Intro to Set Theory, Logic, and Proof Techniques.

Gudfit

Contents

Page

1 Naive Set Theory **3**

1.1 Axiom of Extension . . . 3

1.1.1 Subset . . . 4

1.2 Axiom of Specification . . . 4

1.2.1 Russell's Paradox . . . 6

1.3 Unordered Pairs . . . 6

1.3.1 Axiom of pairing . . . 7

1.3.2 Language . . . 8

1.4 Unions and Intersections . . . 9

1.4.1 Axiom of Unions . . . 9

1.4.2 Intersection . . . 10

1.4.3 Distributive Laws . . . 11

1.5 Complements and Powers . . . 11

1.5.1 De Morgan Laws . . . 12

1.5.2 Axiom of Powers . . . 13

1.6 Ordered Pairs . . . 14

1.6.1 Cartesian Product . . . 15

1.7 Relations . . . 16

1.7.1 Domain and Range . . . 17

1.7.2 Properties of relations . . . 17

1 Naive Set Theory

The mathematical concept of a set can be used as the foundation for all known mathematics. In the typical conception, sets consists of *elements*, or *members*. An element of a set can truly be anything. *Crucially*, a set itself can also serve as an element of another set, such as a line being a set of points. Also it doesn't matter how we specify the set, or how its elements are order or how many times we count its elements, all that matters is **What its elements are**.

1.1 Axiom of Extension

The fundamental notion in set theory is that of **belonging.** If x belongs to A; expressed as

$$x \in A,$$

it means x is an element of A, x is contained within A.

In set theory, there is no convention regarding alphabetic usage. But there is an informal convention for using letters to represent a particular hierarchy of set. Letters at the start of the alphabet usually represent elements, while those at the end denote sets containing them; In a similar vein, basic letters represent individual elements, while more elaborate and bold fonts symbolize sets encompassing these elements.

A more fundamental notion in set theory, is **equality**, usually denoted as

$$A = B;$$

but if A and B are not equal, it is expressed as

$$A \neq B;$$

Putting both belonging and equality together we get the axiom of extension which states

Definition. Axiom of extension. Two sets are equal if and only if they have the same elements.

The axiom of extension is not just a logical property of equality but also a non-trivial statement about belonging. To understand this better, consider an analogy with family relationships instead of sets.

Example 1. Family Relationships.

Let x and y represent individual people. Write "x is a descendant of y" instead of "x belongs to y" $x \in y$ (for sets). If x and y have the same descendants, then x and y are the same person. (The "if" part). If x and y are the same person, then they have the same descendants. (The "only if" part).

In this analogy:

The "only if" part is true: If two people are the same, they must have the same descendants. However, the "if" part is false: Two people can have the same descendants but still be different individuals. This analogy helps illustrate that the axiom of extension for sets is a non-trivial statement about belonging or membership, not just a logical necessity.

1.1.1 Subset

If A and B are sets and if every element of A is an element of B, then we say that A is a **subset** of B or B includes A

$$A \subset B.$$

The definition states that every set is a subset of itself ($X \subset X$); this is called reflexivity of set inclusion.

Remark. In the same sense of the word, equality also is reflexive.

If for sets A and B such that $A \subset B$ and $A \neq B$, the word *'proper'* is used (proper subset, proper inclusion). Set inclusion is also transitive: if $X \subset Y$ and $Y \subset Z$, then $X \subset Z$.

Remark. This transitive property is also shared with equality.

If A and B are sets such that $A \subset B$ and $B \subset A$, then A and B have the same elements. By the axiom of extension, this implies $A = B$. This property is called antisymmetry of set inclusion. (In contrast, equality is symmetric: if $A = B$, then necessarily $B = A$.) The axiom of extension can be reformulated in these terms:

Definition. For sets A and B, a necessary and sufficient condition for $A = B$ is that both $A \subset B$ and $B \subset A$ hold. Consequently, proofs of equality between two sets A and B often involve showing $A \subset B$ and $B \subset A$ separately.

Observe that belonging ($\in$) and inclusion ($\subset$) are conceptually distinct notions. One key difference is that inclusion is always reflexive ($A \subset A$ is always true), whereas it is unclear if belonging is ever reflexive (i.e., if $A \in A$ is ever true for reasonable sets). Similarly, inclusion is transitive, but belonging is not.

