised by plotting the set of points $(x, \sin x)$ and $(x, \cos x)$. These graphs provide insight into the periodic nature and fundamental properties of the functions.

The Graph of the Sine Function We trace the value of $\sin x$ by considering the y-coordinate of a point on the unit circle as the angle x increases from 0 to 2π.

- As x increases from 0 to $\pi/2$ (Quadrant I), the y-coordinate increases from 0 to its maximum value of 1.
- From $\pi/2$ to π (Quadrant II), the y-coordinate decreases from 1 back to 0.
- From π to $3\pi/2$ (Quadrant III), the y-coordinate continues to decrease from 0 to its minimum value of -1.
- From $3\pi/2$ to 2π (Quadrant IV), the y-coordinate increases from -1 back to 0.

This completes one full cycle. Due to the periodic nature of the sine function, where $\sin(x + 2\pi) = \sin x$, this wave pattern repeats indefinitely in both directions along the x-axis. The resulting graph is shown in Figure 7.10.

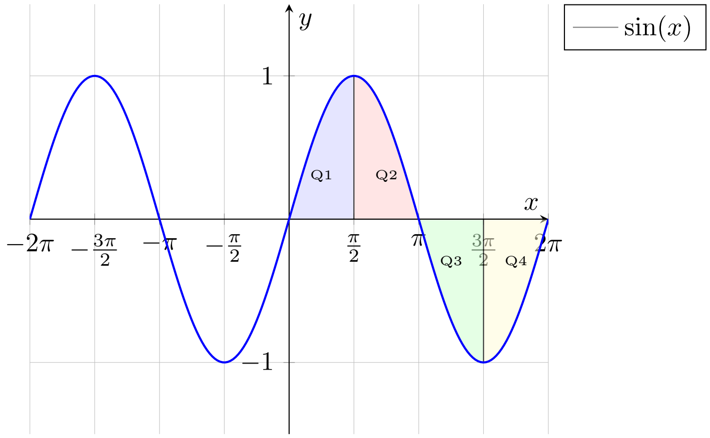


Figure 7.10: The graph of $y = \sin x$. The shaded regions correspond to the four quadrants of a single cycle.

The Graph of the Cosine Function Similarly, we trace the value of $\cos x$ by considering the x-coordinate of a point on the unit circle.

- As x increases from 0 to $\pi/2$ (Quadrant I), the x-coordinate decreases from 1 to 0.
- From $\pi/2$ to π (Quadrant II), the x-coordinate continues to decrease from 0 to -1.
- From π to $3\pi/2$ (Quadrant III), the x-coordinate increases from -1 back to 0.
- From $3\pi/2$ to 2π (Quadrant IV), the x-coordinate increases from 0 back to 1.

The graph of the cosine function, shown in Figure 7.11, exhibits the same periodic wave shape as the sine function. It is, in fact, a horizontal shift of the sine graph. This relationship is captured by the identity $\cos x = \sin(x + \pi/2)$.

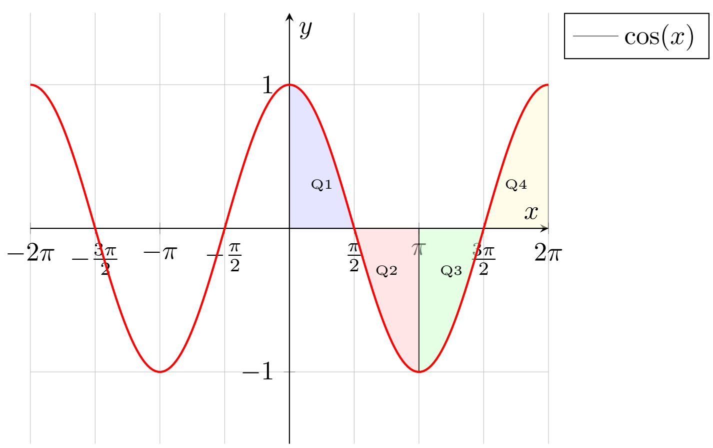


Figure 7.11: The graph of $y = \cos x$.

7.5.1 The Tangent and Cotangent Functions

Building upon the definitions of sine and cosine, we introduce two additional fundamental trigonometric functions.

Definition 7.5.1. ***Tangent***. The tangent of a number x, denoted $\tan x$, is defined as the ratio:

$$\tan x = \frac{\sin x}{\cos x}$$

This function is defined for all real numbers x except those for which $\cos x = 0$. This occurs when $x = \frac{\pi}{2} + k\pi$ for any integer k.

Definition 7.5.2. ***Cotangent***. The cotangent of a number x, denoted $\cot x$, is defined as the ratio:

$$\cot x = \frac{\cos x}{\sin x} = \frac{1}{\tan x}$$

This function is defined for all real numbers x except those for which $\sin x = 0$. This occurs when $x = k\pi$ for any integer k.

Geometrically, for an angle θ in standard position with a point (x, y) on its terminal ray, the tangent is the ratio y/x. This ratio is independent of the distance r from the origin, as r cancels in the division of $\sin\theta = y/r$ by $\cos\theta = x/r$.

***Proposition* 7.5.1.** The tangent of the angle that a non-vertical line makes with the positive x-axis is equal to the slope of the line.

Proof. Let a line pass through the origin with slope m. The equation of the line is $y = mx$. A point on this line is $(1, m)$. For the angle θ this line makes with the x-axis, we have $\tan\theta = y/x = m/1 = m$. ■

The Graph of the Tangent Function The graph of $y = \tan x$ can be understood by analysing the ratio $\sin x / \cos x$.

- When $x = 0$, $\tan x = 0/1 = 0$.
- As x approaches $\pi/2$ from the left, $\sin x$ approaches 1 and $\cos x$ approaches 0 through positive values. The ratio $\tan x$ thus increases without bound, approaching $+\infty$. This indicates a vertical asymptote at $x = \pi/2$.
- As x approaches $-\pi/2$ from the right, $\sin x$ approaches -1 and $\cos x$ approaches 0 through positive values, so $\tan x$ approaches $-\infty$.

The tangent function is periodic. If we consider $\tan(x + \pi)$, we can use the angle addition identities to show $\tan(x + \pi) = \tan x$. The period of the tangent function is π, which is half that of sine and cosine. The graph, with its characteristic repeating branches separated by vertical asymptotes, is shown in Figure 7.12.

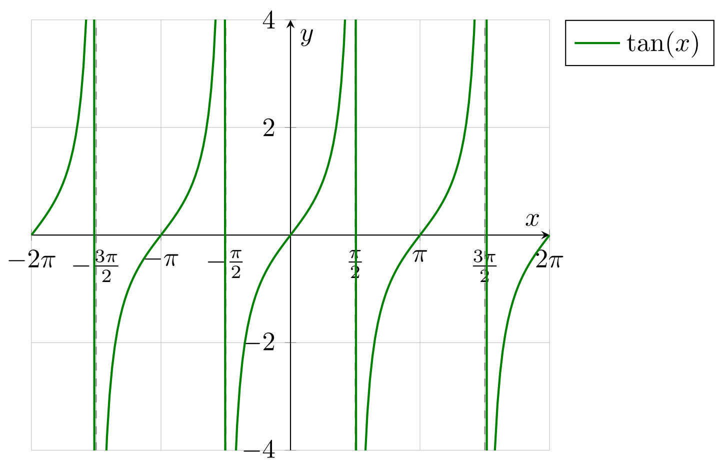


Figure 7.12: The graph of $y = \tan x$ with vertical asymptotes at $x = \frac{\pi}{2} + k\pi$.

The Graph of the Cotangent Function The graph of $y = \cot x$ is derived from the ratio $\cos x/\sin x$. Its vertical asymptotes occur where the denominator, $\sin x$, is zero, which is at integer multiples of π (i.e., at $x = k\pi$). Within the fundamental interval $(0, \pi)$, as x approaches 0 from the right, $\cot x$ approaches $+\infty$. As x approaches π from the left, $\cot x$ approaches $-\infty$. The function is decreasing over each of its branches. Like the tangent, the cotangent function has a period of π. Its graph is shown in Figure 7.13.

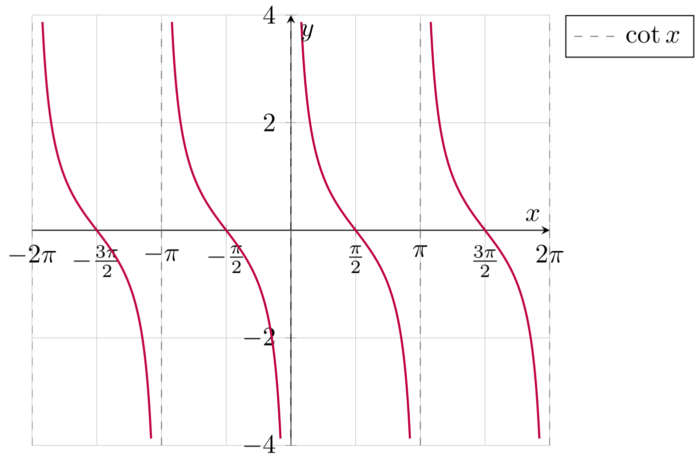


Figure 7.13: The graph of $y = \cot x$ with vertical asymptotes at $x = k\pi$.

Example 7.5.1. (Solving a Trigonometric Equation). Determine all possible values of $\cos x$ given that $\tan x = 2$. The condition $\tan x = y/x = 2$ implies that the terminal ray of the angle x passes through a point where the y-coordinate is twice the x-coordinate, such as $(1, 2)$ or $(-1, -2)$. The tangent is positive in Quadrants I and III.

- **Quadrant I:** The point is $(1, 2)$. The distance from the origin is $r = \sqrt{1^2 + 2^2} = \sqrt{5}$. The cosine is $\cos x = x/r = 1/\sqrt{5}$.
- **Quadrant III:** The point is $(-1, -2)$. The distance is $r = \sqrt{(-1)^2 + (-2)^2} = \sqrt{5}$. The cosine is $\cos x = x/r = -1/\sqrt{5}$.

The two possible values for $\cos x$ are $1/\sqrt{5}$ and $-1/\sqrt{5}$.

Example 7.5.2. (Application). To measure the height of a tower, an observer stands at a distance of 100 ft from its base. The angle of elevation to the top of the tower is measured to be $\pi/3$ radians. Find the height of the tower. Let the height be h. From the right-angled triangle formed, we have:

$$\tan(\pi/3) = \frac{\text{opposite}}{\text{adjacent}} = \frac{h}{100}$$

We know that $\tan(\pi/3) = \sqrt{3}$. Therefore:

$$\sqrt{3} = \frac{h}{100} \implies h = 100\sqrt{3} \text{ ft}.$$

The Secant and Cosecant Functions

The remaining two fundamental trigonometric functions are the reciprocals of the cosine and sine functions.

Definition 7.5.3. ***Secant***. The secant of a number x, denoted $\sec x$, is the reciprocal of the cosine:

$$\sec x = \frac{1}{\cos x}$$

The function is defined for all real numbers x except where $\cos x = 0$, namely $x \neq \frac{\pi}{2} + k\pi$ for any integer k.

Definition 7.5.4. ***Cosecant*.** The cosecant of a number x, denoted $\csc x$, is the reciprocal of the sine:

$$\csc x = \frac{1}{\sin x}$$

The function is defined for all real numbers x except where $\sin x = 0$, namely $x \neq k\pi$ for any integer k.

The Graphs of the Secant and Cosecant Functions The graphs of the secant and cosecant functions are directly related to the graphs of their reciprocal functions.

- The graph of $y = \sec x$ has vertical asymptotes wherever $\cos x = 0$. The local minimum values of $\sec x$ (which are 1) occur where $\cos x$ has its maximum values, and the local maximum values of $\sec x$ (which are -1) occur where $\cos x$ has its minimum values. The range of the secant function is $(-\infty, -1] \cup [1, \infty)$.
- Similarly, the graph of $y = \csc x$ has vertical asymptotes wherever $\sin x = 0$. Its local extrema also correspond to the extrema of the sine function. Its range is also $(-\infty, -1] \cup [1, \infty)$.

