Homework 2

1. **Remainder Arithmetic.** Suppose an integer m leaves a remainder of i when divided by 3, and an integer n leaves a remainder of j when divided by 3, where $0 \leqslant i, j < 3$. Prove that $m+n$ and $i+j$ leave the same remainder when divided by 3.

2. **Quadratic Residues mod 3.** What are the possible remainders of n^2 when divided by 3, where $n \in \mathbb{Z}$? Prove your answer.

3. **Absolute Value Identities.** Prove the following.
   (i) $|xy| = |x| \cdot |y|$.
   (ii) $\left|\frac{1}{x}\right| = \frac{1}{|x|}$, if $x \neq 0$. (The best way to do this is to remember what $|x|^{-1}$ is.)
   (iii) $\frac{|x|}{|y|} = \left|\frac{x}{y}\right|$, if $y \neq 0$.
   (iv) $|x - y| \leqslant |x| + |y|$. (Give a very short proof.)
   (v) $|x| - |y| \leqslant |x - y|$. (A very short proof is possible, if you write things in the right way.)
   (vi) $\big||x| - |y|\big| \leqslant |x - y|$. (Why does this follow immediately from (v)?)
   (vii) $|x + y + z| \leqslant |x| + |y| + |z|$. Indicate when equality holds, and prove your statement.

4. **Max and Min.** The maximum of two numbers x and y is denoted by $\max(x, y)$. Thus $\max(-1, 3) = \max(3, 3) = 3$ and $\max(-1, -4) = \max(-4, -1) = -1$. The minimum of x and y is denoted by $\min(x, y)$. Prove that

$$\max(x, y) = \frac{x + y + |y - x|}{2}, \qquad \min(x, y) = \frac{x + y - |y - x|}{2}.$$

   Derive a formula for $\max(x, y, z)$ and $\min(x, y, z)$, using, for example, $\max(x, y, z) = \max\big(x, \max(y, z)\big)$.

5. **Absolute Value Foundations.**
   (a) Prove that $|a| = |-a|$. (The trick is not to become confused by too many cases. First prove the statement for $a \geqslant 0$. Why is it then obvious for $a \leqslant 0$?)
   (b) Prove that $-b \leqslant a \leqslant b$ if and only if $|a| \leqslant b$. In particular, it follows that $-|a| \leqslant a \leqslant |a|$.
   (c) Use this fact to give a new proof that $|a + b| \leqslant |a| + |b|$.

6. **Positive Definite Forms.** Prove that if x and y are not both 0, then

$$x^2 + xy + y^2 > 0, \qquad x^4 + x^3y + x^2y^2 + xy^3 + y^4 > 0.$$

   Hint: for the first, complete the square. For the second, consider the factorisation of $x^5 - y^5$.

7. **When Does $(x + y)^n = x^n + y^n$?**
   (a) Show that $(x + y)^2 = x^2 + y^2$ only when $x = 0$ or $y = 0$, and $(x + y)^3 = x^3 + y^3$ only when $x = 0$ or $y = 0$ or $x = -y$.
   (b) Using the fact that $x^2 + 2xy + y^2 = (x + y)^2 \geqslant 0$, show that $4x^2 + 6xy + 4y^2 > 0$ unless x and y are both 0.
   (c) Use part (b) to find out when $(x + y)^4 = x^4 + y^4$.
   (d) Find out when $(x + y)^5 = x^5 + y^5$.

8. **Digit Replacement.** Write a programme that reads three integers from the user: a non-negative integer `n`, a non-negative integer `k`, and a single digit `d` ($0 \leqslant d \leqslant 9$). Print the number formed by replacing the k-th digit of `n` (counting from the right, starting at 0) with `d`. If `k` exceeds the number of digits in `n`, the digit `d` is placed in the new position (as if `n` had leading zeros).

`n`	`k`	`d`	Output
468	0	1	461
468	1	1	418
468	2	1	168
468	3	1	1468

9. **Nearest Odd.** Write a programme that reads an integer `n` from the user and prints the nearest odd integer. If `n` is already odd, print it unchanged. If `n` is even, it is equidistant from two odd numbers; print the smaller one.

Input	Output
3	3
4	3
−4	−5
0	−1

10. **Date Validation.** Write a programme that reads three integers `day`, `month`, and `year` from the user and prints `’Valid date’` or `’Invalid date’`. A date is valid if all of the following hold:
    - The year is positive.
    - The month is between 1 and 12.
    - The day is between 1 and the maximum for that month: 31 for months 1, 3, 5, 7, 8, 10, 12; 30 for months 4, 6, 9, 11; and 28 or 29 for month 2, where 29 is permitted only in a leap year.

Input	Output
29, 2, 2024	Valid date
29, 2, 1900	Invalid date
31, 4, 2025	Invalid date

11. **Base-b Digit Interpreter.** Write a programme that reads a string `s` of exactly three characters and an integer `b` ($2 \leqslant b \leqslant 36$) from the user. Interpret `s` as a three-digit base-b number and print its decimal value. Use the convention that `’0’`–`’9’` represent values 0–9 and `’A’`–`’Z’` (case-insensitive) represent values 10–35. If any character of `s` is not a valid digit in base b (i.e., its value is $\geqslant b$), print `’Invalid digit’`. Use `ord` and conditionals; do not use `int(s, b)`.

s	b	Output
`’F0A’`	16	3850
`’1A3’`	12	267
`’1G0’`	16	Invalid digit

12. **Colour Mixer.** An RGB colour is specified by three integers between 0 and 255: red (R), green (G), and blue (B). The *floor* of a real number x, written $\lfloor x \rfloor$, is the greatest integer less than or equal to x. In Python, `int(x)` truncates a non-negative float toward zero, which coincides with the floor for $x \geqslant 0$.

    Write a programme that reads two colours (six integers `r1 g1 b1 r2 g2 b2`) and a blending factor `t` ($0 \leqslant t \leqslant 1$, a float) from the user. Print:

    (a) The blended colour: each channel computed as $\lfloor R_1(1-t) + R_2 \cdot t \rfloor$ (and similarly for G, B).
    (b) The brightness: $(R + G + B)/3$, formatted to one decimal place.
    (c) The classification: `’Warm’` if $R > B + 50$, `’Cool’` if $B > R + 50$, or `’Neutral’` otherwise.
    (d) The greyscale value: $\lfloor 0.299R + 0.587G + 0.114B \rfloor$.

```
Enter r1 g1 b1: 255 100 50
Enter r2 g2 b2: 0 200 255
Enter t: 0.5
Blended colour: (127, 150, 152)
Brightness: 143.0
Classification: Neutral
Greyscale: 143
```