Intro to Set Theory, Logic, and Proof Techniques Exercises.

Gudfit

1 Exercises

(Try it yourself - ★'ed Exercises require some knowledge of proof before attempting).

Exercise 1.1. *Consider the following sequence of sets:* $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$, *and so on. Now consider all possible unordered pairs formed from these sets. Additionally, consider "mixed pairs" formed by pairing any singleton with any pair from the above constructions. This process can be continued indefinitely. Prove that all the sets obtained in this way are distinct from one another.*

Exercise 1.2. *Given a set* $A = \{1, 2, 3, \ldots\}$, *all unordered pairs formed from distinct elements of* A *are distinct from one another.* ★.

Exercise 1.3. *Prove that* $(A \cap B) \cup C = A \cap (B \cup C)$ *if and only if* $C \subseteq A$.
⋆ *Observe that the condition has nothing to do with the set* B.

Exercise 1.4. *Prove that* $\mathscr{P}(G) \cup \mathscr{P}(F) \subseteq \mathscr{P}(G \cup F)$.

Exercise 1.5. *Show that* $\mathscr{P}(G) \cap \mathscr{P}(F) = \mathscr{P}(G \cap F)$.

Exercise 1.6. *Prove that if* $G \subseteq F$ *then* $\mathscr{P}(G) \subseteq \mathscr{P}(F)$.

Exercise 1.7. *If* $A, B, X,$ *and* Y *are sets, then prove*

1. $(A \cup B) \times X = (A \times X) \cup (B \times X)$
2. $(A \cap B) \times (X \cap Y) = (A \times X) \cap (B \times Y)$
3. $(A - B) \times X = (A \times X) - (B \times X)$

Exercise 1.8. *If either* $A = \emptyset$ *or* $B = \emptyset$, *then* $A \times B = \emptyset$ *(and conversely).*

Exercise 1.9. *If* $A \subseteq X$ *and* $B \subseteq Y$ *then* $A \times B \subseteq X \times Y$ *if* $A \times B \neq \emptyset$ *(and conversely).*

1.1 Fun Exercises

Exercise 1.10. *Prove that* $A - (B \cup C) = (A - B) \cap (A - C)$.

Exercise 1.11. *The symmetric difference* $A \oplus B$ *of two sets* A *and* B *is the set of elements that belong to exactly one of* A *and* B. *Express* $A \oplus B$ *in terms of* $\cup$, $\cap$, *and* $-$. *Prove that* $\oplus$ *is associative.*

Exercise 1.12. *Let* $A_1, A_2, A_3, \ldots$ *be sets such that* $A_1 \cap A_2 \cap \ldots \cap A_n \neq \emptyset$ *for all* n. *Must it be that* $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$?

2 Answers

Solution. ***Proof of Exercise 1.1:***

Proof. We start with the empty set $\emptyset$, which is a well-defined set according to the axiom of extension.

Using the axiom of pairing, we can form the set $\{\emptyset\}$ ($\{x : x = \emptyset \ or \ x = \emptyset\} = \{\emptyset, \emptyset\} = \{\emptyset\}$), which contains the empty set as its sole element (singleton). By the axiom of extension, $\{\emptyset\}$ is distinct from $\emptyset$ because $\emptyset$ does not contain any elements, while $\{\emptyset\}$ contains one element, which is $\emptyset$.

Again, using the axiom of pairing, we can form the set $\{\{\emptyset\}\}$, which contains the set $\{\emptyset\}$ as its sole element. By the axiom of extension, $\{\{\emptyset\}\}$ is distinct from both $\emptyset$ and $\{\emptyset\}$ because it contains a different element, namely $\{\emptyset\}$.

Continuing this process, we can form sets like $\{\{\{\emptyset\}\}\}$, $\{\{\{\{\emptyset\}\}\}\}$, and so on, where each new set contains the previous set as its sole element.

To form unordered pairs, we can use the axiom of pairing repeatedly. For example, we can form the pair $\{x : x = \emptyset \ or \ x = \{\emptyset\}\} = \{\emptyset, \{\emptyset\}\}$, which is distinct from $\emptyset$ and $\{\emptyset\}$ by the axiom of extension, as the pairs contains two distinct elements.

We can also form "mixed pairs" by pairing any singleton set with any pair from the above constructions. For instance, we can form the set $\{x : x = \emptyset \ or \ x = \{\emptyset, \{\emptyset\}\}\} = \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$, which is distinct from all previous sets because it has two distinct elements: $\emptyset$ and the pair $\{\emptyset, \{\emptyset\}\}$.