Both functions have a period of 2π. Their graphs are illustrated in Figure 7.14 and Figure 7.15.

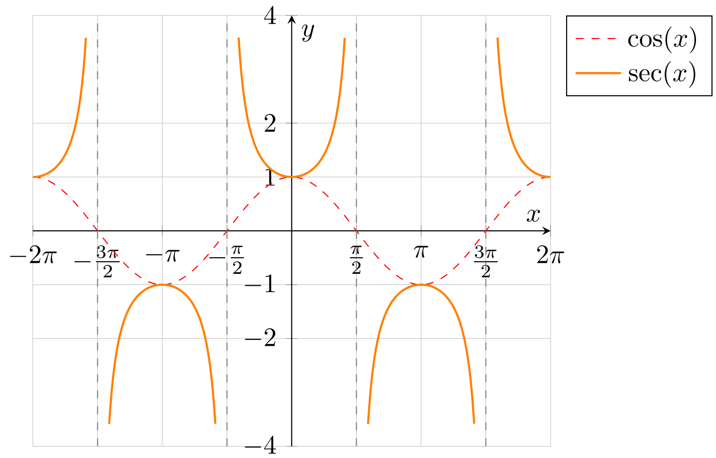


Figure 7.14: The graph of $y = \sec x$, with $y = \cos x$ shown as a guide.

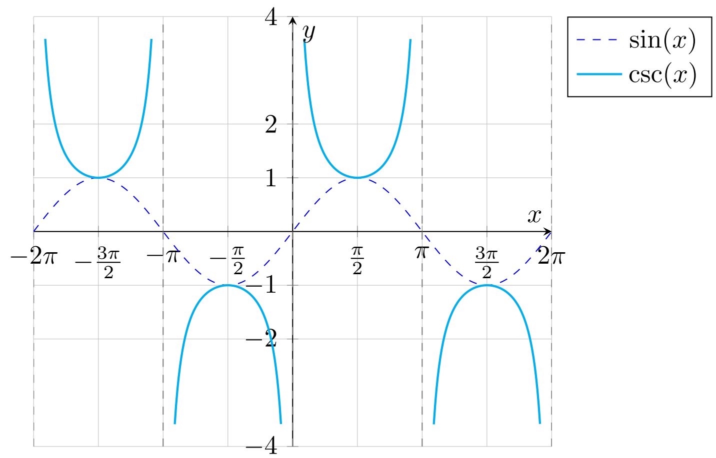


Figure 7.15: The graph of $y = \csc x$, with $y = \sin x$ shown as a guide.

Pythagorean Identities for Tangent, Cotangent, Secant, and Cosecant The fundamental Pythagorean identity from Theorem 7.3.1 gives rise to two other important identities involving the remaining trigonometric functions.

***Theorem* 7.5.1.** . For any number x for which $\cos x \neq 0$,

$$1 + \tan^2 x = \sec^2 x.$$

Proof. We begin with the Pythagorean identity:

$$\cos^2 x + \sin^2 x = 1$$

Since we assume $\cos x \neq 0$, we may divide every term by $\cos^2 x$:

$$\frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

Recognising the definitions of the tangent and secant functions, this simplifies to:

$$1 + \left(\frac{\sin x}{\cos x}\right)^2 = \left(\frac{1}{\cos x}\right)^2$$
$$1 + \tan^2 x = \sec^2 x.$$

■

***Theorem* 7.5.2.** . For any number x for which $\sin x \neq 0$,

$$1 + \cot^2 x = \csc^2 x.$$

Proof. Again, we begin with the Pythagorean identity:

$$\cos^2 x + \sin^2 x = 1$$

Assuming $\sin x \neq 0$, we divide every term by $\sin^2 x$:

$$\frac{\cos^2 x}{\sin^2 x} + \frac{\sin^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$$

Using the definitions of the cotangent and cosecant functions, this simplifies to:

$$\left(\frac{\cos x}{\sin x}\right)^2 + 1 = \left(\frac{1}{\sin x}\right)^2$$
$$\cot^2 x + 1 = \csc^2 x.$$

■

7.6 Exercises

Part I: Graphs of Sine and Cosine

1. Sketch the graph of $y = \sin x$ over the interval $[-2\pi, 2\pi]$. On your graph, clearly label:

(a) The x-intercepts.

(b) The points where the function reaches its maximum value.

(c) The points where the function reaches its minimum value.

(d) The intervals where the function is strictly increasing.

2. Sketch the graph of $y = \cos x$ over the interval $[-2\pi, 2\pi]$.

(a) What is the y-intercept?

(b) Is the function even, odd, or neither? How can you tell from the graph?

(c) On which intervals in $[0, 2\pi]$ is $\cos x > 0$?

3. Use the graph of $y = \sin x$ to find all solutions to the equation $\sin x = 1/2$ in the interval $[0, 2\pi]$.

4. Use the graphs of sine and cosine to solve the inequality $\cos x > \sin x$ on the interval $[0, 2\pi]$.

5. The **amplitude** of a sinusoidal function is half the distance between its maximum and minimum values. The **period** is the length of one full cycle. For the functions $y = \sin x$ and $y = \cos x$, what are the amplitude and period?

6. Use transformations to sketch the graphs of the following functions over one period. State the amplitude and period of each.

(a) $y = 3 \sin x$

(b) $y = \sin(2x)$

(c) $y = \cos(x - \pi/2)$

(d) $y = 2 \cos x + 1$

7. Find the equation of a sine function that has an amplitude of 4 and a period of π.

8. The function $f(x) = A \sin(Bx - C) + D$ describes a general sinusoidal wave.

(a) How do A, B, C, D relate to the amplitude, period, horizontal shift (phase shift), and vertical shift?

(b) Sketch the graph of $y = 2 \sin(x - \pi/4)$.

Part II: The Other Trigonometric Functions

9. Find the value of all six trigonometric functions $(\sin, \cos, \tan, \cot, \sec, \csc)$ for the given angle θ.

(a) $\theta = \pi/4$

(b) $\theta = 2\pi/3$

10. Find the period and sketch the graph of the following functions. Clearly indicate any asymptotes.

(a) $y = \tan x$

(b) $y = \sec x$

(c) $y = \tan(x/2)$

11. If $\tan \theta = -3/4$ and θ is in Quadrant II, find the values of the other five trigonometric functions.

12. Use the fundamental identities to simplify the following expressions.

(a) $\sin x \cot x$

(b) $\frac{1-\sin^2 \theta}{\cos \theta}$

(c) $\sec^2 t - \tan^2 t$

(d) $\frac{\sec x}{\csc x}$

13. For which values of x in $[0, 2\pi]$ is $y = \tan x$ undefined? What about $y = \cot x$, $y = \sec x$, and $y = \csc x$?

14. Prove that the tangent function is an odd function, $\tan(-x) = -\tan(x)$, using the properties of sine and cosine. Is the cotangent function even or odd?

15. Is the secant function even or odd? What about the cosecant function? Justify your answers.

16. Use the graph of $y = \tan x$ to solve the equation $\tan x = 1$ for all values of x.

Part III: Applications and Proofs

17. From a point 200 feet from the base of a building, the angle of elevation to the top of the building is $\pi/4$. Find the height of the building.

18. A 10-metre ladder leans against a wall, making an angle of $\pi/3$ with the ground. How high up the wall does the ladder reach?

19. Prove the identity $\frac{1+\tan x}{1-\tan x} = \frac{\cos x+\sin x}{\cos x-\sin x}$.

20. The average rate of change of $f(x) = \sin x$ on the interval $[0, \pi/2]$ is $\frac{\sin(\pi/2)-\sin(0)}{\pi/2-0} = \frac{1-0}{\pi/2} = 2/\pi$.

(a) Find the average rate of change of $f(x) = \sin x$ on the interval $[\pi/2, \pi]$.

(b) What does this suggest about the shape of the sine curve?

21. Show that the area of a non-right triangle with sides a, b and included angle θ is given by $A = \frac{1}{2}ab\sin\theta$.

Remark. Drop an altitude from one of the vertices to the opposite side.

22. An isosceles triangle has two sides of length 10 and the angle between them is $\pi/6$. Find the area of the triangle.

23. Use the graphs to explain why the equation $\cos x = x$ must have exactly one solution.

Part IV: Challenge Problems

24. $\star$ **Law of Sines.** For a triangle with sides a, b, c opposite angles α, β, γ respectively, prove that:

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

Remark. Use the area formula $A = \frac{1}{2}ab\sin\gamma$ and its permutations.

25. $\star$ **Law of Cosines.** For a triangle with sides a, b, c and angle γ opposite side c, prove that:

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

Remark. Place the triangle in the coordinate plane with vertex γ at the origin and side a along the x-axis. Find the coordinates of the other vertices and use the distance formula.

26. How does the Law of Cosines relate to the Pythagorean theorem?

27. A simple pendulum of length L is released from rest at an angle θ_0 from the vertical. Its angular position as a function of time t is approximately $\theta(t) = \theta_0\cos(\sqrt{g/L}\cdot t)$, where g is the acceleration due to gravity. Describe the motion of the pendulum in terms of amplitude and period.

28. The function $f(x) = \sin(1/x)$ is not defined at $x = 0$. Describe its behaviour as x approaches 0. Sketch the graph for x in the interval $[-1, 1]$.

29. A damped oscillation is described by the function $f(x) = e^{-x/5}\cos(x)$. Sketch this graph. How do the exponential and cosine components interact?

30. Find a function of the form $f(x) = A\cos(Bx - C)$ that has a maximum at $(2, 5)$ and a subsequent minimum at $(4, 1)$.

7.7 Addition Formulas

The addition formulas for sine and cosine are central to trigonometry, enabling the calculation of trigonometric values for sums and differences of angles.

***Theorem* 7.7.1. Addition Formulas**. For any numbers A and B,

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

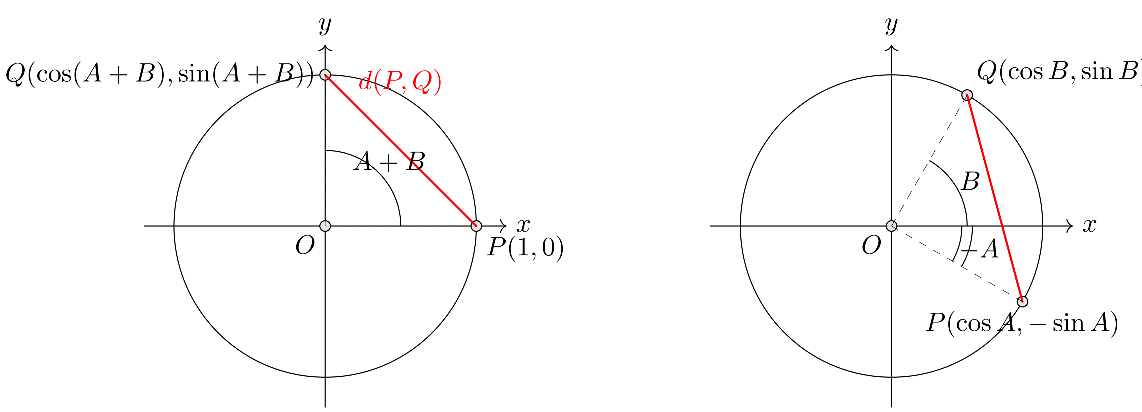
Proof. We first prove the formula for $\cos(A+B)$ using a geometric argument based on the invariance of distance under rotation. Consider a unit circle. We will calculate the squared distance between two points on the circle, P and Q, in two different coordinate orientations.

