

Homework

1. **Inference Rules.** Show the following, stating at each step which rule of inference you used.

(a) Show that the premises

“If I were smart or good-looking, I would be happy and rich.”

“I am not rich.”

lead to the conclusion “I am not smart.”

(b) Show that the premises

$$\forall x (P(x) \vee Q(x))$$

$$\forall x (\neg Q(x) \vee S(x))$$

$$\forall x (R(x) \rightarrow \neg S(x))$$

$$\exists x \neg P(x)$$

lead to the conclusion $\exists x \neg R(x)$.

2. **A Married Observer.** Given that Lars is married, that Jeff is not married, that Lars can only see Lisa, that Lisa can only see Jeff, and that Jeff cannot see anyone, show that there is a married person who can see an unmarried one.

3. **Irrationality.**

(a) Prove that $\sqrt{3}$ is irrational.

(b) Prove that $\log_2 9$ is irrational.

(c) Without using logarithms or the irrationality of $\sqrt{2}$, and following the nonconstructive existence proof given in class, show that there exist irrational numbers x and y such that x^y is rational and at least one of x, y equals $\sqrt{3}$.

(d) Find an irrational number x such that \sqrt{x} is rational, or prove that no such x exists.

4. **Integers on a Circle.** The integers $1, 2, \dots, 13$ are written on a circle, in any order.

(a) Show that there exist four adjacent numbers whose sum is at most 28.

(b) Can 28 be replaced by 27? Prove your answer.

5. **Truth-Table Evaluator.** Write a Python programme that evaluates all $2^3 = 8$ rows of the truth table for the proposition

$$(p \rightarrow (q \rightarrow r)) \rightarrow ((p \wedge q) \rightarrow r)$$

and determines whether it is a tautology, contradiction, or contingency. Recall that $a \rightarrow b$ is equivalent to `not a or b` in Python.

6. **Goldbach Verification.** The Goldbach conjecture asserts that every even integer $n > 2$ can be written as the sum of two primes. Write a Python programme that checks this claim for all even integers from 4 to 10 000. For each n , find one decomposition $n = p + q$ with $p \leq q$, both prime. Report whether any counterexample is found.

7. **Divisibility by Cases.** Suppose you want to prove that every product of consecutive integers of the form $k(k+1)(k+2)$ is divisible by 6. If you want to prove this by cases, which of the following is a set of cases you should use?

A. The product ends in 3; the product ends in 6; the product ends in 9.

B. When k is divided by 3, the remainder is 0; when k is divided by 3, the remainder is 1; when k is divided by 3, the remainder is 2.

C. $k = 3n$; $k \neq 3n$.

D. k is prime; k is not prime.

8. **Without Loss of Generality.** Suppose you want to prove that the following is true for all pairs of distinct real numbers x and y : the average of x and y lies strictly between x and y . Which of the following can you assume, without loss of generality?

- A. x and y are even.
- B. $x < 0$ and $y > 0$.
- C. $x < y$.
- D. x and y are integers.

9. Absolute Value Cases. Suppose you want to prove a theorem about the product of absolute values of real numbers, $|x| \cdot |y|$. If you were to give a proof by cases, what set of cases would probably be best?

- A. Both x and y non-negative; one negative and one non-negative; both negative.
- B. Both x and y rational; one rational and one irrational; both irrational.
- C. Both x and y even; one even and one odd; both odd.
- D. $x > y$; $x < y$; $x = y$.

10. Spot the Errors. We provide the following proof for the statement

$$\forall x \in \mathbb{R} \setminus \{0\} \left(\sqrt{2 - \frac{1}{x^2}} = 1 \leftrightarrow (x = 1 \vee x = -1) \right).$$

Step 1. $\sqrt{2 - \frac{1}{x^2}} = 1$ iff $x\sqrt{2 - \frac{1}{x^2}} = x$, since for $x \neq 0$ we may multiply both sides by x .

Step 2. $x\sqrt{2 - \frac{1}{x^2}} = x$ iff $\sqrt{2x^2 - 1} = x$, since $\forall x \in \mathbb{R} \setminus \{0\}$ $(x\sqrt{2 - \frac{1}{x^2}} = \sqrt{2x^2 - 1})$.

Step 3. $\sqrt{2x^2 - 1} = x$ iff $2x^2 - 1 = x^2$, since $\forall x \in \mathbb{R} \setminus \{0\}$ $(\sqrt{2 - \frac{1}{x^2}} = x \leftrightarrow 2x^2 - 1 = x^2)$.

Step 4. $2x^2 - 1 = x^2$ iff $x^2 - 1 = 0$ (subtract x^2 from both sides).

Step 5. $x^2 - 1 = 0$ iff $(x + 1)(x - 1) = 0$, since $\forall x \in \mathbb{R}$ $(x^2 - 1 = (x + 1)(x - 1))$.

Step 6. $(x + 1)(x - 1) = 0$ iff $x = 1 \vee x = -1$.

This proof contains:

- A. 1 error.
- B. 2 errors.
- C. 3 or more errors.
- D. No errors.