Homework

1. The Shape of e^x. Use the first- and second-derivative tests to show that the graph of

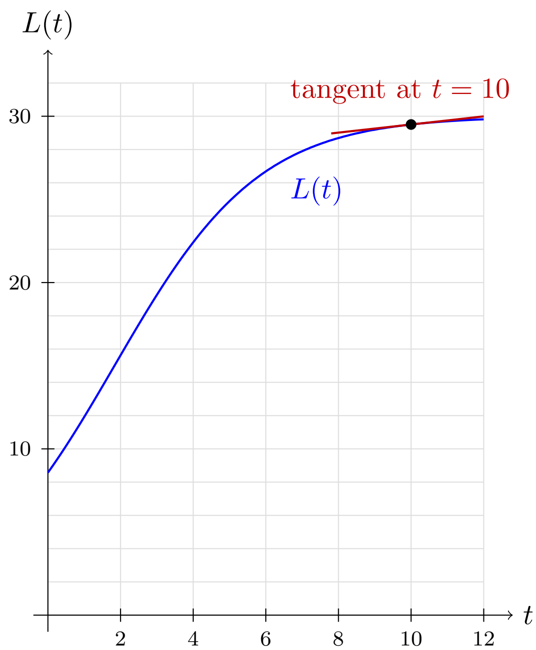
$$y = e^x$$

has no relative extreme points and is concave up for every real x.

2. Height of a Plant. The length of a certain weed, in centimetres, after t weeks is modelled by

$$L(t) = \frac{60}{2 + 5e^{-0.5t}}.$$

The graph below shows $L(t)$ for $0 \le t \le 12$.



(a) How fast is the weed growing after 10 weeks? Give an exact expression and a decimal approximation.

(b) When is the weed 10 centimetres long?

3. Simplifying Exponential and Logarithmic Expressions. Simplify each expression to a number.

(a) $e^{-2\ln 7}$.

(b) $e^{\ln e + \ln 2}$.

(c) $\ln\left(\frac{1}{6}\right)$.

(d) $\ln\left(\frac{2}{9}\right)$.

(e) $\ln\left(\frac{1}{\sqrt{2}}\right)$.

4. Solving Equations. Solve each equation on its natural domain.

(a) $\ln(3x) = 0$.

(b) $(e^x)^2 e^{2-3x} = 4$.

(c) $\ln x - \ln(x^2) + \ln 3 = 0$.

(d) $\ln(x+1) - \ln(x-2) = 1$.

(e) $\ln \sqrt{x} = \sqrt{\ln x}$.

5. Logarithmic Differentiation. Differentiate each function.

(a) $y = \ln(e^x + e^{-x})$.

(b) $y = (\ln x)^2 + \ln(x^2)$, on $x > 0$.

(c) $y = \dfrac{\ln x}{\sqrt{x}}$, on $x > 0$.

6. Maximum of a Logarithmic Curve. The graph of

$$f(x) = \frac{\ln x}{\sqrt{x}}, \qquad x > 0,$$

has one maximum point. Find its coordinates exactly, and confirm by a derivative test that it is a maximum.

7. Domains of Iterated Logarithms. Determine the domain of each function.

(a) $f(t) = \ln(\ln t)$.

(b) $g(t) = \ln(\ln(\ln t))$.

8. Log Laws.

(a) Condense $5 \ln x - \frac{1}{2} \ln y + 3 \ln z$ into one logarithm, assuming $x, y, z > 0$.

(b) Which is larger, $2 \ln 5$ or $3 \ln 3$? Justify without using decimals.

(c) Which expression is the same as $\ln\left(\dfrac{8x^2}{2x}\right)$, for $x > 0$?

(i) $\ln(4x)$ (ii) $4x$ (iii) $\ln(8x^2) - \ln(2x)$ (iv) none of these.

9. Population Growth. Let $P(t)$ be the population, in millions, of a city t years after 1990, and suppose

$$P'(t) = 0.02P(t), \qquad P(0) = 3.$$

(a) Find $P(t)$.

(b) What was the initial population?

(c) What is the growth constant?

(d) What was the population in 1998?

(e) Use the differential equation to determine how fast the population is growing when it reaches 4 million people.

(f) How large is the population when it is growing at the rate of 70,000 people per year?

10. Radioactive Decay. A sample of 8 grams of radioactive material is placed in a vault. Let $P(t)$ be the amount remaining after t years, and suppose

$$P'(t) = -0.021P(t).$$

(a) Find $P(t)$.

(b) What is $P(0)$?

(c) What is the decay constant?

(d) How much material remains after 10 years?

(e) Use the differential equation to determine how fast the sample is disintegrating when just 1 gram remains.

(f) What amount remains when it is disintegrating at the rate of 0.105 grams per year?

(g) If the material has half-life 33 years, how much remains after 33, 66, and 99 years?

11. Carbon Dating. In 1947, a cave with prehistoric wall paintings was discovered in Lascaux, France. Some charcoal found in the cave contained 20% of the ^{14}C expected in living trees. Assuming the decay constant for ^{14}C is 0.00012, estimate the age of the charcoal.

12. Variable Exponents. Suppose $f(x) > 0$ for every real x, and suppose f and g are differentiable. Use logarithmic differentiation to find a formula for

$$\frac{d}{dx}\left(f(x)^{g(x)}\right)$$

in terms of f, g, f', and g'.

13. A Variable-Exponent Function. Consider

$$F(x) = x^{1/(x-1)}, \qquad x > 0, \quad x \neq 1.$$

(a) Find $F'(x)$.

(b) Decide whether F can be extended continuously to $x = 1$. If so, find the limiting value.

(c) After making that continuous extension, decide whether the extended function is differentiable at $x = 1$.