First, let the point P be at $(1,0)$ and the point Q be at $(\cos(A+B), \sin(A+B))$, as shown in Figure 7.16(a). The squared distance between them is:

$$\begin{aligned} d(P,Q)^2 &= (\cos(A+B)-1)^2 + (\sin(A+B)-0)^2 \\ &= \cos^2(A+B) - 2\cos(A+B) + 1 + \sin^2(A+B) \\ &= (\cos^2(A+B) + \sin^2(A+B)) - 2\cos(A+B) + 1 \\ &= 1 - 2\cos(A+B) + 1 = 2 - 2\cos(A+B). \end{aligned}$$

Next, we rotate the coordinate system by an angle of $-A$. In this new orientation, as shown in Figure 7.16(b), point P is now at $(\cos A, \sin(-A)) = (\cos A, -\sin A)$, and point Q is at $(\cos B, \sin B)$. The distance between the points is unchanged by this rotation. The squared distance is now:

$$\begin{aligned} d(P,Q)^2 &= (\cos B - \cos A)^2 + (\sin B - (-\sin A))^2 \\ &= (\cos^2 B - 2\cos A\cos B + \cos^2 A) + (\sin^2 B + 2\sin A \sin B + \sin^2 A) \\ &= (\cos^2 B + \sin^2 B) + (\cos^2 A + \sin^2 A) - 2\cos A \cos B + 2\sin A \sin B \\ &= 1 + 1 - 2\cos A\cos B + 2\sin A \sin B = 2 - 2\cos A \cos B + 2\sin A \sin B. \end{aligned}$$



(a) First orientation. (b) Rotated orientation.

Figure 7.16: Geometric proof of the cosine addition formula.

Equating the two expressions for the squared distance gives:

$$\begin{aligned} 2 - 2\cos(A+B) &= 2 - 2\cos A\cos B + 2\sin A \sin B \\ -2\cos(A+B) &= -2(\cos A \cos B - \sin A \sin B) \\ \cos(A+B) &= \cos A \cos B - \sin A \sin B. \end{aligned}$$

The formula for $\sin(A+B)$ is derived using the cofunction identities:

$$\begin{aligned}\sin(A+B) &= \cos\left(\frac{\pi}{2}-(A+B)\right) = \cos\left(\left(\frac{\pi}{2}-A\right)-B\right)\\ &= \cos\left(\frac{\pi}{2}-A\right)\cos B + \sin\left(\frac{\pi}{2}-A\right)\sin B\\ &= \sin A\cos B + \cos A\sin B.\end{aligned}$$

■

***Corollary* 7.7.1.** For any numbers x and y,

$$\begin{aligned}\cos(x-y) &= \cos x\cos y + \sin x\sin y\\ \sin(x-y) &= \sin x\cos y - \cos x\sin y\end{aligned}$$

Proof. This follows from the main theorem by substituting $-y$ for B and applying the even-odd identities: $\cos(-y) = \cos y$ and $\sin(-y) = -\sin y$. ■

Remark. The addition formulas, along with the Pythagorean and even-odd identities, are the core formulas of trigonometry from which most others can be derived. It is most efficient to commit these to memory.

Example 7.7.1. (Calculating an Exact Value). Find the exact value of $\sin(\pi/12)$. We write $\pi/12$ as a difference of angles with known trigonometric values, $\pi/12 = \pi/3 - \pi/4$.

$$\begin{aligned}\sin(\pi/12) &= \sin(\pi/3-\pi/4)\\ &= \sin(\pi/3)\cos(\pi/4) - \cos(\pi/3)\sin(\pi/4)\\ &= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}\end{aligned}$$

Double-Angle and Power-Reducing Formulas

The addition formulas lead directly to several other essential identities.

***Theorem* 7.7.2. Double-Angle and Power-Reducing Formulas.** For any number x,

1. $\sin(2x) = 2\sin x\cos x$
2. $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
3. $\cos^2 x = \frac{1+\cos(2x)}{2}$
4. $\sin^2 x = \frac{1-\cos(2x)}{2}$

Proof. For (1), we begin with the sine addition formula:

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

Let $A = x$ and $B = x$. Substituting these into the formula gives:

$$\begin{aligned}\sin(x+x) &= \sin x\cos x + \cos x\sin x\\ \sin(2x) &= 2\sin x\cos x.\end{aligned}$$

For (2), let $A = B = x$ in the cosine addition formula. The other forms of $\cos(2x)$ are found by substituting $\sin^2 x = 1 - \cos^2 x$ or $\cos^2 x = 1 - \sin^2 x$. Formulas (3) and (4) are rearrangements of the alternative forms of the $\cos(2x)$ identity. ■

Example 7.7.2. (Double-Angle Calculation). Suppose x is a number such that $0 < x < \pi/2$ and $\sin x = 0.8$. Find $\sin(2x)$. First, we find $\cos x$. Since x is in Quadrant I, $\cos x$ is positive.

$$\cos x = \sqrt{1 - \sin^2 x} = \sqrt{1 - (0.8)^2} = \sqrt{1 - 0.64} = \sqrt{0.36} = 0.6.$$

Using the double-angle formula for sine:

$$\sin(2x) = 2 \sin x \cos x = 2(0.8)(0.6) = 0.96.$$

Example 7.7.3. (Half-Angle Calculation). Find the exact value of $\cos(\pi/8)$. We use the power-reducing formula for cosine, $\cos^2 \theta = \frac{1+\cos(2\theta)}{2}$, with $\theta = \pi/8$.

$$\cos^2(\pi/8) = \frac{1 + \cos(2 \cdot \pi/8)}{2} = \frac{1 + \cos(\pi/4)}{2} = \frac{1 + 1/\sqrt{2}}{2} = \frac{\sqrt{2} + 1}{2\sqrt{2}}.$$

Since $\pi/8$ is in Quadrant I, $\cos(\pi/8)$ is positive. Taking the square root gives:

$$\cos(\pi/8) = \sqrt{\frac{\sqrt{2} + 1}{2\sqrt{2}}}.$$

To find $\sin(\pi/8)$, we may use the Pythagorean identity:

$$\sin^2(\pi/8) = 1 - \cos^2(\pi/8) = 1 - \frac{\sqrt{2} + 1}{2\sqrt{2}} = \frac{2\sqrt{2} - \sqrt{2} - 1}{2\sqrt{2}} = \frac{\sqrt{2} - 1}{2\sqrt{2}}.$$

Therefore, $\sin(\pi/8) = \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}$.

Product-to-Sum Formulas

By combining the addition and subtraction formulas, we can derive identities that convert products of trigonometric functions into sums, which is a useful technique in calculus and other areas of mathematics.

***Theorem* 7.7.3. Product-to-Sum Formulas**. For any numbers x and integers m, n:

1. $\sin(mx)\sin(nx) = \frac{1}{2}[\cos((m - n)x) - \cos((m + n)x)]$
2. $\sin(mx)\cos(nx) = \frac{1}{2}[\sin((m + n)x) + \sin((m - n)x)]$
3. $\cos(mx)\cos(nx) = \frac{1}{2}[\cos((m + n)x) + \cos((m - n)x)]$

Proof. For (1) we start with the cosine formulas:

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$
$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

Subtracting the second equation from the first gives:

$$\cos(A - B) - \cos(A + B) = 2 \sin A \sin B$$

Letting $A = mx$ and $B = nx$, and dividing by 2, yields the first identity.

(2) and (3) follow the same formula with $\sin(A + B) + \sin(A - B) = 2 \sin A \cos B$ and for (2) $\cos(A - B) + \cos(A + B) = 2 \cos A \cos B$ for (3). ■

7.8 Exercises

Part I: Applying the Addition and Subtraction Formulas

1. Use the addition or subtraction formulas to find the exact value of each expression.

(a) $\cos(75°)$

Remark. Use $75° = 45° + 30°$.

(b) $\sin(15°)$

(c) $\cos(105°)$

(d) $\tan(7\pi/12)$

Remark. First find the formula for $\tan(A + B)$.

2. Use the addition or subtraction formulas to simplify the following expressions.

(a) $\cos(2\theta)\cos(\theta) - \sin(2\theta)\sin(\theta)$

(b) $\sin(x - \pi)$

(c) $\cos(x + \pi/2)$

(d) $\sin(A)\cos(2A) + \cos(A)\sin(2A)$

3. Suppose α and β are angles in Quadrant I, with $\sin\alpha = 3/5$ and $\cos\beta = 5/13$. Find:

(a) $\sin(\alpha + \beta)$

(b) $\cos(\alpha + \beta)$

(c) The quadrant in which $\alpha + \beta$ lies.

4. Let $f(x) = \sin(x)$. Show that the difference quotient can be written as:

$$\frac{f(x + h) - f(x)}{h} = \sin(x)\left(\frac{\cos(h) - 1}{h}\right) + \cos(x)\left(\frac{\sin(h)}{h}\right)$$

5. Derive the addition and subtraction formulas for the tangent function:

(a) $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

(b) $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

Remark. Start with $\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$ and divide the numerator and denominator by $\cos A \cos B$.

6. Prove the identity $\sin(x + y) + \sin(x - y) = 2\sin x \cos y$.

7. A function is given by $f(x) = A\cos x + B\sin x$. Show that this function can be written in the form $f(x) = R\cos(x - \alpha)$, where $R = \sqrt{A^2 + B^2}$, $\cos\alpha = A/R$, and $\sin\alpha = B/R$. This is the harmonic form.

8. Using the result from the previous problem, write the function $f(x) = \cos x + \sqrt{3}\sin x$ in the form $R\cos(x - \alpha)$. What is the maximum value of this function?

Part II: Double-Angle, Half-Angle, and Power-Reducing Formulas

9. If $\cos\theta = -2/3$ and θ is in Quadrant II, find the exact values of:

(a) $\sin(2\theta)$

(b) $\cos(2\theta)$

(c) $\tan(2\theta)$

10. Use a double-angle formula to prove the identity $\frac{\sin(2x)}{1+\cos(2x)} = \tan x$.

11. Use a power-reducing formula to rewrite $\cos^4 x$ as an expression with no powers of trigonometric functions greater than 1.

12. Use a half-angle formula (a rearrangement of the power-reducing formulas) to find the exact value of:

(a) $\sin(22.5^\circ)$
(b) $\cos(\pi/12)$
(c) $\tan(3\pi/8)$

13. Find a formula for $\sin(3x)$ in terms of $\sin x$.

14. Find a formula for $\cos(4x)$ in terms of $\cos x$.

15. The area of an isosceles triangle with two sides of length 1 and an included angle θ is $A = \frac{1}{2}\sin\theta$. Use a double-angle formula to show that this is also equal to $\sin(\theta/2)\cos(\theta/2)$.

Part III: Product-to-Sum and Sum-to-Product Formulas

16. Write each of the following products as a sum of trigonometric functions.

(a) $\sin(3x)\cos(2x)$
(b) $\cos(5\theta)\cos(4\theta)$
(c) $\sin(x)\sin(2x)$

17. By letting $A = \frac{u+v}{2}$ and $B = \frac{u-v}{2}$ in the product-to-sum formulas, derive the following sum-to-product formulas:

(a) $\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$
(b) $\cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$

18. Write each of the following sums as a product.

(a) $\sin(5x) + \sin(3x)$
(b) $\cos(4\theta) - \cos(6\theta)$

19. Find all solutions to the equation $\sin(4x) + \sin(2x) = 0$.

20. In acoustics, the phenomenon of "beats" occurs when two sound waves of slightly different frequencies interfere. This can be modelled by the sum of two sine functions, e.g., $f(t) = \sin(10t) + \sin(11t)$. Use a sum-to-product formula to rewrite this as a single product, representing a fast oscillation modulated by a slow one.

Part IV: Geometric Proofs and Applications

21. Ptolemy's Theorem. For a cyclic quadrilateral, the sum of the products of the opposite sides equals the product of the diagonals: $AB \cdot CD + BC \cdot DA = AC \cdot BD$. Let a quadrilateral be inscribed in a circle of diameter 1. Let the vertices be chosen such that the diagonals are diameters, or use the vertices corresponding to angles $A, B, -B, 90^\circ + A$. Use Ptolemy's Theorem to derive the formula for $\sin(A + B)$.

