Exercises

1. **Proposition Check.** For each sentence below, determine whether it is a proposition. If it is, state whether its truth value is true, false, or currently unknown.
   (a) 9 is divisible by 3.
   (b) $x + 1 = 4$.
   (c) Solve $x^2 - 1 = 0$.
   (d) There are infinitely many twin primes.
   (e) Is 17 prime?

2. **Classification.** For each of the following sentences, determine if it is a proposition, a term, or neither. If it is a proposition, state its truth value ($\top$ or $\bot$). If it is neither, briefly explain why.
   (a) The sum of the angles in any triangle is 180 degrees.
   (b) $(x + y)^2$.
   (c) Will you pass this course?
   (d) There exists a prime number greater than 100.
   (e) This statement is false.
   (f) Add 5 to both sides of the equation.

3. **Truth Values.** Determine the truth value of each proposition.
   (a) $19 - 4 = 12$ if and only if 3 is a prime number.
   (b) If $1 + 1 = 5$, then $1 + 1 = 3$.
   (c) If the moon is a star, then so is the sun.
   (d) If 5 is a prime number, then the Earth is flat.
   (e) $0 > 1$ if and only if $2 > 1$.
   (f) Either Toronto is the capital of Canada or Hamburg is the capital of Germany.

4. **Your Own Examples.** Give:
   (a) a true proposition,
   (b) a false proposition,
   (c) a proposition whose truth value you do not currently know,
   (d) a sentence that is not a proposition.

5. **Atomic Statements.** Identify the atomic statements in each compound proposition.
   (a) "7 is prime and 12 is even."
   (b) "If n is divisible by 6, then n is even."
   (c) "$x > 0$ or $x = 0$."

6. **Symbol to Words.** Let p: "12 is even" and q: "12 is prime". Write in words:
   (a) $p \wedge q$
   (b) $p \vee q$
   (c) $\neg q$
   (d) $q \rightarrow p$

7. **Words to Symbols.** Let p: "John is a mathematician" and q: "John lives in Pittsburgh". Express in symbolic form:
   (a) John is a mathematician and lives in Pittsburgh.
   (b) John is a mathematician or lives in Pittsburgh.
   (c) John does not live in Pittsburgh.
   (d) If John is a mathematician, then he lives in Pittsburgh.

8. **Translation.** Let p be the proposition "It is raining," let q be "The wind is blowing," and let r be "The ground is wet." Translate the following English sentences into propositional formulae using these variables and the logical connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$.
   (a) It is not raining, but the wind is blowing.

(b) If it is raining, then the ground is wet.
(c) The ground is wet only if it is raining.
(d) For the ground to be wet, it is sufficient that it is raining.
(e) Either it is raining or the wind is blowing, but not both.
(f) The ground is wet if and only if it is raining.

9. Basic Truth Tables. Construct a truth table for each proposition:

(a) $\neg p \vee q$
(b) $p \wedge \neg q$
(c) $(p \vee q) \wedge \neg p$

10. XOR Truth Tables. Construct a truth table for each proposition:

(a) $p \oplus (p \vee q)$
(b) $p \wedge (q \oplus u)$
(c) $(p \wedge q) \oplus (p \wedge u)$
(d) $(p \leftrightarrow q) \oplus (p \to q)$

11. The Contrapositive. For an implication $p \to q$, we can define three related conditional statements:

- The converse: $q \to p$.
- The inverse: $\neg p \to \neg q$.
- The contrapositive: $\neg q \to \neg p$.

Use truth tables to prove that an implication is logically equivalent to its contrapositive, but is not logically equivalent to its converse or its inverse.

12. Tautologies. A proposition that is true for every possible assignment of truth values to its variables is called a *tautology*. A proposition that is always false is a *contradiction*. By constructing a truth table, determine whether the following formula is a tautology, a contradiction, or neither.

$$((p \to q) \wedge \neg q) \to \neg p$$

13. De Morgan's Laws. In the text, a brief mention was made of the duality between conjunction and disjunction. This relationship is formalised by De Morgan's Laws.

(a) Use a truth table to prove the logical equivalence $\neg(p \wedge q) \equiv (\neg p \vee \neg q)$.
(b) Consider the English sentence: "It is not the case that the test is both easy and long." Using part (a), write an equivalent sentence that does not use the word "both".

14. Conditional Language. Let p: "x is a multiple of 4" and q: "x is even". Write each statement symbolically:

(a) If x is a multiple of 4, then x is even.
(b) x is even only if x is a multiple of 4.
(c) x is a multiple of 4 if and only if x is even.

15. Necessary vs Sufficient. In each statement below, identify which condition is necessary and which is sufficient.

(a) If n is divisible by 6, then n is divisible by 3.
(b) If a number is a square, then it is nonnegative.
(c) A function is continuous only if it is defined at every point of its domain.

16. Necessary and Sufficient Conditions. Translate the following statements into the form $p \to q$. Clearly define what propositions your variables p and q represent.

(a) A necessary condition for a number to be divisible by 6 is that it is divisible by 3.
(b) Being over 18 is a sufficient condition for being eligible to vote.
(c) You can access the network only if you have a password.

17. The Biconditional. The text defines the biconditional $p \leftrightarrow q$ as being true when p and q have the same truth value. Prove, using a truth table, that this is logically equivalent to the conjunction of an implication and its converse. That is, prove:

$$(p \leftrightarrow q) \equiv ((p \to q) \land (q \to p))$$

18. Parentheses and Meaning. Insert parentheses to make the intended meaning explicit:

(a) $\neg p \lor q$
(b) $p \to q \lor r$
(c) $p \land q \to r$
(d) $p \leftrightarrow q \lor r$

19. Fully Parenthesised Form. Using precedence rules, rewrite each formula with full parentheses:

(a) $\neg p \land q$
(b) $p \lor q \to r$
(c) $\neg p \lor q \leftrightarrow r$

20. Functional Completeness. A set of logical connectives is called *functionally complete* if all other connectives can be expressed using only connectives from that set. Consider the "NAND" connective, denoted by the Sheffer stroke $|$, with the following truth table:

p	q	$p \vert q$
$\top$	$\top$	$\bot$
$\top$	$\bot$	$\top$
$\bot$	$\top$	$\top$
$\bot$	$\bot$	$\top$

Notice that $p|q$ is equivalent to $\neg(p \land q)$. Show that the set $\{|\}$ is functionally complete by finding formulae, using only the NAND connective, that are logically equivalent to:

(a) $\neg p$
(b) $p \land q$
(c) $p \lor q$