1.2 Axiom of Specification

There are two basic types of sentences, namely assertions of belonging,

$$x \in A.$$

and assertions of equality,

$$A = B;$$

all other compound sentences are derived from *atomic* sentences by repeated applying of the usual logical operators. The List of seven common logical operators include:

- *and* ($\wedge$)
- *or* (either—or—or both, $\vee$)
- *not* ($\sim$)
- *if—then—* (or implies, $\implies$)

- *if and only if* ($\Leftrightarrow$)
- *for some* (or there exists, $\exists$)
- *for all* ($\forall$)

Theses, here and below, are used to guarantee unambiguity. They make all other punctuation marks unnecessary. In normal mathematical practice, to be followed in this note book, several different sizes and shapes of parentheses are used, but that is for visual convenience only.

To form sentences, we can put "and" or "or" or "if and only if" between two sentences and enclose the result between parentheses, like

$$(S_1) \text{ and/or/if and only if } (S_2)$$

. We can also replace the dashes in "if—then—" by sentences and enclose the result in parentheses, giving

$$\text{if } (S_1) \text{ then } (S_2).$$

Additionally, we can replace the dash in "for some—" or in "for all—" by a letter, follow the result by a sentence, and enclose the whole in parentheses, resulting in

$$\text{for some } x(S)$$

or

$$\text{for all } x(S)$$

.

If the letter used does not occur in the sentence, no harm is done. According to the usual and natural convention, for some $y(x \in A)$ just means $x \in A$. It is equally harmless if the letter used has already been used with "for some—" or "for all—."

Remark. Recall that

$$\text{for some } x(x \in A)$$

means the same as for some $y(y \in A)$; it follows that a judicious change of notation will always avert alphabetic collisions.

This allows us to get the definition a major principle of set theory,

Definition. Axiom of Specification. For any set A and any property $S(x)$, there exists a subset B of A such that for every element x, $x \in B$ if and only if $x \in A$ and $S(x)$ holds.

Remark. A "condition" here is just a sentence.

What this basically means is that x is *"free"* as in; x already existed before being called/introduced by our phrases/sentence. Thus to indicate the way B is obtained from A using the condition $S(x)$ it is customary to write.

$$B = \{x \in A : S(x)\}.$$

An example of this could be a set A that contains "chickens" as elements. If we want a set B that only has "male chickens" (roosters) as elements, instead of building a new set from scratch, we can specify set B in terms of set A as:

$$B = \{x \in A \mid x \text{ is male}\}.$$

This notation reads as "the set B consists of all elements x in set A such that x is male." It allows us to define set B as a subset of A by selecting only those elements that satisfy the condition "is male."

1.2.1 Russell's Paradox

A rather more amusing and instructive application is this: consider, in the role of $S(x)$, the sentence:

$$\text{not } (x \in x).$$

or more conveniently known as $x \notin x$. Now it follows, that whatever the set A may be, if set $B = \{x \in A : x \notin x\}$, then, for all y, where

$$y \in B \text{ if and only if } (y \in A \text{ and } y \notin y).$$

Can it be that $B \in A$?

If $B \in A$, then either $B \in B$ or $B \notin B$. If $B \in B$, then, by the assumption $B \in A$, we have $B \notin B$, which is a contradiction. But if $B \notin B$, then, again by the assumption $B \in A$, we have $B \in B$, which is also a contradiction.

This the **Russell's Paradox** that proves that there exists something that doesn't belong to A, or rather we have proved that

> *nothing contains everything.*.

or, more spectacularly,

> *there is no universe*.

Remark. "Universe" here is used in the sense of "universe of discourse," meaning, in any particular discussion, a set that contains all the objects that enter into that discussion. In older (pre-axiomatic) approaches to set theory, the existence of a universe was taken for granted, and the argument in the preceding paragraph.

The moral is that it is impossible, especially in mathematics, to get something for nothing. To specify a set, it is not enough to pronounce some magic words (which may form a sentence such as $x \notin x$); it is necessary also to have at hand a set to whose elements the magic words apply.

Definition (Russell's Paradox)**.** A universal set (a set of all sets) leads to a contradiction. Specifically, there cannot exist a set R such that for every set x, $x \in R$ if and only if $x \notin x$. This paradox indicates the necessity of restricting set formation to avoid such contradictions in set theory.

1.3 Unordered Pairs

For this subsection let us create a temporary assumption that

$$\textit{there exists a set}.$$

Already, we can come to the conclusion from this assumption that there must exist a set without any elements at all. If we apply the axiom of specification to a set A with the sentence "$x \neq x$" (or any other universally false sentence), the result is the set $\{x \in A : x \neq x\}$, a set with no elements. This implies that there can be only one set with no elements, denoted as: $\emptyset$ called the **empty set**, or another proof is:

Proposition: Prove that there is at most one empty set, i.e., show that if A and B are sets without elements, then A = B.

