Exercises

1. **Quantifier Negation.**
   (a) What is the negation of $\forall x\,(P(x) \to Q(x))$?
   (b) What is the negation of $\forall x\,\exists y\,\forall z\; Q(x,y,z)$?
   (c) Negate the statement $\forall x\,\exists y\,(P(x,y) \wedge (\exists z\; R(x,y,z)))$.

2. **Quantifier Translation.**
   (a) Assume the universe of discourse consists of all real numbers. Write a formal statement that says "Every number has exactly one additive inverse."
   (b) Let $C(x)$ denote "x is a comedian" and $F(x)$ denote "x is funny," where the domain consists of all people. Express the statement "Every comedian is funny" using quantifiers and the predicates C and F.
   (c) Let the domain of m be all students, and let $P(m)$ be the statement "m spends more than 2 hours playing polo." Express $\forall m\,\neg P(m)$ in English.
   (d) The statement "If x and y are real numbers, then $|x+y| \leqslant |x| + |y|$" contains two variables with implicit quantifiers. What are these quantifiers?

3. **Negation (Multiple Choice).** Which of the following is the negation of the statement "Everyone in the class except Lee has a computer"?
   A. Someone in the class other than Lee does not have a computer, or Lee has a computer.
   B. Lee and someone else in the class have a computer.
   C. Someone in the class other than Lee does not have a computer, and Lee does not have a computer.
   D. Someone in the class other than Lee does not have a computer.

4. **Validity and Quantifier Equivalence.**
   (a) Determine the validity of the following rule of inference:

$$\frac{p \to (q \to r), \qquad q \to (p \to r)}{(p \vee q) \to r}$$

   (b) Which of the following expressions are equivalent to $\neg(\forall x\,\exists y\; P(x,y))$? Explain.
      (i) $\exists x\,\forall y\,\neg P(x,y)$
      (ii) $\exists x\,\exists y\,\neg P(x,y)$

5. **Logical Equivalences.** Prove or disprove the following: give a proof if it is indeed a logical equivalence, give a counterexample if not.
   (a) $(p \to q) \to r \equiv p \to (q \to r)$
   (b) $(p \to q) \wedge (p \to r) \equiv p \to (q \vee r)$
   (c) $(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$

6. **Sum of Two Squares.** Write a Python programme that determines, for each integer n from 1 to 100, whether n can be expressed as $n = a^2 + b^2$ for some non-negative integers a and b, using exhaustive enumeration. For each representable n, print n alongside one valid pair (a, b).