This process can be continued indefinitely, and each new set formed will be distinct from all previously constructed sets by the axiom of extension, as it will contain a unique combination of elements.

Therefore, by successively applying the axioms of extension, specification, and pairing, we can construct an infinite sequence of distinct sets, proving that all the sets obtained in this way are indeed distinct from one another.

(A sorta fun introduction to Induction) ■

Solution. ***Proof of Exercise 1.2:***

Proof. Let $A = \{1, 2, 3, \ldots\}$ be the set of positive integers. We will consider unordered pairs of the form $\{a, b\}$ where a and $b \in A$ and $a \neq b$.

By the axiom of pairing, for any a and $b \in A$ with $a \neq b$, we can form the set $\{a, b\}$. Now, consider two distinct pairs $\{a, b\}$ and $\{c, d\}$ where a, b, c and $d \in A$ and $a \neq b$, $c \neq d$.

We want to show that $\{a, b\}$ and $\{c, d\}$ are distinct sets. By the axiom of extension, two sets are equal if and only if they have the same elements. That is: $\{a, b\} = \{c, d\}$ if and only if all elements ($x \in \{a, b\}$ is equal to all elements $x \in \{c, d\}$)

Suppose $\{a, b\} = \{c, d\}$. Then, by the axiom of extension, we have:

$$\text{All elements } (x \in \{a, b\} \text{ is equal to all elements } x \in \{c, d\})$$

By the definition of $\{a, b\}$ and $\{c, d\}$, we can rewrite this as:

$$\text{For all elements } x \text{ such that } ((x = a \text{ or } x = b) \text{ and is equal to } (x = c \text{ or } x = d))$$

Since $a \neq b$ and $c \neq d$, there are only two possibilities for the elements in $\{a, b\}$ and $\{c, d\}$:

Case 1: $a = c$ and $b = d$
Case 2: $a = d$ and $b = c$

By the axiom of specification, we can define two new sets:

$$B = \{x \in A : x = a \text{ or } x = b\}$$

$$C = \{x \in A : x = c \text{ or } x = d\}$$

If Case 1 holds, then $B = C$, which implies $\{a, b\} = \{c, d\}$ by the axiom of extension. If Case 2 holds, then $B = C$, which implies $\{a, b\} = \{c, d\}$ by the axiom of extension. Therefore, if $\{a, b\} = \{c, d\}$, then either Case 1 or Case 2 must hold, which contradicts our assumption that $a \neq b$ and $c \neq d$. Hence, by contradiction, $\{a, b\}$ and $\{c, d\}$ are distinct sets.

Since A is infinite, we can form infinitely many such distinct pairs. For example, $\{1, 2\}$, $\{3, 4\}$, and so on, are all distinct from one another. Thus, we have proven that all unordered pairs formed from distinct elements of A are distinct from one another, using only the axioms of extension, specification, and pairing. ■

Solution. ***Proof of Exercise 1.3:***

Proof. We need to show that $(A \cap B) \cup C = A \cap (B \cup C)$ if and only if $C \subseteq A$.

($\Rightarrow$) Assume $(A \cap B) \cup C = A \cap (B \cup C)$. Let $x \in C$. Then $x \in (A \cap B) \cup C$, which implies $x \in A \cap (B \cup C)$. Therefore, $x \in A$. Hence, $C \subseteq A$.

($\Leftarrow$) Assume $C \subseteq A$. Then:

$$(A \cap B) \cup C = (A \cap B) \cup (A \cap C) = A \cap (B \cup C)$$

Thus, $(A \cap B) \cup C = A \cap (B \cup C)$.

Therefore, the equality holds if and only if $C \subseteq A$. ■

Solution. ***Proof of Exercise 1.4:***

Proof. Let $X \in \mathscr{P}(G) \cup \mathscr{P}(F)$. Then, either $X \subseteq G$ or $X \subseteq F$. In either case, $X \subseteq G \cup F$. Hence, $X \in \mathscr{P}(G \cup F)$.

Therefore, $\mathscr{P}(G) \cup \mathscr{P}(F) \subseteq \mathscr{P}(G \cup F)$. ■

Solution. ***Proof of Exercise 1.5:***

Proof. Let $X \in \mathscr{P}(G) \cap \mathscr{P}(F)$. Then, $X \subseteq G$ and $X \subseteq F$. Therefore, $X \subseteq G \cap F$, which implies $X \in \mathscr{P}(G \cap F)$.