Proof. Let A and B be sets without elements, i.e., A and B are empty sets.
By the axiom of extensionality (equality of sets), we have: $A = B$ if and only if $(A \subset B)$ and $(B \subset A)$.

1. Prove $A \subset B$:
   - Let x be an arbitrary element in A.
   - But A is empty, so there are no elements in A.
   - Therefore, the statement "$x \in A$ implies $x \in B$" is vacuously true for all x.
   - Hence, $A \subset B$.
2. Prove $B \subset A$:
   - Let y be an arbitrary element in B.
   - But B is empty, so there are no elements in B.
   - Therefore, the statement "$y \in B$ implies $y \in A$" is vacuously true for all y.
   - Hence, $B \subset A$.

Since $A \subset B$ and $B \subset A$, we conclude that $A = B$.
Therefore, any two empty sets are equal, which means there is at most one empty set. ■

The empty set is a subset of every set, $\emptyset \subset A$ for every A.

Proposition The empty set $\emptyset$ is a subset of every set A

Proof. Recall the definition of a subset: $B \subset A$. To show $\emptyset \subset A$, we need to prove that for all $x \in \emptyset$, we have $x \in A$. But there are no elements in $\emptyset$. So, the statement "for all $x \in \emptyset$, we have $x \in A$" is vacuously true. Therefore, $\emptyset \subset A$ for any set A. ■

1.3.1 Axiom of pairing

From the set theory developed so far, we know:

1. There is only one empty set.
2. This empty set is a subset of every set.

However, this leaves us with many unanswered questions. For instance, how do we construct sets with elements? The best place to start addressing this is with the following axiom:

Definition. Axiom of Pairing: For any two sets A and B, there exists a set C such that $A \in C$ and $B \in C$.

A consequence of this axiom (in fact, an equivalent formulation) is that for any two sets, there exists a set that contains exactly those two sets as elements and nothing else. To see this, let a and b be sets such that $a \in A$ and $b \in A$. By applying the axiom of specification to A with the sentence "x = a *or* x = b", we get the resulting set: $\{x \in A : x = a \text{ or } x = b\}$; This set clearly contains just a and b. The axiom of extension implies that only one set can have this property; we call it the unordered pair of a and b, symbolized as: $\{a, b\}$.

Temporarily if we refer to the sentence *"x = a or x = b"* as $S(x)$, we may express the axiom of paring by saying there exists a set B such that

$$x \in B \textit{ if and only if } S(x). \qquad (*)$$

The axiom of specification, applied on a set A, asserts the existence of a set B such that

$$x \in B \text{ if and only if } (x \in A \textit{ and } S(x)). \qquad (**)$$

All remaining principles of set construction are essentially special cases of the axiom of specification, as they all assert the existence of a set specified by a certain condition.

If there exists a set a, we may form the unordered pair $\{a, a\}$. This set is denoted by:

$$\{a\}$$

and is called a *singleton* of a. It is uniquely characterized by the fact that a is its only element. Thus, $\{\emptyset\}$ and $\emptyset$ are two distinct sets (one has is a box no elements, the other is a box with a box inside it that has no elements). Therefore to say that $a \in A$ this is equivalent to saying that $\{a\} \subset A$.

$$\boxed{a \in A \equiv \{a\} \subset A}$$

1.3.2 Language

Set-builder notation: For a condition $S(x)$ on x, we denote:

- Set: $\{x : S(x)\}$ or $\{x \in A : S(x)\}$
- Example: $\{x : x = a \text{ or } x = b\} = \{a, b\}$

Special cases:

- $\{x : x \in A\} = A$
- $\{x : x \neq x\} = \emptyset$
- $\{x : x = a\} = \{a\}$

Non-sets: If $S(x)$ is $(x \notin x)$ or $(x = x)$, $\{x : S(x)\}$ is not a set.
Classes: Some theories use "classes" for these non-sets.

1.4 Unions and Intersections

Let's say we have sets A and B, and we want to combine them into one comprehensive set. One way to do this is to define the comprehensive set as containing all the elements that belong to at least one of the sets A or B.

1.4.1 Axiom of Unions

Definition. Axiom of unions For every collection $\mathscr{C}$ of sets there exists a set that contains all the elements that belong to at least one of the set given collection.

The comprehensive set U can be described using the axiom of specification as the set:

$$U = \{x : x \in X \textit{ for some } X \in \mathscr{C}\}$$

In this definition, the statement can be translated to: *"For some X $(x \in X)$ and $X \in \mathscr{C}$."* Thus, for this set, for every x to be included, it is necessary that it belongs to some set X in the collection $\mathscr{C}$.