22. Consider the rotation mapping G_A which maps a point (x, y) to (x', y') where $x' = x\cos A - y\sin A$ and $y' = x\sin A + y\cos A$.

(a) Apply the mapping G_B to the point (x', y') to find the coordinates (x'', y'').
(b) We know that $G_B \circ G_A = G_{A+B}$. Write the formula for G_{A+B} applied to (x, y).

(c) Equate the coefficients of x and y in your two expressions for (x'', y'') to prove the addition formulas for sine and cosine.

23. ⋆ Let a triangle have vertices at the origin O, $P_A = (\cos A, \sin A)$, and $P_B = (\cos B, \sin B)$.

(a) Find the square of the distance between P_A and P_B using the distance formula.

(b) Now use the Law of Cosines on $\triangle OP_AP_B$ to find the square of the same distance. The angle at the origin is $A - B$.

(c) Equate the two expressions to derive the formula for $\cos(A - B)$.

24. Find the angle between two lines with slopes m_1 and m_2.

Remark. Let $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$. The angle between the lines is $\theta_2 - \theta_1$. Use the subtraction formula for tangent.

25. Find the angle between the lines $y = 2x + 1$ and $y = -3x + 5$.

Part V: Challenge Problems

26. Prove that for any triangle $\triangle ABC$, $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

Remark. $A + B + C = \pi$.

27. Find the exact value of $\sin(18°)$.

28. Find the exact value of $\cos(\pi/5)$.

29. Show that $\frac{1-\cos x}{\sin x} = \frac{\sin x}{1+\cos x} = \tan(x/2)$. These are the tangent half-angle formulas.

30. ⋆ If $t = \tan(x/2)$, show that $\cos x = \frac{1-t^2}{1+t^2}$ and $\sin x = \frac{2t}{1+t^2}$. This is the Weierstrass substitution, which connects the trigonometric functions to the rational parameterisation of the unit circle.

Appendix A

Mass Points and the Centre of Mass

Many geometric facts become vivid when viewed through the physical notion of a centre of mass. We shall use this as a motivating device for ratio problems in triangles. The underlying idea can be made precise, but for our purposes we work intuitively and state our assumptions explicitly.

A.1 Mass–Point Notation and Assumptions

Definition A.1.1. ***Mass–Point***. A mass–point mA denotes a point A together with a (positive) mass m. A finite system is written $m_1A_1, \ldots, m_kA_k$, with total mass $m_1 + \cdots + m_k$.

Assumption. *(Two–point centre).* Let m_1A and m_2B be two mass–points. Their centre of mass is the unique point P on the segment $\overline{AB}$ (cf. 2.1.1) satisfying

$$m_1\, d(P, A) \;=\; m_2\, d(P, B).$$

Equivalently, the lengths of the segments are in the ratio $d(P, A) : d(P, B) = m_2 : m_1$. This is illustrated in Figure A.1.

Assumption. *(Replacement and uniqueness).* The centre of mass of a system is unique. One may compute it in stages: any subset of mass–points may be replaced by a single mass–point, equivalent to their total mass located at their collective centre of mass, without altering the final centre of the whole system.

Figure A.1: Two masses m_1 at A and m_2 at B balance at P such that $d(P, A) : d(P, B) = m_2 : m_1$.

These assumptions allow us to encode a division of a segment by writing the mass opposite the point: the label above $\overline{AP}$ is m_2, and above $\overline{PB}$ is m_1, so the ratio is read directly.

A.1.1 The Centroid of a Triangle

Theorem **A.1.1. Medians meet at the centroid and divide each other in ratio** 2:1. In any triangle, the three medians are concurrent at a point G, called the centroid. This point lies two–thirds of the way from each vertex along its median.

Proof. Place a unit mass ($m = 1$) at each vertex A, B, C. By the two–point rule, the pair $1A$ and $1B$ may be replaced by a single mass of 2 at the midpoint of $\overline{AB}$, which we denote M_{AB}. The centre of mass of the original system is now the centre of mass of the system $\{2M_{AB}, 1C\}$. This point, G, must lie on the median $\overline{CM_{AB}}$ and divide it such that $d(G, C) : d(G, M_{AB}) = 2 : 1$.

By symmetry, the same point G would be found by first combining masses at B and C, or at C and A. By the uniqueness of the centre of mass, G must lie on all three medians. Therefore, the medians concur at G, which divides each in a $2 : 1$ ratio as shown in Figure A.2. ■

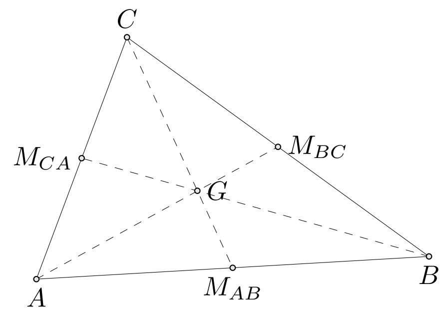


Figure A.2: The medians concur at the centroid G, which satisfies $d(C, G) : d(G, M_{AB}) = 2 : 1$.

Remark. We use braces ($\{\dots\}$) to denote a collection, or system, of mass–points.

A.1.2 Ceva's Theorem via Mass Points

The lines connecting the vertices of a triangle to the opposite sides are called cevians.

***Theorem* A.1.2. Ceva's Theorem**. Let $A_1 \in \overline{BC}$, $B_1 \in \overline{CA}$, and $C_1 \in \overline{AB}$. The cevians $\overline{AA_1}$, $\overline{BB_1}$, and $\overline{CC_1}$ are concurrent if and only if

$$\frac{d(B, A_1)}{d(A_1, C)} \cdot \frac{d(C, B_1)}{d(B_1, A)} \cdot \frac{d(A, C_1)}{d(C_1, B)} = 1.$$

Proof. (Mass–point argument.) Suppose the cevians meet at a point P, as in Figure A.3. We can assign masses $m, n, p > 0$ to vertices A, B, C such that P is the centre of mass of the system $\{mA, nB, pC\}$. The point A_1 on $\overline{BC}$ must be the centre of mass for the subsystem $\{nB, pC\}$. From the two-point rule, this implies $d(B, A_1) : d(A_1, C) = p : n$. Similarly, B_1 on $\overline{AC}$ balances $\{mA, pC\}$, so $d(C, B_1) : d(B_1, A) = m : p$. Finally, C_1 on $\overline{AB}$ balances $\{mA, nB\}$, giving $d(A, C_1) : d(C_1, B) = n : m$. Multiplying these ratios gives:

$$\frac{p}{n} \cdot \frac{m}{p} \cdot \frac{n}{m} = 1.$$

Conversely, if the product of ratios is 1, one can choose positive masses m, n, p that satisfy these three ratio conditions. For example, set $d(A, C_1) : d(C_1, B) = n : m$ and $d(B, A_1) : d(A_1, C) = p : n$. The product condition then forces $d(C, B_1) : d(B_1, A) = m : p$. The centre of mass of the system $\{mA, nB, pC\}$ must lie on all three cevians by construction, forcing their concurrence. ■

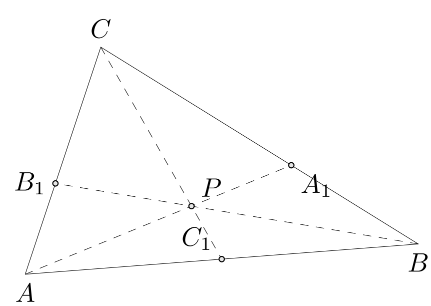


Figure A.3: Cevians $\overline{AA_1}, \overline{BB_1}, \overline{CC_1}$ concurrent at P. Mass–point ratios yield Ceva's criterion.

Worked Ratio Example

Example A.1.1. (Intersecting cevians). In $\triangle ABC$, let $P \in \overline{AB}$ with $d(A,P) : d(P,B) = 5 : 1$, and $Q \in \overline{PC}$ with $d(P,Q) : d(Q,C) = 2 : 3$. The line containing $\overline{BQ}$ meets $\overline{AC}$ at R. Find the ratios $d(A,R) : d(R,C)$ and $d(B,Q) : d(Q,R)$.

We assign masses to balance each given ratio. From $d(A,P) : d(P,B) = 5 : 1$, we place mass 1 at A and mass 5 at B. The combined mass at P is $1+5=6$. From $d(P,Q) : d(Q,C) = 2 : 3$, we require the ratio of masses at C and P to be $2 : 3$. Since mass at P is 6, we need mass at C to be 4 (as $4 : 6 = 2 : 3$). The system is now defined by masses $1, 5, 4$ at vertices A, B, C respectively. The point R on $\overline{AC}$ is the centre of mass of $\{1A, 4C\}$. Thus, $d(A,R) : d(R,C) = 4 : 1$. The point Q lies on the line segment $\overline{BR}$. It must balance the mass at B with the combined mass at R. The mass at R is the sum of masses at A and C, which is $1+4=5$. The mass at B is 5. Thus, Q balances two masses of 5, which means Q is the midpoint of $\overline{BR}$, and $d(B,Q) : d(Q,R) = 5 : 5 = 1 : 1$. This configuration is shown in Figure A.4.

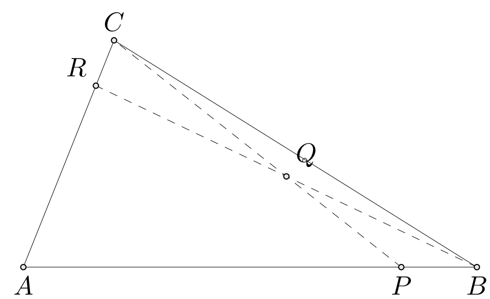


Figure A.4: With masses $1, 5, 4$ at A, B, C, one reads $d(A,R) : d(R,C) = 4 : 1$ and $d(B,Q) : d(Q,R) = 1 : 1$.

A.1.3 Extensions to Solids: Centroid of a Tetrahedron

***Theorem* A.1.3. Centroid of a tetrahedron**. Let $ABCD$ be a tetrahedron. The four segments joining a vertex to the centroid of the opposite face are concurrent at the tetrahedron's centroid M. Moreover, M lies three–quarters of the way from each vertex to the centroid of the opposite face. M also bisects every segment joining the mid–points of opposite edges.

Proof. Assign a unit mass to each vertex $\{1A, 1B, 1C, 1D\}$. To find the centre of mass M, we can first replace the subsystem $\{1A, 1B, 1C\}$ with an equivalent mass of 3 located at the centroid of face $\triangle ABC$, call it G_{ABC}. The total system is now $\{3G_{ABC}, 1D\}$. Their centre of mass M lies on the segment $\overline{DG_{ABC}}$ and divides it in the ratio $3 : 1$, i.e., $d(D,M) : d(M,G_{ABC}) = 3 : 1$. So M is three-quarters of the way from D to G_{ABC}. By symmetry and uniqueness, this point M is the same regardless of which face we start with, so all four such segments concur at M.

Alternatively, consider opposite edges, e.g., $\overline{AB}$ and $\overline{CD}$. Replace $\{1A, 1B\}$ with mass 2 at the midpoint M_{AB}, and $\{1C, 1D\}$ with mass 2 at the midpoint M_{CD}. The centre of mass M of $\{2M_{AB}, 2M_{CD}\}$ must be the midpoint of the segment $\overline{M_{AB}M_{CD}}$. Since M is unique, it must bisect all three such segments connecting midpoints of opposite edges, as depicted in Figure A.5. ■

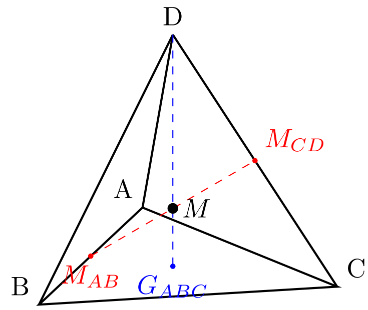


Figure A.5: The centroid M of a tetrahedron lies on the segment from a vertex D to the centroid of the opposite face G_{ABC}, with $d(D, M) : d(M, G_{ABC}) = 3 : 1$. It also bisects the segment $\overline{M_{AB}M_{CD}}$ joining midpoints of opposite edges.