Conversely, let $X \in \mathscr{P}(G \cap F)$. Then, $X \subseteq G \cap F$, so $X \subseteq G$ and $X \subseteq F$. Hence, $X \in \mathscr{P}(G) \cap \mathscr{P}(F)$.

Thus, $\mathscr{P}(G) \cap \mathscr{P}(F) = \mathscr{P}(G \cap F)$. ■

Solution. ***Proof of Exercise 1.6:***

Proof. Assume $G \subseteq F$. Let $X \in \mathscr{P}(G)$. Then, $X \subseteq G$. Since $G \subseteq F$, it follows that $X \subseteq F$. Therefore, $X \in \mathscr{P}(F)$.

Hence, $\mathscr{P}(G) \subseteq \mathscr{P}(F)$. ■

Solution. ***Proof of Exercise 1.7:***

Proof. We will prove each part separately.

1. $(A \cup B) \times X = (A \times X) \cup (B \times X)$

   Consider an arbitrary element $(a, x) \in (A \cup B) \times X$. By the definition of the Cartesian product, this means that $a \in A \cup B$ and $x \in X$. Then, either $a \in A$ or $a \in B$, and $x \in X$.

   (a) If $a \in A$, then $(a, x) \in A \times X \subseteq (A \times X) \cup (B \times X)$.
   (b) If $a \in B$, then $(a, x) \in B \times X \subseteq (A \times X) \cup (B \times X)$.

   Thus, $(A \cup B) \times X \subseteq (A \times X) \cup (B \times X)$. Conversely, let $(a, x) \in (A \times X) \cup (B \times X)$. Then, either $(a, x) \in A \times X$ or $(a, x) \in B \times X$.

   (a) If $(a, x) \in A \times X$, then $a \in A$ and $x \in X$, so $a \in A \subseteq A \cup B$, hence $(a, x) \in (A \cup B) \times X$.
   (b) If $(a, x) \in B \times X$, then $a \in B$ and $x \in X$, so $a \in B \subseteq A \cup B$, hence $(a, x) \in (A \cup B) \times X$.

   Therefore, $(A \times X) \cup (B \times X) \subseteq (A \cup B) \times X$. Thus, $(A \cup B) \times X = (A \times X) \cup (B \times X)$.

2. $(A \cap B) \times (X \cap Y) = (A \times X) \cap (B \times Y)$

   First, let $(a, x) \in (A \cap B) \times (X \cap Y)$. Then, $a \in A \cap B$ and $x \in X \cap Y$. Thus, $a \in A$ and $a \in B$, and $x \in X$ and $x \in Y$. Therefore, $(a, x) \in A \times X$ and $(a, x) \in B \times Y$, so $(a, x) \in (A \times X) \cap (B \times Y)$.

   Conversely, let $(a, x) \in (A \times X) \cap (B \times Y)$. Then, $(a, x) \in A \times X$ and $(a, x) \in B \times Y$. So $a \in A$ and $x \in X$, and $a \in B$ and $x \in Y$. Therefore, $a \in A \cap B$ and $x \in X \cap Y$. Hence, $(a, x) \in (A \cap B) \times (X \cap Y)$. Thus, $(A \cap B) \times (X \cap Y) = (A \times X) \cap (B \times Y)$.

3. $(A - B) \times X = (A \times X) - (B \times X)$

   First, let $(a, x) \in (A - B) \times X$. Then, $a \in A - B$ and $x \in X$. So $a \in A$ and $a \notin B$, and $x \in X$. Therefore, $(a, x) \in A \times X$, and since $a \notin B$, $(a, x) \notin B \times X$. Thus, $(a, x) \in (A \times X) - (B \times X)$.

   Conversely, let $(a, x) \in (A \times X) - (B \times X)$. Then, $(a, x) \in A \times X$, so $a \in A$ and $x \in X$, and $(a, x) \notin B \times X$, so it must be that $a \notin B$ (since $x \in X$). Therefore, $a \in A - B$, so $(a, x) \in (A - B) \times X$.

   Thus, $(A - B) \times X = (A \times X) - (B \times X)$.

■

Solution. ***Proof of Exercise 1.8:***

Proof. We need to show that if either $A = \emptyset$ or $B = \emptyset$, then $A \times B = \emptyset$, and conversely, if $A \times B = \emptyset$, then either $A = \emptyset$ or $B = \emptyset$.

($\Rightarrow$) Assume that $A = \emptyset$ or $B = \emptyset$.

   (a) If $A = \emptyset$, then there are no elements $a \in A$, so there are no pairs (a, b) where $a \in A$ and $b \in B$. Therefore, $A \times B = \emptyset$.