This set U is called the *union* of the collection $\mathscr{C}$ of sets.

Remark. The axiom of extension guarantees its uniqueness.

The symbol for U can be expressed as:

$$\bigcup\{X : X \in \mathscr{C}\}.$$

The simplest facts we can deduce from this notation are (these should be intuitive enough without proof):

$$\bigcup\{X : X \in \emptyset\} = \bigcup\emptyset = \emptyset.$$

and

$$\bigcup\{X : X \in \{A\}\} = \bigcup\{A\} = A.$$

It's a little different when trying to get the union of a pair of sets:

$$\bigcup\{X : X \in \{A, B\}\} = A \cup B..$$

The general rule of unions shows that in case $x \in A \cup B$, then x must belong to either A or B, or both, as follows:

$$A \cup B = \{x : x \in A \textit{ or } x \in B\}.$$

Some easily proven facts about unions of pairs (Try these yourself):

$$\begin{aligned} A \cup \emptyset &= A \\ A \cup A &= A \ (idempotence) \\ A \cup B &= B \cup A \ (commutativity) \\ A \cup B \cup C &= (A \cup B) \cup C = A \cup (B \cup C) \ (associativity) \\ B &= A \cup B \text{ if and only if } A \subset B \bigstar. \end{aligned}$$

Proposition Suppose $B = A \cup B$ if and only if $A \subset B$. Then this can be written as $A \subset B$ if and only if $A \cup B = B$.

Proof. Suppose $B = A \cup B$ if and only if $A \subset B$. This can be written as $A \subset B$ if and only if $A \cup B = B$. We will prove both directions of this statement.

($\Rightarrow$) First, let's assume that $A \subset B$.

By the definition of a subset, every element of A is also an element of B. Therefore, $A \cup B = B$, since adding the elements of A to B does not change the set B.

($\Leftarrow$) Now, let's assume that $A \cup B = B$. We want to show that $A \subset B$. By the definition of the union, every element in $A \cup B$ belongs to either A or B (or both). Since $A \cup B = B$, every element of A must also be an element of B. Thus, $A \subset B$. ■

An equally suggestive fact is

$$\{a, b, c, \ldots\} = \{a\} \cup \{b\} \cup \{c\} \cup \ldots.$$

1.4.2 Intersection

If A and B are sets, the intersection of A and B is the set

$$A \cap B.$$

defined by

$$A \cap B = \{x \in A : x \in B\}.$$

In other words, the intersection of sets A and B is the set containing all elements that belong to both A and B.

Remark. The definition is symmetric as $A \cap B = \{x \in B : x \in A\}$.

The general rule of intersections shows that in case $x \in A \cap B$, then x must belong to both A and B, as follows:

$$A \cap B = \{x : x \in A \textit{ and } x \in B\}$$

The basic facts for intersections are similar to the basic facts about unions, including their proofs.

$$\begin{aligned}
A \cap \emptyset &= \emptyset \\
A \cap A &= A \ (idempotence) \\
A \cap B &= B \cap A \ (commutativity) \\
A \cap B \cap C = (A \cap B) \cap C &= A \cap (B \cap C) \ (associativity) \\
A = A \cap B &\text{ if and only if } A \subset B \bigstar.
\end{aligned}$$

But what if the pairs of sets have an empty intersection? Well, this happens frequently enough to justify a term called "disjoint," which occurs when $A \cap B = \emptyset$.

1.4.3 Distributive Laws

These are identities built up with the combination of both unions and intersections.

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Lets prove the first identify:

Proof. If x belongs on the left equation then $x \in A$ and $x \in B$ or $x \in C$; if $x \in A$ and $x \in B$, then $x \in (A \cap B)$, and therefore $x \in (A \cap B) \cup (A \cap C)$. If $x \in A$ and $x \in C$, then $x \in (A \cap C)$, and therefore $x \in (A \cap B) \cup (A \cap C)$. This proves the equality

Conversely if x belongs to the right equation, then $x \in A \cap B$ or $x \in A \cap C$, then x belongs in A and B or C. ■

Lets prove the second identity:

Proof. If x belongs to the left equation, then $x \in A$ or both $x \in B$ and $x \in C$; if $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$. This proves the equality. Conversely, if x belongs to the right equation, then $x \in A \cup B$ and $x \in A \cup C$, then either x belongs to A or both B and C. ■

For each collection $\mathscr{C}$, other than $\emptyset$, there exists a set V such that $x \in V$ if and only if $x \in X$ for ever X in $\mathscr{C}$. This can be proven with the set:

$$V = \{x : x \in X \textit{ for every } X \in \mathscr{C}\}.$$

which means *"for all X (if $X \in \mathscr{C}$, then $x \in X$)"*

Remark. Just like the axiom of union, the axiom of extension guarantees the uniqueness of the intersection.