A.2 Centre of Mass Axioms

We now formalise the rules used so far. Capital letters $A, B, \ldots, P, Q, \ldots$ denote points; lower-case letters $a, b, \ldots, m, n, \ldots$ denote positive numbers (masses). A mass–point is written mP. We introduce a binary operation $(+)$ on mass–points, where the sum of two mass–points is the single mass–point concentrated at their centre of mass.

***Axiom* A.2.1. *(Closure)*.** For any mass–points mP and nQ, there is a unique mass–point of total mass $m + n$, written $mP + nQ = (m + n)R$, where R is the centre of mass on $\overline{PQ}$.

***Axiom* A.2.2. *(Commutativity)*.** $mP + nQ = nQ + mP$.

***Axiom* A.2.3. *(Associativity)*.** $mP + (nQ + rR) = (mP + nQ) + rR$.

***Axiom* A.2.4. *(Idempotence)*.** $mP + nP = (m + n)P$.

***Axiom* A.2.5. *(Homogeneity)*.** For $k > 0$, $k(mP + nQ) = kmP + knQ$. This states that the centre of mass depends only on the ratio of masses.

***Axiom* A.2.6. *(Subtraction)*.** If $m > n$, the equation $mP = nQ + xX$ has a unique solution for the mass–point xX (where $x = m - n > 0$).

Remark. The commutative and associative laws formalise the assumption that a system has only one centre of mass, regardless of how the calculation is grouped. The associative law is geometrically equivalent to Menelaus's theorem.

These axioms justify algebraic manipulation.

Definition A.2.1. *Difference of mass–points*. For $m > n$, the difference $mP - nQ$ is defined as the unique mass–point xX which solves the equation $mP = nQ + xX$.

Definition A.2.2. *Betweenness*. A point P is between A and B if there exist $m, n > 0$ such that $mA + nB = (m + n)P$.

Definition A.2.3. *Insidedness*. A point P is inside triangle ABC if there exist $m, n, p > 0$ such that $mA + nB + pC = (m + n + p)P$.

***Theorem* A.2.1. Cancellation Law.** If $mA + nB = mA + pC$, then $nB = pC$. In particular, this implies $n = p$ and $B = C$.

Proof. Let $mA + nB = qD$. By hypothesis, $mA + pC = qD$ as well. By A.2.6, the equation $mA + xX = qD$ has a unique solution. Since both nB and pC are solutions, they must be equal: $nB = pC$. ■

A.2.1 An Algebraic Attack on Geometry

We now recast the mass–point techniques in a purely algebraic form using the axioms. An identity such as $mP + nQ = (m + n)R$ encodes the geometric fact that R lies on the segment $\overline{PQ}$ dividing it in the ratio

$n : m$.

Example A.2.1. (Solving for an intersection point). In $\triangle ABC$, let $D \in \overline{AB}$ and $P \in \overline{DC}$ be defined by the relations

$$3D = 1A + 2B, \qquad 4P = 1D + 3C.$$

The line $\overline{BP}$ intersects $\overline{AC}$ at a point F. Find the ratio $d(A, F) : d(F, C)$. This setup is shown in Figure A.6. To find P in terms of the vertices, we eliminate D. Multiply the second equation by 3:

$$12P = 3D + 9C.$$

Substitute the expression for $3D$ from the first equation:

$$12P = (1A + 2B) + 9C = A + 2B + 9C.$$

To find F, we seek a point on the line through B and P that also lies on $\overline{AC}$. A point on the line $\overline{BP}$ can be expressed as a combination of B and P. Algebraically, we can isolate the terms involving A and C using the difference operation from A.2.6:

$$12P - 2B = A + 9C.$$

The right side, $A + 9C$, represents a mass-point of mass 10 located on $\overline{AC}$. Let this point be $10F$. The left side represents a point on the line through P and B. Since these are equal, the point F must be the intersection of $\overline{BP}$ and $\overline{AC}$. The relation $10F = A + 9C$ implies that F divides $\overline{AC}$ in the ratio $9 : 1$. Thus, $d(A, F) : d(F, C) = 9 : 1$.

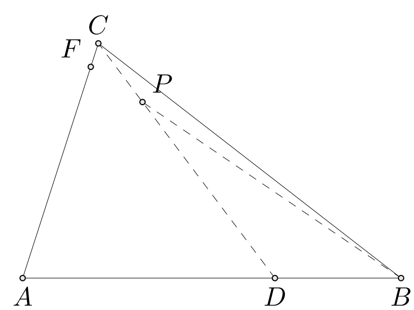


Figure A.6: From $3D = A+2B$ and $4P = D+3C$ one finds $12P = A+2B+9C$ and $d(A, F) : d(F, C) = 9 : 1$.

Example A.2.2. (Interpreting a mass–point identity). Interpret the equation $(1A + 2B) + 1C = (1A + 1B) + (1B + 1C)$ geometrically. Let us define new points from the terms in parentheses.

$$1A + 2B = 3D, \quad 1A + 1B = 2E, \quad 1B + 1C = 2F.$$

These definitions imply D divides $\overline{AB}$ in ratio $2 : 1$, E is the midpoint of $\overline{AB}$, and F is the midpoint of $\overline{BC}$. Substituting these into the original equation gives:

$$3D + 1C = 2E + 2F.$$

Let $4X$ be the resulting mass-point. The left side, $3D + 1C = 4X$, implies X lies on $\overline{DC}$ and divides it in ratio $1 : 3$. The right side, $2E + 2F = 4X$, implies X is the midpoint of $\overline{EF}$. The identity thus asserts that the segment connecting D to C and the segment connecting the midpoints E and F intersect at a point X which is the midpoint of $\overline{EF}$ and divides $\overline{DC}$ in the ratio $1 : 3$. This is illustrated in Figure A.7.

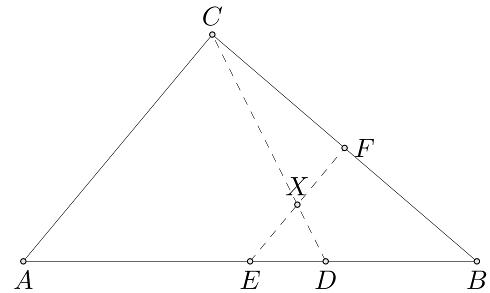


Figure A.7: The identity yields concurrence at X, with X bisecting $\overline{EF}$ and dividing $\overline{DC}$ in ratio 1 : 3.

Remark. Equating coefficients in a relation like $\alpha A+\beta B+\gamma C=\alpha' A+\beta' B+\gamma' C$ is legitimate only when A, B, C are not collinear. If they are collinear, a point on the line can have multiple representations.

A.3 Triangles and Art?

We can reinterpret the axioms of mass points through the analogy of paint–mixing. Let a "point" be a "colour" and a mass–point mP be "m litres of colour P". The operation + corresponds to mixing paints.

- Closure, Commutativity, Associativity: Mixing paints produces a unique new colour, regardless of order or grouping.
- Idempotence: Mixing 7 litres of purple with 4 litres of purple yields 11 litres of the same purple.
- Homogeneity: Doubling all quantities of paint doubles the final amount but does not change the resulting shade.
- Subtraction (A.2.6) fails. One cannot obtain 5 litres of orange by adding some unknown colour to 3 litres of purple.

For paint-mixing, subtraction is replaced by a weaker axiom.

***Axiom* A.3.1. *(Cancellation* VI′*)*.** The equation $mP = nQ + xX$ has at most one solution xX.

With this model, we can "paint" a triangle whose vertices are the primary colours R, Y, B. A point is coloured according to its algebraic representation in terms of the vertices. For example, a pure purple P is the midpoint of red and blue: $2P = 1R + 1B$. As shown in Figure A.8, any interior point Q can be coloured by mixing a vertex colour (e.g., R) with the colour of the point E on the opposite side. The associative law guarantees that the resulting colour for Q is independent of the chosen vertex.

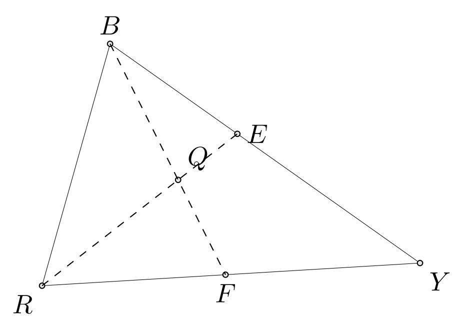
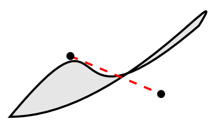


Figure A.8: Colouring a point Q by mixing along $\overline{RQ}$ with the boundary point E (or along $\overline{BQ}$ with F) gives the same shade.

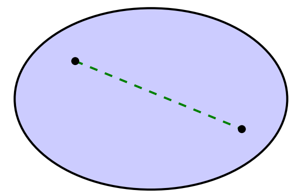
The failure of subtraction corresponds to the fact that to balance masses, one may need to place a point outside the original triangle, leaving the "colour space" of the triangle's interior.

Definition A.3.1. ***Convex set***. A set V is convex if for any two points $P, Q \in V$, the entire segment $\overline{PQ}$ lies within V.

As Figure A.9 illustrates, a triangle is a convex set. The set of points within a convex body satisfies the axioms of mass points with cancellation (VI′), but not necessarily subtraction (VI).



(a) Non-convex: a segment leaves the set.



(b) Convex: segments remain inside.

Figure A.9: Convex vs. non-convex sets.

Remark. A set satisfying axioms A.2.1–A.2.6 (with subtraction) contains the entire line through any two of its points; such a set is an affine space. A set satisfying axioms A.2.1–A.2.5 and A.3.1 (cancellation) contains the segment between any two of its points; such a set is a convex space.

A.4 Barycentric Coordinates

By algebraic convention, we identify a unit mass–point with its location: $1P \equiv P$. The equation for the midpoint M of $\overline{AB}$ becomes

$$2M = 1A + 1B \quad \Rightarrow \quad M = \tfrac{1}{2}(A+B) = \tfrac{1}{2}A + \tfrac{1}{2}B.$$

The midpoint is the algebraic average of the endpoints.

Definition A.4.1. ***Barycentric Coordinates***. Let ABC be a triangle. Every point P inside or on the triangle can be uniquely expressed as

$$P = aA + bB + cC, \qquad \text{where } a, b, c \geq 0 \text{ and } a + b + c = 1.$$

The triple (a, b, c) constitutes the barycentric coordinates of P with respect to $\{A, B, C\}$.

The condition $a + b + c = 1$ treats all three vertices symmetrically. Points on the edges have one coordinate equal to zero; the vertices themselves are $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

Parallelogram Method for Construction

Given barycentric coordinates (a, b, c), one can construct the point P geometrically. Fix a vertex, say C. On $\overline{AC}$ find the point D such that $d(C, D) = a \cdot d(C, A)$. On $\overline{CB}$ find E such that $d(C, E) = b \cdot d(C, B)$. Then P is the fourth vertex of the parallelogram formed by D, C, E, as shown in Figure A.10.

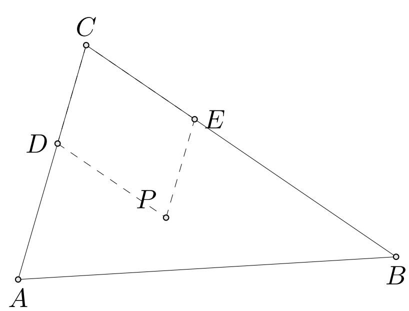


Figure A.10: Parallelogram method for finding $P = aA + bB + cC$ where $a + b + c = 1$.

