Recitation 2

1. **YouTube Base-64.** The video-sharing website YouTube assigns each video a unique identifier: a string of 11 characters drawn from the 64-character set

$$\{A, B, \ldots, Z,\ a, b, \ldots, z,\ 0, 1, \ldots, 9,\ \texttt{-},\ \texttt{\_}\}.$$

   This string encodes a natural number in base 64, where the characters represent values 0 through 63 in the listed order ($A = 0, B = 1, \ldots, Z = 25, a = 26, \ldots, z = 51, 0 = 52, \ldots, 9 = 61, \texttt{-} = 62, \texttt{\_} = 63$).

   (a) How many distinct videos can YouTube host under this scheme?
   (b) Find the natural number whose base-64 expansion is `dQw4w9WgXcQ`.
   (c) Find the base-64 expansion of 7,159,047,702,620,056,984.

2. **Division Theorem Computations.** For each pair below, find the unique quotient q and remainder r satisfying $a = qb + r$ with $0 \leqslant r < |b|$.

   (a) $a = 2024,\ b = 13$.
   (b) $a = -2024,\ b = 13$.
   (c) $a = 2024,\ b = -13$.
   (d) Verify all three answers using Python's `divmod`. Explain why (a) and (c) have the same remainder but opposite-sign quotients.

3. **Absolute Value Equations and Inequalities.** Find all real numbers x satisfying each of the following.

   (a) $|2x - 5| = 3$.
   (b) $|x + 4| < 2$.
   (c) $|3x - 1| \geqslant 7$.
   (d) $|x - 1| + |x + 1| = 4$.

4. **Divisibility.** Prove each of the following.

   (a) For every integer n, the product $n(n + 1)$ is even.
   (b) For every integer n, the expression $n^2 + n + 4$ is even.
   (c) If $a \mid b$ and $c \mid d$, then $ac \mid bd$.

5. **Absolute Value Simplification.** Express each of the following without absolute value signs, treating cases separately where necessary.

   (a) $|a + b|$ where $a > 0$ and $b > 0$.
   (b) $|x^2 - 9|$, considering $|x| > 3$, $|x| = 3$, and $|x| < 3$.
   (c) Prove that $|ab| = |a| \cdot |b|$ for all real numbers a and b.
   (d) Using (c), simplify $|x^2 - 2xy + y^2|$.

6. **Fewer Absolute Value Signs.** Express each of the following with at least one fewer pair of absolute value signs.

   (i) $|\sqrt{2} + \sqrt{3} - \sqrt{5} + \sqrt{7}|$.
   (ii) $\big|\, |a + b| - |a| - |b| \,\big|$.
   (iii) $\big|\, |a + b| + |c| - |a + b + c| \,\big|$.
   (iv) $|x^2 - 2xy + y^2|$.
   (v) $\big|\, |\sqrt{2} + \sqrt{3}| - |\sqrt{5} - \sqrt{7}| \,\big|$.

7. **Absolute Value by Cases.** Express each of the following without absolute value signs, treating various cases separately when necessary.

   (i) $|a + b| - |b|$.
   (ii) $\big|\, |x| - 1 \,\big|$.
   (iii) $|x| - |x^2|$.
   (iv) $a - \big|\, a - |a| \,\big|$.

8. **Absolute Value Equations.** Find all numbers x for which the following hold.

(i) $|x-3|=8$.
(ii) $|x-3|<8$.
(iii) $|x+4|<2$.
(iv) $|x-1|+|x-2|>1$.
(v) $|x-1|+|x+1|<2$.
(vi) $|x-1|+|x+1|<1$.
(vii) $|x-1|\cdot|x+1|=0$.
(viii) $|x-1|\cdot|x+2|=3$.

9. Powers and Signs.

(b) Prove that if $x<y$ and n is odd, then $x^n<y^n$.
(c) Prove that if $x^n=y^n$ and n is odd, then $x=y$.
(d) Prove that if $x^n=y^n$ and n is even, then $x=y$ or $x=-y$.

10. Time Decomposition. Write a programme that reads a non-negative integer representing a total number of seconds from the user and prints the equivalent duration in hours, minutes, and seconds.

```
Enter total seconds: 7384
2 hours, 3 minutes, 4 seconds
```

11. Extracting a Digit. Write a programme that reads a non-negative integer `n` and a non-negative integer `k` from the user, and prints the k-th digit of `n`, counting from the right and starting at 0. Use the Division Theorem, not string conversion. Then, on a second line, verify the result using string indexing on `str(n)`.

```
Enter n: 7291
Enter k: 2
Digit: 2
Verification: 2
```

12. Caesar Shift. Write a programme that reads a single uppercase letter and a positive integer `k` from the user, and prints the letter obtained by shifting forward by `k` positions in the alphabet, wrapping from `Z` back to `A`.

```
Enter letter: X
Enter shift: 5
Shifted letter: C
```

13. Leap Year. Write a programme that reads a positive integer year from the user and prints whether it is a leap year. A year is a leap year if it is divisible by 4, except that years divisible by 100 are not, unless they are also divisible by 400.

```
Enter year: 1900
Not a leap year

Enter year: 2000
Leap year
```

14. Making Change. Write a programme that reads a non-negative integer `total` representing an amount in pence. Using the denominations £2 (200p), £1 (100p), 50p, 20p, 10p, 5p, 2p, and 1p, compute the minimum number of coins required and print the result.

```
Enter amount in pence: 347
Minimum coins: 6
```

(Breakdown: $1\times200+1\times100+0\times50+2\times20+0\times10+1\times5+1\times2+0\times1$)