(b) If $B = \emptyset$, similarly, there are no elements $b \in B$, so there are no pairs (a, b) where $a \in A$ and $b \in B$. Therefore, $A \times B = \emptyset$.

($\Leftarrow$) Conversely, assume that $A \times B = \emptyset$. We need to show that either $A = \emptyset$ or $B = \emptyset$.

Suppose, for contradiction, that $A \neq \emptyset$ and $B \neq \emptyset$. Then there exist elements $a \in A$ and $b \in B$. Therefore, the pair $(a, b) \in A \times B$, which contradicts the assumption that $A \times B = \emptyset$. Therefore, if $A \times B = \emptyset$, then either $A = \emptyset$ or $B = \emptyset$.

■

Solution. ***Proof of Exercise 1.9:***

Proof. We need to show that if $A \subset X$ and $B \subset Y$, and $A \times B \neq \emptyset$, then $A \times B \subset X \times Y$, and conversely, if $A \times B \subset X \times Y$ and $A \times B \neq \emptyset$, then $A \subset X$ and $B \subset Y$.

($\Rightarrow$) Assume that $A \subset X$, $B \subset Y$, and $A \times B \neq \emptyset$.

Let $(a, b) \in A \times B$. Then $a \in A \subset X$, so $a \in X$, and $b \in B \subset Y$, so $b \in Y$. Therefore, $(a, b) \in X \times Y$. Thus, $A \times B \subset X \times Y$.

($\Leftarrow$) Conversely, assume that $A \times B \subset X \times Y$, and $A \times B \neq \emptyset$.

We need to show that $A \subset X$ and $B \subset Y$. Since $A \times B \neq \emptyset$, there exists at least one pair $(a_0, b_0) \in A \times B$.

Let $a \in A$ be arbitrary. Since $B \neq \emptyset$ (otherwise $A \times B = \emptyset$), choose any $b \in B$. Then $(a, b) \in A \times B \subset X \times Y$, so $a \in X$. Thus, every $a \in A$ is in X, so $A \subset X$.

Similarly, let $b \in B$ be arbitrary. Since $A \neq \emptyset$, choose any $a \in A$. Then $(a, b) \in A \times B \subset X \times Y$, so $b \in Y$. Thus, $B \subset Y$.

■

2.1 Fun Answers

Solution. *Proof of Exercise 1.10.* We show mutual inclusion.

($\subseteq$) Let $x \in A - (B \cup C)$. Then $x \in A$ and $x \notin B \cup C$, hence $x \notin B$ and $x \notin C$. Therefore $x \in A - B$ and $x \in A - C$, so $x \in (A - B) \cap (A - C)$.

($\supseteq$) Let $x \in (A - B) \cap (A - C)$. Then $x \in A$, $x \notin B$, and $x \notin C$, hence $x \notin B \cup C$. Thus $x \in A - (B \cup C)$.

Therefore $A - (B \cup C) = (A - B) \cap (A - C)$. ■

Solution. *Solution to Exercise 1.11. Expression.* By definition of symmetric difference,

$$A \oplus B = (A - B) \cup (B - A).$$

Using elementary identities this is equivalent to

$$A \oplus B = (A \cup B) - (A \cap B).$$

Associativity. For any set S and element x, let $\chi_S(x) \in \{0, 1\}$ be the indicator of membership. Then for all sets X, Y,

$$\chi_{X \oplus Y}(x) \equiv \chi_X(x) + \chi_Y(x) \pmod 2.$$

Hence, for all x,

$$\chi_{(A \oplus B) \oplus C}(x) \equiv (\chi_A(x)+\chi_B(x))+\chi_C(x) \equiv \chi_A(x)+(\chi_B(x)+\chi_C(x)) \equiv \chi_{A \oplus (B \oplus C)}(x) \pmod 2.$$

Therefore $(A \oplus B) \oplus C = A \oplus (B \oplus C)$. ■

Solution. *Solution to Exercise 1.12.* No. Let $A_n = \left(0, \frac{1}{n}\right)$. For each finite n,

$$\bigcap_{k=1}^{n} A_k = (0, 1/n) \neq \emptyset.$$

But

$$\bigcap_{n=1}^{\infty} A_n = \emptyset.$$

Indeed, if $x > 0$ then choose $N > \frac{1}{x}$, so $1/N < x$ and hence $x \notin (0, 1/N)$; also $0 \notin (0, 1/n)$ for any n. Thus no element lies in all A_n. ■