This construction is verified algebraically. The midpoint of $\overline{PC}$ must coincide with the midpoint of $\overline{DE}$, so $P + C = D + E$. By construction, $D = aA + (1 - a)C$ and $E = bB + (1 - b)C$. Substituting gives

$$P + C = (aA + (1 - a)C) + (bB + (1 - b)C) = aA + bB + (2 - a - b)C.$$

Solving for P yields $P = aA + bB + (1 - a - b)C$. Since $a + b + c = 1$, we have $c = 1 - a - b$, confirming $P = aA + bB + cC$.

Relation to Cartesian Coordinates

If we place the vertices in a Cartesian plane at $A = (1, 0)$, $B = (0, 1)$, and $C = (0, 0)$, the parallelogram method shows that a point $P = (x, y)$ has barycentric coordinates $(x, y, 1 - x - y)$. As Figure A.11 shows, this provides a direct mapping between the two coordinate systems for points within this standard reference triangle.

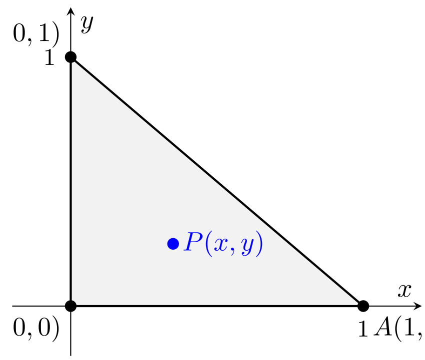


Figure A.11: A point with Cartesian coordinates (x, y) has barycentric coordinates $(x, y, 1 - x - y)$ with respect to vertices $A(1, 0), B(0, 1), C(0, 0)$.

A.4.1 Algebraic Anticipation

We conclude with two algebraic conveniences.

Negative Masses and Directed Ratios

The equation $3P = 5Q - 2R$ is interpreted as $3P + 2R = 5Q$. More generally, if we allow masses m, n to be negative, the relation

$$(m + n)P = mQ + nR, \tag{A.1}$$

places P on the line through Q and R, but not necessarily on the segment $\overline{QR}$. This is equivalent to the directed-segment ratio

$$\frac{\overline{QP}}{\overline{PR}} = \frac{n}{m}, \tag{A.2}$$

where $\overline{XY}$ denotes a directed length (so $\overline{YX} = -\overline{XY}$). A negative ratio implies P is external to the segment, as seen in Figure A.12. When $m + n = 0$, (A.1) does not define a unique point P.

(a) Negative mass: P lies outside $\overline{QR}$; signs record orientation.

(b) Positive masses: P on segment with $\overline{QP}/\overline{PR} = n/m$.

Figure A.12: Directed ratios extend the relation $d(Q, P) : d(P, R) = n : m$ to cases with negative coefficients.

Mnemonic for Ratios: “Multiplying” by Points

To read ratios quickly from an algebraic identity, one can formally “multiply” the identity by a point and interpret the results as directed lengths. For example, from $3P = 2B + 1A$, "multiplying" by P gives

$$3\,\overline{PP} = 2\,\overline{PB} + 1\,\overline{PA} \quad \Rightarrow \quad 0 = 2\,\overline{PB} + \overline{PA}.$$

Since $\overline{PA} = -\overline{AP}$, this becomes $2\,\overline{PB} = \overline{AP}$, so the ratio of directed lengths $\overline{AP}/\overline{PB} = 2$. Multiplying the original equation by A gives $3\,\overline{AP} = 2\,\overline{AB} + 1\,\overline{AA}$, which simplifies to $3\,\overline{AP} = 2\,\overline{AB}$, yielding $\overline{AP}/\overline{AB} = 2/3$. This device serves as a useful memory aid.

Appendix B

Vectors

This chapter revisits the geometric concepts of the previous one, but with an emphasis on parallelism and ratios of lengths along parallel lines. The algebra of vectors presented here provides a more complete framework that can be used to justify the algebraic manipulations of mass points introduced in Appendix A.

Vector algebra is more general. For instance, the subtraction of mass–points was not always possible, leading to cumbersome notions like "negative mass". An equation such as $3A = 5B - 2C$ was defined only through rearrangement to $3A + 2C = 5B$. Furthermore, an expression like $3Z = 2T + 2S$ was meaningless because the total mass on each side of the equation had to be equal. As we shall see, vector algebra has no such restrictions.

B.1 The Definition of a Vector

Definition B.1.1. ***Directed Line Segment***. A directed, or oriented, line segment is a line segment where the endpoints are given in a specific order. The directed segment from point P to point Q is denoted $\overrightarrow{PQ}$. P is the initial point, and Q is the terminal point.

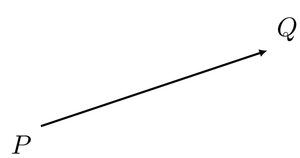


Figure B.1: A directed line segment $\overrightarrow{PQ}$.

A vector is an abstraction of a directed line segment. We consider two directed segments to represent the same vector if they can be translated onto one another without rotation.

Definition B.1.2. ***Vector***. A vector is the collection of all directed line segments having the same length and the same direction. We say that two directed segments $\overrightarrow{PQ}$ and $\overrightarrow{P'Q'}$ are equivalent, and write $\overrightarrow{PQ} = \overrightarrow{P'Q'}$, if they satisfy the following three conditions:

1. The lines L_{PQ} and $L_{P'Q'}$ are parallel ($L_{PQ} \parallel L_{P'Q'}$). This includes the case where the lines are identical.
2. The lengths are equal: $d(P, Q) = d(P', Q')$.
3. They have the same orientation.

The relationship is illustrated in Figure B.2. We denote vectors by bold-face letters, such as $\mathbf{u}, \mathbf{v}$, or by specifying the endpoints, e.g., $\mathbf{u} = \overrightarrow{PQ}$.

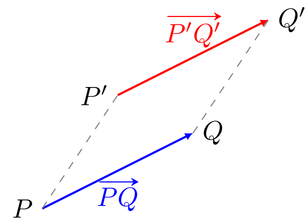


Figure B.2: Equivalent directed segments $\overrightarrow{PQ}$ and $\overrightarrow{P'Q'}$ represent the same vector.

The notion of "same orientation" is intuitive. For two non-collinear, parallel segments $\overrightarrow{PQ}$ and $\overrightarrow{P'Q'}$, they have the same orientation if the figure $PQQ'P'$ forms a parallelogram, which is equivalent to the segments $\overline{PQ'}$ and $\overline{P'Q}$ intersecting at their midpoint.

Definition B.1.3. ***The Zero Vector***. If $P = Q$, the directed segment $\overrightarrow{PP}$ has zero length. All such segments are equivalent and define the zero vector, denoted $\mathbf{0}$. Thus, $\overrightarrow{AB} = \mathbf{0}$ is a concise way of writing $A = B$.

A key property of vectors is that they are not bound to a specific location.

***Proposition* B.1.1.** *Uniqueness of Vector Representation.* Given a vector $\mathbf{v}$ and any point P', there exists one and only one point Q' such that $\mathbf{v} = \overrightarrow{P'Q'}$.

This proposition states that any vector can be "placed" to start at any arbitrary point in the plane.

The equality of vectors, being an equivalence relation, must satisfy three fundamental properties:

- **Reflexivity:** For any directed segment $\overrightarrow{PQ}$, we have $\overrightarrow{PQ} = \overrightarrow{PQ}$.
- **Symmetry:** If $\overrightarrow{PQ} = \overrightarrow{P'Q'}$, then $\overrightarrow{P'Q'} = \overrightarrow{PQ}$.
- **Transitivity:** If $\overrightarrow{PQ} = \overrightarrow{P'Q'}$ and $\overrightarrow{P'Q'} = \overrightarrow{P''Q''}$, then $\overrightarrow{PQ} = \overrightarrow{P''Q''}$.

The transitivity property can be paraphrased as: "Directed segments equivalent to the same segment are equivalent to each other."

B.1.1 Vector Addition

We now introduce the algebra of vectors. This framework allows many properties of space to be stated concisely and provides powerful algebraic methods for solving geometric problems. In what follows, $\mathbf{u}, \mathbf{v}, \mathbf{w}$ represent vectors.

Definition B.1.4. ***Vector Addition (Triangle Rule)***. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors. To compute their sum $\mathbf{u} + \mathbf{v}$, choose points P, Q, R such that $\mathbf{u} = \overrightarrow{PQ}$ and $\mathbf{v} = \overrightarrow{QR}$. The sum is then defined as the vector represented by the directed segment from the initial point of $\mathbf{u}$ to the terminal point of $\mathbf{v}$:

$$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}.$$

This is often called the "tip-to-tail" method, as illustrated in Figure B.3.

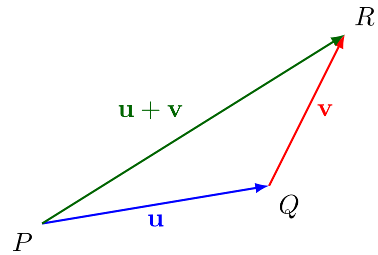


Figure B.3: The Triangle Rule for vector addition: $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$.

The definition of vector addition is independent of the initial point chosen. If one starts at a different point P' such that $\mathbf{u} = \overrightarrow{P'Q'}$ and $\mathbf{v} = \overrightarrow{Q'R'}$, the resulting sum $\overrightarrow{P'R'}$ is equal to $\overrightarrow{PR}$. This is a consequence of congruent triangles, as shown in Figure B.4. This property is known as the **Closure Law**: if $\mathbf{u}$ and $\mathbf{v}$ are vectors, $\mathbf{u} + \mathbf{v}$ is a uniquely determined vector.

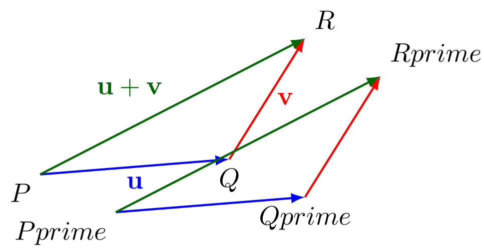


Figure B.4: The sum $\mathbf{u} + \mathbf{v}$ is unique, regardless of the starting point.

Vector addition satisfies several fundamental algebraic properties.

***Proposition* B.1.2.** *Commutative Law.* $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

Proof. Let $\mathbf{u} = \overrightarrow{PQ}$ and $\mathbf{v} = \overrightarrow{PS}$ share the same initial point P. Complete the parallelogram $PQRS$, as shown in Figure B.5. Since opposite sides of a parallelogram are parallel and equal in length, $\overrightarrow{QR} = \overrightarrow{PS} = \mathbf{v}$ and $\overrightarrow{SR} = \overrightarrow{PQ} = \mathbf{u}$. Applying the Triangle Rule:

$$\mathbf{u} + \mathbf{v} = \overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}.$$

$$\mathbf{v} + \mathbf{u} = \overrightarrow{PS} + \overrightarrow{SR} = \overrightarrow{PR}.$$

Thus, $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$. ■

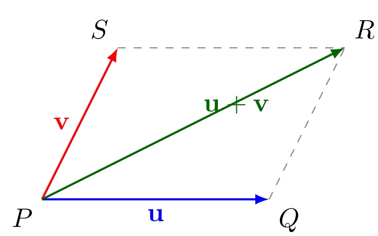


Figure B.5: The Commutative Law illustrated by a parallelogram. The sum $\mathbf{u} + \mathbf{v}$ is the diagonal $\overrightarrow{PR}$.

***Proposition* B.1.3.** *Associative Law.* $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

Proof. Let $\mathbf{u} = \overrightarrow{PQ}$, $\mathbf{v} = \overrightarrow{QR}$, and $\mathbf{w} = \overrightarrow{RS}$, as in Figure B.6.