The set V is usually denoted by:

$$\bigcap\{X : X \in \mathscr{C}\} = \bigcap_{X \in \mathscr{C}} X.$$

1.5 Complements and Powers

If A and B are sets, the *relative complement or difference* of B in A, is the set defined by

$$A - B = \{x \in A : x \notin B\}.$$

Remark. Not a necessity that $B \subset A$.

An often used symbol for the temporarily absolute complement of a set A is A^c These are some basic (easily proved) facts about complementation:

$$(A^c)^c = A$$
$$\emptyset^c = U$$
$$U^c = \emptyset$$
$$A \cap A^c = \emptyset$$
$$A \cup A^c = U.$$

$$\star B^c \subset A^c \text{ if and only if } A \subset B$$

Remark. For this section only we say that all sets to be mentioned are subsets of one and the same set U also known as the universal set, and that all complements (unless stated otherwise) are formed relative to U.

1.5.1 De Morgan Laws

De Morgan's laws are statements about complements in set theory:

$(A \cup B)^c = A^c \cap B^c$

$(A \cap B)^c = A^c \cup B^c$.

These laws illustrate the principle of duality, which implies that set-theoretic theorems often come in dual pairs. These pairs are obtained by interchanging unions and intersections, complementing sets, and reversing inclusions.

Here are some easy exercises on complementation.

$$A - B = A \cap B^c$$
$$A \subset B if\ and\ only\ if\ A - B = \emptyset$$
$$A - (A - B) = A \cap B \star$$
$$A \cap (B - C) = (A \cap B) - (A \cap C) \star$$
$$A \cap B \subset (A \cap C) \cup (B \cap C^c)$$
$$(A \cup C) \cap (B \cup C^c) \subset A \cup B$$

If there exist sets A and B, $A + B$ (*symmetric difference*) is defined by

$$A + B = (A - B) \cup (B - A)$$

$$A - B = \{x \in A : x \notin B\}, B - A = \{x \in B : x \notin A\}$$

This operation is commutative and associative, and it is also so that: $A + \emptyset = A$ and $A + A = \emptyset$

In set theory, we often talk about intersections of collections of sets. When a collection is non-empty, the definition is straightforward: an element is in the intersection if it's in every set of the collection.

But what about the intersection of an empty collection? It seems odd because there are no sets to intersect. To understand this, let's think about it logically.

$$x \in X\ for\ every\ X\ in\ \emptyset$$

We say an element x is in the intersection if it's in every set of the collection. Now, if the collection is empty, which elements do not satisfy this? Well, for x to fail this condition, there must be a set in the collection that doesn't contain x. But there are no sets in an empty collection! So, no x can fail the condition.

This means every x satisfies the condition. In other words, the intersection of an empty collection includes every element in our universe. It might seem strange, but it's just a technicality to keep our definitions consistent, even when a set in our work turns out to be empty.

It's not a deep problem, just a little quirk we have to deal with in set theory. We can't avoid it because we always need to account for the possibility that a set might be empty.

The point of all of this is so we can define the intersection of a collection where $\mathscr{C}$ (is a collection of subsets U) as the set:

$$U = \{x : x \in X \ for \ every \ X \in \mathscr{C}\}.$$

This new definition isn't revolutionary. For any non-empty collection, it gives the same result as the old definition. The difference lies in how they treat an empty collection:

- Old definition: The intersection of an empty collection is undefined or problematic.
- New definition: The intersection of an empty collection is the entire set U.

Why? Because for an empty collection, there are no sets X to check, so the condition "x is in every X" is always true for all x in U. This difference is just a matter of language. If you think about it, the "new" definition for the intersection of a collection $\mathscr{C}$ of subsets of U is really the same as the old definition applied to the collection $\mathscr{C}$ plus the set U itself ($\mathscr{C} \cup \{U\}$). And this collection is never empty because it always includes U.

1.5.2 Axiom of Powers

Up until now, we've been discussing the subsets of a given set E. However, a natural question arises: can we consider all these subsets as forming a set themselves?

Definition. Axiom of powers For each set there exists a collection of sets that contains among its elements all the subsets of a given set.

thus if U is a set, then there is a set $\mathscr{P}$ such that if $X \subset U$, then $X \in \mathscr{P}$, or applying the axiom of specification gives us $\{X \in \mathscr{P} : X \subset U\}$ or

$$\mathscr{P} = \{X : X \subset U\}.$$

thus the set $\mathscr{P}$ is called the *power set* of U.

