Exercises

1. **Logical Equivalence.** Show that $(p \to q) \vee (p \to r)$ and $p \to (q \vee r)$ are logically equivalent.

2. **Algebraic Identities.** Prove the following, using only the axioms (A1)–(A9).
   (a) If $ax = a$ for some number $a \neq 0$, then $x = 1$.
   (b) $x^2 - y^2 = (x-y)(x+y)$.
   (c) If $x^2 = y^2$, then $x = y$ or $x = -y$.
   (d) $x^3 - y^3 = (x-y)(x^2+xy+y^2)$.
   (e) $x^3 + y^3 = (x+y)(x^2-xy+y^2)$.

3. **Properties of Inequalities.** Prove the following.
   (i) If $a < b$ and $c < d$, then $a + c < b + d$.
   (ii) If $a < b$, then $-b < -a$.
   (iii) If $a < b$ and $c > d$, then $a - c < b - d$.
   (iv) If $a < b$ and $c > 0$, then $ac < bc$.
   (v) If $a < b$ and $c < 0$, then $ac > bc$.
   (vi) If $a > 1$, then $a^2 > a$.
   (vii) If $0 < a < 1$, then $a^2 < a$.
   (viii) If $0 \leq a < b$ and $0 \leq c < d$, then $ac < bd$.
   (ix) If $0 \leq a < b$, then $a^2 < b^2$. (Use (viii).)
   (x) If $a, b \geq 0$ and $a^2 < b^2$, then $a < b$. (Use (ix), backwards.)

4. **Introduction to Number Theory.** For each statement below, decide whether it is true or false. Prove your claim.
   (a) If $n \in \mathbb{N}$ and $n^2 + (n+1)^2 = (n+2)^2$, then $n = 3$.
   (b) For all $n \in \mathbb{N}$, it is false that $(n-1)^3 + n^3 = (n+1)^3$.

5. **Solving Inequalities.** Find all real numbers x for which the following hold.
   (i) $5 - x^2 < 8$.
   (ii) $5 - x^2 < -2$.
   (iii) $(x-1)(x-3) > 0$.
   (iv) $x^2 - 2x + 2 > 0$.
   (v) $x^2 + x + 1 > 2$.
   (vi) $x^2 - x + 10 > 16$.
   (vii) $x^2 + x + 1 > 0$.
   (viii) $(x - \pi)(x+5)(x-3) > 0$.
   (ix) $(x - \sqrt[3]{2})(x - \sqrt{2}) > 0$.
   (x) $\dfrac{1}{x} + \dfrac{1}{1-x} > 0$.
   (xi) $\dfrac{x-1}{x+1} > 0$.

6. **Closure.** A set X is *closed* under an operation $\odot$ if, whenever a and b are elements of X, $a \odot b$ is also an element of X. For each operation below, determine which of the number sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ are closed under $\odot$.
   (a) $a \odot b = a + b$
   (b) $a \odot b = a - b$
   (c) $a \odot b = (a-b)(a+b)$
   (d) $a \odot b = (a-1)(b-1) + 2(a+b)$
   (e) $a \odot b = \dfrac{a}{b^2+1}$
   (f) $a \odot b = \dfrac{a}{\sqrt{b^2+1}}$

7. **Irrationality.** Which of the following numbers are irrational for *every* choice of numbers r, a and b,

such that r is rational and a and b are irrational?

$$a+r \qquad a+b \qquad ar \qquad ab \qquad a^r \qquad r^a \qquad a^b$$

Prove your claims, either by proving that the number must always be irrational or by providing a counterexample.

8. **Products Involving $\sqrt{2}$.** Let $a, b, c, d \in \mathbb{Z}$. Under what conditions is $(a+b\sqrt{2})(c+d\sqrt{2})$ an integer?

9. **Algebraic Numbers.** A complex number α is called *algebraic* if $p(\alpha) = 0$ for some nonzero polynomial $p(x)$ over $\mathbb{Q}$.
   (a) Let x be a rational number. Prove that x is algebraic.
   (b) Prove that $\sqrt{2}$ is algebraic.
   (c) Prove that $\sqrt{2} + \sqrt{3}$ is algebraic.
   (d) Prove that $x + yi$ is algebraic, where x and y are any two rational numbers.

10. **Pythagorean Triples.** A Pythagorean triple is a triple of positive integers (a, b, c) such that $a^2 + b^2 = c^2$. Let (x, y, z) be a Pythagorean triple, and let $P = x + y + z$ and $A = \frac{1}{2}xy$ be the perimeter and area, respectively, of the right-angled triangle whose side lengths are x, y and z.
    (a) Find the possible values of (x, y, z) when $P = A$.
    (b) Find the possible values of (x, y, z) when $P = 2A$.

11. **The AM–GM Inequality.** Prove that if $0 < a < b$, then

$$a < \sqrt{ab} < \frac{a+b}{2} < b.$$

Bonus Problem

The fact that $a^2 \geq 0$ for all numbers a, elementary as it may seem, is nevertheless the fundamental idea upon which most important inequalities are ultimately based. The great-granddaddy of all inequalities is the *Schwarz inequality*:

$$x_1y_1 + x_2y_2 \leq \sqrt{x_1^2 + x_2^2}\,\sqrt{y_1^2 + y_2^2}.$$

The three proofs outlined below have only one thing in common: their reliance on the fact that $a^2 \geq 0$ for all a.

(a) Prove that if $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$ for some number λ, then equality holds in the Schwarz inequality. Prove the same thing if $y_1 = y_2 = 0$. Now suppose that y_1 and y_2 are not both 0, and that there is no number λ such that $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$. Then

$$0 < (\lambda y_1 - x_1)^2 + (\lambda y_2 - x_2)^2 = \lambda^2(y_1^2 + y_2^2) - 2\lambda(x_1y_1 + x_2y_2) + (x_1^2 + x_2^2).$$

Using the discriminant, complete the proof of the Schwarz inequality.

(b) Prove the Schwarz inequality by using $2xy \leq x^2 + y^2$ (how is this derived?) with

$$x = \frac{x_i}{\sqrt{x_1^2 + x_2^2}}, \qquad y = \frac{y_i}{\sqrt{y_1^2 + y_2^2}},$$

first for $i = 1$ and then for $i = 2$.

(c) Prove the Schwarz inequality by first proving that

$$(x_1^2 + x_2^2)(y_1^2 + y_2^2) = (x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2.$$

(d) Deduce, from each of these three proofs, that equality holds only when $y_1 = y_2 = 0$ or when there is a number λ such that $x_1 = \lambda y_1$ and $x_2 = \lambda y_2$.