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = (\overrightarrow{PQ} + \overrightarrow{QR}) + \overrightarrow{RS} = \overrightarrow{PR} + \overrightarrow{RS} = \overrightarrow{PS}.$$

$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = \overrightarrow{PQ} + (\overrightarrow{QR} + \overrightarrow{RS}) = \overrightarrow{PQ} + \overrightarrow{QS} = \overrightarrow{PS}.$$

Both groupings result in the same vector $\overrightarrow{PS}$. ∎

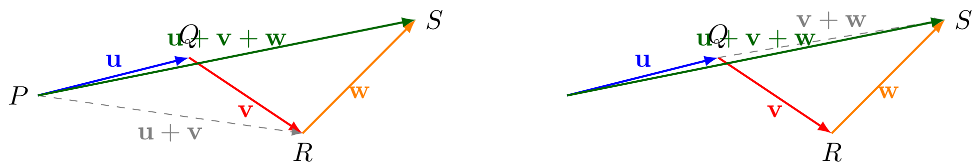


Figure B.6: The Associative Law for vector addition.

***Proposition* B.1.4.** *Additive Identity.* $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$.

Proof. Let $\mathbf{u} = \overrightarrow{PQ}$. Since $\mathbf{0} = \overrightarrow{PP} = \overrightarrow{QQ}$, we have $\mathbf{u} + \mathbf{0} = \overrightarrow{PQ} + \overrightarrow{QQ} = \overrightarrow{PQ} = \mathbf{u}$. Similarly, $\mathbf{0} + \mathbf{u} = \overrightarrow{PP} + \overrightarrow{PQ} = \overrightarrow{PQ} = \mathbf{u}$. ∎

Definition B.1.5. ***Negative Vector***. For any vector $\mathbf{u} = \overrightarrow{PQ}$, its negative, denoted $-\mathbf{u}$, is the vector with the same length but opposite direction, $\overrightarrow{QP}$.

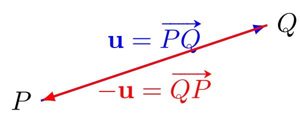


Figure B.7: A vector $\mathbf{u}$ and its negative $-\mathbf{u}$.

***Proposition* B.1.5.** *Additive Inverse.* For any vector $\mathbf{u}$, there exists a vector $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

Proof. Let $\mathbf{u} = \overrightarrow{PQ}$. Then $-\mathbf{u} = \overrightarrow{QP}$. Their sum is $\mathbf{u} + (-\mathbf{u}) = \overrightarrow{PQ} + \overrightarrow{QP} = \overrightarrow{PP} = \mathbf{0}$. ∎

These five laws establish that vector addition has the structure of an algebraic group, which justifies the standard rules of manipulation for addition and subtraction. For instance, the equation $\mathbf{x} + \mathbf{a} = \mathbf{b}$ has the unique solution $\mathbf{x} = \mathbf{b} - \mathbf{a}$.

Definition B.1.6. ***Vector Subtraction***. The difference $\mathbf{v} - \mathbf{u}$ is defined as the sum $\mathbf{v} + (-\mathbf{u})$.

A useful geometric interpretation arises when vectors share an initial point. If $\mathbf{u} = \overrightarrow{PQ}$ and $\mathbf{v} = \overrightarrow{PR}$, then the difference is the vector from the tip of $\mathbf{u}$ to the tip of $\mathbf{v}$:

$$\mathbf{v} - \mathbf{u} = \overrightarrow{PR} - \overrightarrow{PQ} = \overrightarrow{QR}.$$

This follows directly from the triangle rule, since $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$, which rearranges to $\overrightarrow{QR} = \overrightarrow{PR} - \overrightarrow{PQ}$. This is illustrated in Figure B.8.

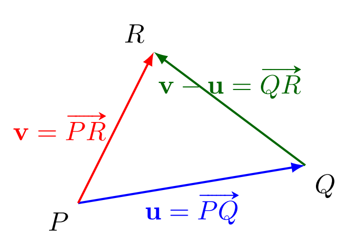


Figure B.8: Geometric interpretation of vector subtraction.

Remark. Vector addition is also commonly visualised using the Parallelogram Rule. If $\mathbf{u}$ and $\mathbf{v}$ are represented with a common initial point, $\mathbf{u} = \overrightarrow{PQ}$ and $\mathbf{v} = \overrightarrow{PS}$, their sum $\mathbf{u} + \mathbf{v}$ is the diagonal of the completed parallelogram starting from the common point, $\overrightarrow{PR}$, as shown in Figure B.5. This is entirely equivalent to the Triangle Rule.

B.2 Scalar Multiplication

This section introduces the multiplication of a vector by a real number, known as a scalar. This operation scales the length of a vector and possibly reverses its direction.

Definition B.2.1. ***Scalar Multiplication***. Let $\mathbf{u}$ be a vector and b be a real number (a scalar). The scalar product $b\mathbf{u}$ is a vector defined as follows:

(a) If $b > 0$, $b\mathbf{u}$ is the vector with the same direction and orientation as $\mathbf{u}$, but with length $b \cdot d(\mathbf{u})$.

(b) If $b < 0$, $b\mathbf{u}$ is the vector with the same direction as $\mathbf{u}$ but opposite orientation, and with length $|b| \cdot d(\mathbf{u})$.

(c) If $b = 0$ or $\mathbf{u} = \mathbf{0}$, then $b\mathbf{u} = \mathbf{0}$.

The various cases are illustrated in Figure B.9.

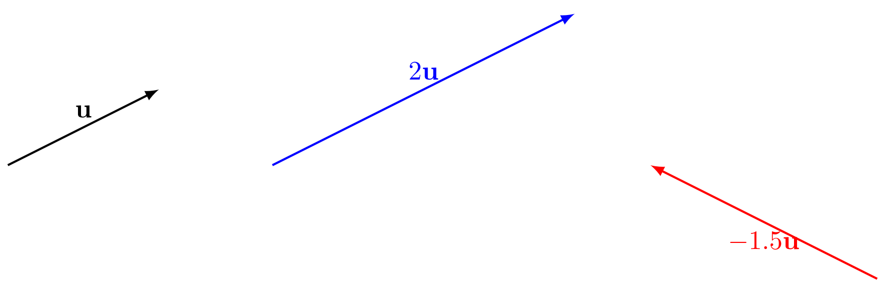


Figure B.9: Scalar multiplication of a vector $\mathbf{u}$ by a positive scalar (e.g., 2) and a negative scalar (e.g., -1.5).

Scalar multiplication follows familiar algebraic laws, which are numbered to continue from the laws of vector addition.

6. **Identity**: $1\mathbf{u} = \mathbf{u}$.
7. **Associative Law**: $a(b\mathbf{u}) = (ab)\mathbf{u}$. This law can be verified by considering the length and orientation of the vectors on both sides. For example, $(-2)(-3\mathbf{u}) = 6\mathbf{u}$. Reversing the orientation twice returns it to the original orientation, and the length is scaled by 3 and then by 2, which is equivalent to a single scaling by 6.

Two distributive laws connect scalar multiplication with vector and scalar addition.

8. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$.
9. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$.

The first distributive law, $a(\mathbf{u}+\mathbf{v}) = a\mathbf{u}+a\mathbf{v}$, is a statement about similar triangles. As shown in Figure B.10, if $\mathbf{u} = \overrightarrow{PQ}$ and $\mathbf{v} = \overrightarrow{QR}$, then $\mathbf{u} + \mathbf{v} = \overrightarrow{PR}$. Scaling the entire triangle PQR by a factor of a from point P produces a new triangle PST. The sides of the new triangle are $\overrightarrow{PS} = a\mathbf{u}$, $\overrightarrow{ST} = a\mathbf{v}$, and $\overrightarrow{PT} = a(\mathbf{u} + \mathbf{v})$. The triangle rule applied to $\triangle PST$ gives $\overrightarrow{PS} + \overrightarrow{ST} = \overrightarrow{PT}$, which proves the law.

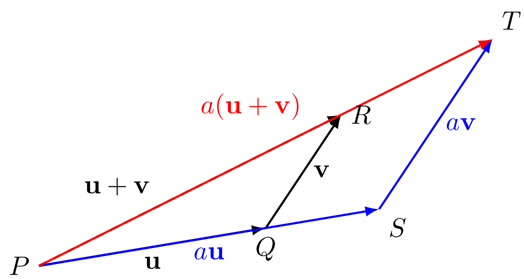


Figure B.10: The distributive law $a(\mathbf{u}+\mathbf{v}) = a\mathbf{u} + a\mathbf{v}$ is a consequence of similar triangles PQR and PST.

The second distributive law, $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$, is also intuitively clear, as it corresponds to combining scalings along the same direction.

While vector algebra shares many properties with real number algebra, some operations are not defined. The product of two vectors, $\mathbf{uv}$, is not defined in this context. Division by a vector is also meaningless. However, we can "divide" by a non-zero scalar. The equation $a\mathbf{x} = \mathbf{u}$ for an unknown vector $\mathbf{x}$ has the solution $\mathbf{x} = \frac{1}{a}\mathbf{u}$ provided $a \neq 0$. The equation $x\mathbf{u} = \mathbf{v}$ for an unknown scalar x has a solution only if $\mathbf{u}$ and $\mathbf{v}$ are parallel.

***Theorem* B.2.1. Zero Product Property**. If $a\mathbf{u} = \mathbf{0}$, then either $a = 0$ or $\mathbf{u} = \mathbf{0}$.

Proof. Suppose $a\mathbf{u} = \mathbf{0}$ and $a \neq 0$. Then we can multiply by the scalar $1/a$.

$$\frac{1}{a}(a\mathbf{u}) = \frac{1}{a}\mathbf{0}$$

$$\left(\frac{1}{a}a\right)\mathbf{u} = \mathbf{0}$$

$$1\mathbf{u} = \mathbf{0}$$

$$\mathbf{u} = \mathbf{0}$$

Thus, if $a \neq 0$, it must be that $\mathbf{u} = \mathbf{0}$. The only other possibility is that $a = 0$. ∎

B.2.1 Physical and Other Applications

The algebra of vectors provides a powerful tool for describing physical quantities and geometric transformations.

Velocity

Velocity is a vector quantity, possessing both magnitude (speed) and direction. A particle moving at a speed of v km/h in a certain direction is described by a velocity vector $\mathbf{v}$ of length v pointing in that direction. If a boat has a velocity $\mathbf{w}$ relative to the water, and the water (stream) has a velocity $\mathbf{v}$, the boat's resulting velocity relative to the ground is the vector sum $\mathbf{v} + \mathbf{w}$, as shown in Figure B.11.

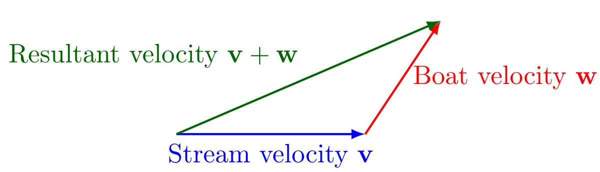


Figure B.11: Resultant velocity as a vector sum.

Force

Force is also a vector. A force of magnitude F acting in a specific direction is represented by a force vector $\mathbf{F}$ of length F. When multiple forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ act on a single point, the net or resultant force is their vector sum $\mathbf{F}_{\text{net}} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n$. This is often visualised with the parallelogram of forces, where the sum of two forces is the diagonal of the parallelogram they form. A system of forces is in equilibrium if their vector sum is the zero vector, $\sum \mathbf{F}_i = \mathbf{0}$. Geometrically, this means that if the force vectors are placed tip-to-tail, they form a closed polygon, as shown in Figure B.12.

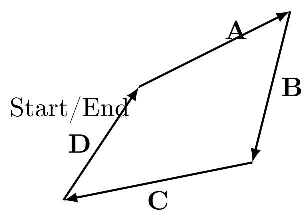


Figure B.12: Forces $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are in equilibrium as they form a closed polygon: $\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D} = \mathbf{0}$.