Remark. the axiom of extension guarantee its uniqueness and the dependence of $\mathscr{P}$ on U is denoted by writing $\mathscr{P}(U)$.

If $U = \emptyset$, then the set $\mathscr{P}(\emptyset) = \{\emptyset\}$. The power set of singleton are described as

$$\mathscr{P}(\{x\}) = \{\emptyset, \{x\}\}$$

and for pairs

$$\mathscr{P}(\{x,y\}) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}.$$

$\star$ What do you notice about the power sets? and can you prove the statement "the power set of a finite set with n elements is 2^n elements"

If a collection $\mathscr{C}$ as a subcollection of $\mathscr{P}(U)$, where $\mathscr{P}(U)$ is the power set of U. This means $\mathscr{C}$ is a collection of subsets of the set U. Then, it defines a new collection D as:

$$D = \{X \in \mathscr{P}(U) : X^c \in \mathscr{C}\}.$$

This can be read as: "D is the set of all X, where X is a subset of U (i.e., $X \in \mathscr{P}(E)$), such that the complement of X (denoted as X^c) is in the collection $\mathscr{C}$."

Now to be precise we suggest rewriting the condition $X^c \in \mathscr{C}$ in a more precise form:

$$\textit{for some } Y[Y \in \mathscr{C} \textit{ and for all } x(x \in X \textit{ if and only if } (x \in U \textit{ and } x \in Y))]$$

This rewritten form is more explicit and can be understood as:
- There exists a set Y in the collection $\mathscr{C}$
- For every element x, x is in X if and only if two conditions are met: (a). x is an element of U ($x \in U$), and (b). x is not an element of $Y (x \notin Y)$.

This is just a more verbose way of saying "$X^c = Y$" or "X is the complement of Y with respect to U".

Thus in this notion the general forms of the De morgan laws become:

$$(\bigcup_{x \in \mathscr{C}} X)^c = \bigcap_{x \in \mathscr{C}} X^c.$$

which states that "The complement of the union of all sets X in the collection $\mathscr{C}$ is equal to the intersection of the complements of each set X in $\mathscr{C}$", and

$$(\bigcap_{x \in \mathscr{C}} X)^c = \bigcup_{x \in \mathscr{C}} X^c.$$

which translates to "The complement of the intersection of all sets X in the collection $\mathscr{C}$ is equal to the union of the complements of each set X in $\mathscr{C}$."

1.6 Ordered Pairs

Remember at the start of Chapter 2 "It doesn't matter how we specify the set, or how its elements are order or how many times we count its elements, all that matters is", but what if for this instance, that we want a set $A = \{rat, duck, chicken, cat\}$ of distinct elements and we want to consider its elements in an order

$$cat, chicken, duck, rat.$$

we can intuitively solve this by creating a collection of sets that state its order such as

$$\mathscr{C} = \{\{cat\}, \{cat, chicken\}, \{cat, chicken, duck\}, \{cat, chicken, duck, rat\}\}$$

now lets scramble it and see if we can still find the order we want

$$\mathscr{C} = \{\{chicken, cat, duck\}, \{cat\}, \{rat, chicken, cat, duck\}, \{cat, chicken\}\}.$$

To determine the order, we can check which element is present in all other sets in $\mathscr{C}$. Since the element 'cat' is present in all the sets in $\mathscr{C}$, it must be the first element in the desired order. After identifying 'cat' as the first element, we can then look at the remaining sets in $\mathscr{C}$ and find the element that is present in all of them except the first set (which only contains 'cat'). This element is 'chicken', indicating that it is the second element in the desired order. Continuing this process, we can determine that 'duck' is the third element, and 'rat' is the fourth element in the desired order, since they appear in the remaining sets of $\mathscr{C}$. Therefore, by examining the collection of sets $\mathscr{C}$, we can deduce that the order of the elements in the set A is $cat, chicken, duck, rat$ (or alphabetical).

The key point is that while sets themselves are unordered collections, we can associate an ordering with the elements of a set by constructing a specific collection of subsets. Let's illustrate this idea with the pair $A = \{a,b\}$: if the desired order means a comes first then $\mathscr{C} = \{\{a\},\{a,b\}\}$ but if its the opposite then $\mathscr{C} = \{\{b\},\{a,b\}\}$.