Translations

A translation is a rigid motion that moves every point of a space by the same distance in the same direction. A translation can be completely described by a single vector $\mathbf{v}$. For any point P, its translated image is the point Q such that $\overrightarrow{PQ} = \mathbf{v}$. If we apply a translation $\mathbf{v}$ followed by a translation $\mathbf{w}$, the result is a single translation given by the vector sum $\mathbf{v} + \mathbf{w}$.

B.3 Geometric Applications of Vectors

Vector algebra provides a powerful framework for addressing geometric problems involving ratios of lengths along parallel lines. While some foundational vector properties rely on geometric concepts like similar triangles, establishing these properties allows us to build a self-contained algebraic system. We can then use this system to derive geometric results.

A crucial tool for this is the principle of linear independence for non-collinear vectors.

***Proposition* B.3.1.** *Uniqueness of Representation.* Let P, Q, R be non-collinear points. Let $\mathbf{u} = \overrightarrow{PQ}$ and $\mathbf{v} = \overrightarrow{PR}$. If an equation of the form

$$a\mathbf{u} + b\mathbf{v} = c\mathbf{u} + d\mathbf{v}$$

holds, where a, b, c, d are scalars, then we can equate coefficients: $a = c$ and $b = d$.

Proof. Rearranging the equation gives $(a - c)\mathbf{u} = (d - b)\mathbf{v}$. If $a \neq c$, we could write $\mathbf{u} = \frac{d-b}{a-c}\mathbf{v}$. This would imply that $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$, meaning $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are parallel. Since they share the point P, the points P, Q, R must be collinear, which contradicts our assumption. Therefore, we must have $a - c = 0$, so $a = c$. This implies $(d - b)\mathbf{v} = \mathbf{0}$, and since $\mathbf{v} \neq \mathbf{0}$, it must be that $d - b = 0$, so $d = b$. ∎

Example B.3.1. The opposite sides of a parallelogram are equal in length. Let $PQRS$ be a parallelogram. Let $\mathbf{u} = \overrightarrow{PQ}$ and $\mathbf{v} = \overrightarrow{PS}$. By definition of a parallelogram, $\overline{QR} \parallel \overline{PS}$ and $\overline{SR} \parallel \overline{PQ}$. Therefore, $\overrightarrow{QR}$ must be a scalar multiple of $\mathbf{v}$, say $\overrightarrow{QR} = b\mathbf{v}$, and $\overrightarrow{SR}$ a multiple of $\mathbf{u}$, say $\overrightarrow{SR} = a\mathbf{u}$. We can express the diagonal vector $\overrightarrow{PR}$ in two ways using the triangle rule:

$$\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR} = \mathbf{u} + b\mathbf{v}$$

$$\overrightarrow{PR} = \overrightarrow{PS} + \overrightarrow{SR} = \mathbf{v} + a\mathbf{u}$$

Equating these two expressions gives $\mathbf{u} + b\mathbf{v} = a\mathbf{u} + \mathbf{v}$. Since P, Q, S are not collinear, $\mathbf{u}$ and $\mathbf{v}$ are not parallel, so we can equate coefficients: $a = 1$ and $b = 1$. Thus, $\overrightarrow{SR} = \mathbf{u} = \overrightarrow{PQ}$ and $\overrightarrow{QR} = \mathbf{v} = \overrightarrow{PS}$. This implies their lengths are equal: $d(S, R) = d(P, Q)$ and $d(Q, R) = d(P, S)$.

Example B.3.2. The diagonals of a parallelogram bisect each other. Let $PQRS$ be a parallelogram with $\mathbf{u} = \overrightarrow{PQ}$ and $\mathbf{v} = \overrightarrow{PS}$, as shown in Figure B.13. Let X be the intersection of the diagonals $\overline{PR}$ and $\overline{SQ}$. Since X lies on $\overline{PR}$, $\overrightarrow{PX}$ is a scalar multiple of $\overrightarrow{PR}$.

$$\overrightarrow{PX} = a\overrightarrow{PR} = a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$$

for some scalar a. Since X also lies on $\overline{SQ}$, $\overrightarrow{SX}$ is a scalar multiple of $\overrightarrow{SQ}$.

$$\overrightarrow{SX} = b\overrightarrow{SQ}$$

for some scalar b. We can express $\overrightarrow{PX}$ using P as an origin:

$$\overrightarrow{PX} = \overrightarrow{PS} + \overrightarrow{SX} = \mathbf{v} + b\overrightarrow{SQ}$$

The diagonal $\overrightarrow{SQ}$ can be written as $\overrightarrow{PQ} - \overrightarrow{PS} = \mathbf{u} - \mathbf{v}$. Substituting this gives:

$$\overrightarrow{PX} = \mathbf{v} + b(\mathbf{u} - \mathbf{v}) = b\mathbf{u} + (1 - b)\mathbf{v}$$

Equating our two expressions for $\overrightarrow{PX}$:

$$a\mathbf{u} + a\mathbf{v} = b\mathbf{u} + (1 - b)\mathbf{v}$$

By the principle of unique representation, we equate coefficients:

$$a = b \quad \text{and} \quad a = 1 - b$$

Solving this system gives $a = b = 1/2$. Therefore, $\overrightarrow{PX} = \frac{1}{2}\overrightarrow{PR}$ and $\overrightarrow{SX} = \frac{1}{2}\overrightarrow{SQ}$, which means X is the midpoint of both diagonals.

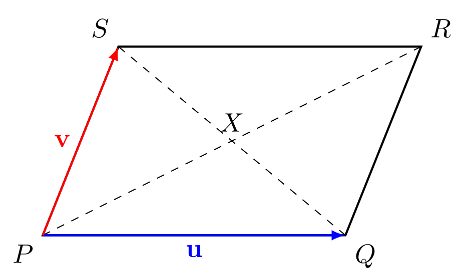


Figure B.13: Diagonals of a parallelogram bisect each other at X.

Example B.3.3. Ceva's Theorem point. In $\triangle ABC$, let $D \in \overline{AB}$ and $E \in \overline{AC}$ such that $d(A, D) : d(D, B) = 3 : 2$ and $d(A, E) : d(E, C) = 4 : 1$. Let $\overline{BE}$ and $\overline{CD}$ intersect at X. Find the ratios $d(C, X) : d(X, D)$ and $d(B, X) : d(X, E)$. Let's choose A as the origin. Let $\mathbf{b} = \overrightarrow{AB}$ and $\mathbf{c} = \overrightarrow{AC}$. From the given ratios:

$$\overrightarrow{AD} = \frac{3}{5}\overrightarrow{AB} = \frac{3}{5}\mathbf{b} \quad \text{and} \quad \overrightarrow{AE} = \frac{4}{5}\overrightarrow{AC} = \frac{4}{5}\mathbf{c}$$

Since X lies on $\overline{CD}$, its position vector $\overrightarrow{AX}$ can be written as a linear combination of $\overrightarrow{AC}$ and $\overrightarrow{AD}$:

$$\overrightarrow{AX} = (1 - a)\overrightarrow{AC} + a\overrightarrow{AD} = (1 - a)\mathbf{c} + a\left(\frac{3}{5}\mathbf{b}\right) = \frac{3a}{5}\mathbf{b} + (1 - a)\mathbf{c}$$

for some scalar a representing the ratio $d(C, X) : d(C, D)$. Since X also lies on $\overline{BE}$, we can write $\overrightarrow{AX}$ as a combination of $\overrightarrow{AB}$ and $\overrightarrow{AE}$:

$$\overrightarrow{AX} = (1 - b)\overrightarrow{AB} + b\overrightarrow{AE} = (1 - b)\mathbf{b} + b\left(\frac{4}{5}\mathbf{c}\right) = (1 - b)\mathbf{b} + \frac{4b}{5}\mathbf{c}$$

for some scalar b representing the ratio $d(B,X) : d(B,E)$. Equating the coefficients of $\mathbf{b}$ and $\mathbf{c}$ from both expressions for $\overrightarrow{AX}$:

$$\frac{3a}{5} = 1 - b \quad \text{and} \quad 1 - a = \frac{4b}{5}$$

Solving this system of linear equations yields $a = 5/13$ and $b = 10/13$. Therefore, $d(C,X) : d(C,D) = a = 5 : 13$, which implies $d(C,X) : d(X,D) = 5 : 8$. And $d(B,X) : d(B,E) = b = 10 : 13$, which implies $d(B,X) : d(X,E) = 10 : 3$.

B.4 A Vector Approach to the Centre of Mass

We can connect the algebra of vectors to the system of mass points from Appendix A. To do this, we fix an arbitrary point O in space, called the origin. Any point P can then be uniquely identified with its **position vector** $\mathbf{p} = \overrightarrow{OP}$. This creates a one-to-one correspondence between points and vectors.

With a chosen origin, any operation on vectors can be interpreted as an operation on points. For instance, we can define the sum of two points P and Q as the point R whose position vector is $\mathbf{r} = \mathbf{p} + \mathbf{q}$. A key formula is the expression for a vector between two points:

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \mathbf{q} - \mathbf{p}$$

In this context, we can simply write $\overrightarrow{PQ} = Q - P$, where P and Q are understood as their position vectors.

Let's revisit the definition of the centre of mass. The point R that divides the segment $\overline{PQ}$ in the ratio $n : m$ is defined by the vector equation $m\overrightarrow{PR} = n\overrightarrow{RQ}$. Using position vectors, this becomes:

$$m(\mathbf{r} - \mathbf{p}) = n(\mathbf{q} - \mathbf{r})$$

$$m\mathbf{r} - m\mathbf{p} = n\mathbf{q} - n\mathbf{r}$$

$$(m + n)\mathbf{r} = m\mathbf{p} + n\mathbf{q}$$

This equation is the vector equivalent of the mass-point summation. It shows that the axioms for the centre of mass in section A.2 are direct consequences of the laws of vector algebra.

An important feature of this formulation is that the resulting point R is independent of the choice of origin O. An equation involving position vectors that has this property is called an affine equation.

Definition B.4.1. ***Affine Equation***. An equation of the form

$$m_1P_1 + m_2P_2 + \cdots + m_jP_j = n_1Q_1 + n_2Q_2 + \cdots + n_kQ_k$$

is an **affine equation** if the sum of the scalar coefficients on both sides is equal:

$$\sum_{i=1}^{j} m_i = \sum_{i=1}^{k} n_i$$

***Theorem* B.4.1.** . An affine equation is independent of the choice of origin.

Proof. Let the equation hold for an origin O, meaning $\sum m_i\overrightarrow{OP_i} = \sum n_i\overrightarrow{OQ_i}$. Let O' be another origin. We have $\overrightarrow{OP_i} = \overrightarrow{OO'} + \overrightarrow{O'P_i}$. Substituting this into the equation:

$$\sum m_i(\overrightarrow{OO'} + \overrightarrow{O'P_i}) = \sum n_i(\overrightarrow{OO'} + \overrightarrow{O'Q_i})$$

$$\left(\sum m_i\right)\overrightarrow{OO'} + \sum m_i\overrightarrow{O'P_i} = \left(\sum n_i\right)\overrightarrow{OO'} + \sum n_i\overrightarrow{O'Q_i}$$

Since $\sum m_i = \sum n_i$, the $\overrightarrow{OO'}$ terms cancel, leaving $\sum m_i\overrightarrow{O'P_i} = \sum n_i\overrightarrow{O'Q_i}$. The equation holds for the origin O'. ■

All the equations in mass point geometry were affine equations. For example, the barycentric coordinates of a point P with respect to $\triangle ABC$,

$$P = aA + bB + cC, \quad \text{with } a + b + c = 1,$$

is an affine equation since $1 = a+b+c$. This confirms that the point P is defined geometrically, independent of any coordinate system or origin. Choosing A as the origin, the equation becomes $\overrightarrow{AP} = b\overrightarrow{AB} + c\overrightarrow{AC}$, which is the vector form of the parallelogram method for constructing P.