Thus the **ordered pair** with *first* coordinate a and *second coordinate* b, is the set (a,b) defined by:

$$(a,b) = \{\{a\},\{a,b\}\}.$$

But this doesnt really mean anything unless we can prove the crucial idea that ordered pairs are identical if and only if (iff) they share the first element and share the second element, i.e.:

$$(a,b) = (c,d) \text{ iff both } a = c \text{ and } b = d.$$

Proof. Firstly if $a = b$ then the result of the pair is the singleton $(a,b) = \{\{a\},\{a,a\}\} = \{\{a\}\}$, and conversely if (a,b) is a singleton then $\{a\} = \{a,b\}$, such that $b \in \{a\}$ thus $a = b$.
Now suppose if $(a,b) = (x,y)$:

Case 1: $a = b$. If $a = b$, then both (a,b) and (x,y) are singletons, then $x = y$, thus since $\{x\} \in (a,b)$ and $\{a\} \in (x,y)$ then x,y,a,b must be all equal.

Case 2: $a \neq b$. If $a \neq b$, then both equations contain only one singleton $\{a\}$ and $\{x\}$, thus respectively $a = x$. Since both equations contain one unordered pair that is not a singleton then respectively it follows that $\{a,b\} = \{x,y\}$, and thus $b \in \{x,y\}$. Since $a = x$ already then $b \neq x$ then b must equal y. ■

Having a fixed definition (and proof) of an ordered pair, we can use it to define further sets. Ordered sequences of more than two elements, also known as *ordered n-tuples*, can be constructed recursively by treating them as special ordered pairs. The first two coordinates form an ordered pair, and the remaining coordinates are paired with the previous pair recursively $(n-1)$ times, e.g.:

$$(a,b,c) = ((a,b),c), (a,b,c,d) = (((a,b),c),d).$$

1.6.1 Cartesian Product

Definition. Cartesian Product of sets A and B is characterized by the set

$$A \times B = \{(x,y) : x \in A \textit{ and } y \in B\}$$

.

To prove this exist lets take the question as defined: does a set exist such that it contains all the ordered pairs (a,b) and $a \in A$, $b \in B$? Here's the corrected version of the proof:

Proof. Given sets A and B, we aim to show that $A \times B = \{(a,b) \mid a \in A, b \in B\}$ exists.

By the Axiom of Pairing for each $a \in A$ and $b \in B$, the ordered pair $(a,b) = \{\{a\},\{a,b\}\}$ exists.

Using the Axiom of Specification, we can form the set $A \times B$ by specifying all such ordered pairs:

$$A \times B = \{(a,b) \mid a \in A, b \in B\} = \{\{\{a\},\{a,b\}\} \mid a \in A, b \in B\}$$

Therefore, $A \times B$ exists as a set by the Axiom of Specification. ∎

1.7 Relations

A relation is (sorta) a mathematical concept formulated using ordered pairs, representing associations like belonging (between elements and sets), and is commonly referred to as a binary relation (in this notes, there exists others like ternary relation or worse).

Let's consider an example of an ordered pair (x,y), where x represents a cat and y represents a mouse. And for the sake of the example the relation between x and y is that x hunts y. Although this example does not fully explore the concept of a relation, it suggests that every relation should uniquely determine the set of all ordered pairs for which the first coordinate stands in relation to the second coordinate. In other words, if we know the relation, we should be able to determine the set of ordered pairs (a,b) that satisfy that relation, and vice versa.

This idea implies that there is a one-to-one correspondence between a relation and the set of ordered pairs that satisfy that relation. By understanding the relation, we can identify the set of ordered pairs (a,b) that fall under that relation, and conversely, by examining the set of ordered pairs (a,b), we can infer the underlying relation that governs their association.

The example of a cat hunting a mouse illustrates this concept in a concrete way. The relation "hunts" determines the set of ordered pairs (x,y) where x is a cat and y is a mouse. Alternatively, if we are given a set of ordered pairs (x,y) where the first coordinate is a cat and the second coordinate is a mouse, we can reasonably deduce that the relation between them is that the cat hunts the mouse. Thus even if we are programmed to not know what "hunt" may mean we can always tell when a cat x is hunting a mouse y and when not; as we would just have to see whether the ordered pair (x,y) does or does not belong to the set.

Hereby we can define a relation as a set of ordered pairs:

Definition. A set R is a relation if each element of R is an ordered pair; such that if $z \in R$, then there exists x and y such that $z = (x,y)$.

or

Definition 12. *A relation on a set A is a subset of $A \times A$. If $R \subset A \times A$ is a relation on A and $x,y \in A$, we sometimes write xRy for $(x,y) \in R$.*

Remark. xRy is stated as x stands in relation R to y.

Prove that $\emptyset$ is a set of ordered pairs.

Proof. To prove that the empty set $\emptyset$ is a set of ordered pairs, we must show that $\emptyset$ satisfies the condition $\{(a,b) \mid a,b \in \emptyset\}$. Since $\emptyset$ has no elements, the condition is automatically fulfilled, by specification. ■

Fact: Now let X be any set, and let R be the set of all those pairs (x,y) in $X \times X$ for which $x = y$, if x and y are in X, then xRy means $x = y$.

Fact: Let X be any set, and let R be the set of all those pairs (x,A) in $X \times \mathscr{P}(X)$ for which $x \in A$, if $x \in X$ and $A \in \mathscr{P}(X)$, then xRA means the same as $x \in A$

1.7.1 Domain and Range

Associated with every set R of ordered pairs, there are two sets called *projections* of R onto the first and second coordinates. These sets are known as the *domain* and *range* of R, defined as

$$\text{domain } R = \text{dom } R = \{x : for\ some\ y(xRy)\}$$

and

$$\text{range } R = \text{ran } R = \{y : for\ some\ y(xRy)\}.$$

Remark. Both the domain and range of $\emptyset$ are equal to $\emptyset$.

If $R = X \times Y$, then dom $R = X$, and ran $R = Y$. If R is equality in X, then dom $R =$ ran $R = X$. If R is belonging, between X and $\mathscr{P}(X)$, then dom $R = X$ and ran R $\mathscr{P}(X) - \{\emptyset\}$

When we say that a relation R is included in a Cartesian product $X \times Y$, it means that the domain of $R \subset X$ and the range of $R \subset Y$. For simplicity, we can describe R as a relation from X to Y. If R is a relation where both the domain and range are within X, we refer to R as a relation in X.

1.7.2 Properties of relations

A relation R in X has certain properties.

1. **Reflective**: if every element $x \in X$ is related to itself (i.e., xRx for all $x \in X$)
2. **Symmetric:** if whenever an element x is related to an element y, then y is also related to x (i.e., xRy implies yRx)
3. **Transitive**: if whenever an element x is related to an element y and y is related to an element z, then x is also related to z (i.e., xRy and yRz together imply xRz).

Remark. A relation in a set is an equivalence relation if it is reflexive, symmetric, and transitive. Smallest in a set X is the relation of equality in X , largest in X is $X \times X$.

There is a close connection between equivalence relations in a set X and certain collections called partitions of subsets of X.

Definition (Partition of set)**.** A partition of X is a disjoint collection $\mathscr{C}$ of non-empty subsets of X whose union is X.

If R is an equivalence relation in X, and if $x \in X$, the equivalence class of x with respect to R is the set of all elements $y \in X$ for which xRy. (The weight of tradition makes the use of the word "class" at this point unavoidable.)

For example:

- If R is the equality relation in X, then each equivalence class is a singleton.
- If R is the entire Cartesian product $X \times X$, then the set X itself is the only equivalence class.

There is no standard notation for the equivalence class of x with respect to R; we shall usually denote it by $\frac{x}{R}$, and we shall write $\frac{X}{R}$ or $[X]$ for the set of all equivalence classes. (Pronounce $\frac{X}{R}$ as "X modulo R," or, in abbreviated form, "X mod R.")

Definition (Equivalence class)**.** If $\sim$ is an equivalence relation, then the equivalence class $[x]$ is the set of all elements that are related via $\sim$ to x.

Theorem. If $\sim$ is an equivalence relation on A, then the equivalence classes of $\sim$ form a partition of A.

Proof. By reflexivity, we have $a \in [a]$. Thus the equivalence classes cover the whole set. We must now show that for all $a, b \in A$, either $[a] = [b]$ or $[a] \cap [b] = \emptyset$.

Suppose $[a] \cap [b] \neq \emptyset$. Then $\exists c \in [a] \cap [b]$. So $a \sim c, b \sim c$. By symmetry, $c \sim b$. By transitivity, we have $a \sim b$. For all $b' \in [b]$, we have $b \sim b'$. Thus by transitivity, we have $a \sim b'$. Thus $[b] \subseteq [a]$. By symmetry, $[a] \subseteq [b]$ and $[a] = [b]$. ■

On the other hand, each partition defines an equivalence relation in which two elements are related iff they are in the same partition. Thus partitions and equivalence relations are "the same thing".

Definition (Quotient map)**.** The *quotient map* q maps each element in A to the equivalence class containing a, i.e. $a \mapsto [a]$. e.g. $q(8\heartsuit) = \{\heartsuit\}$.