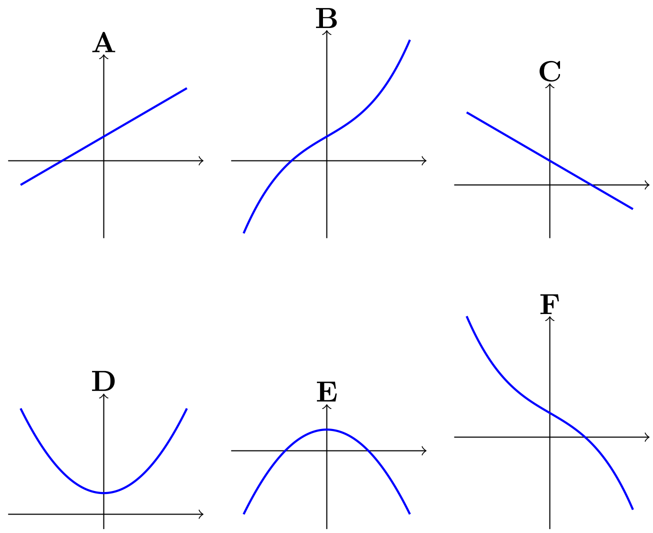
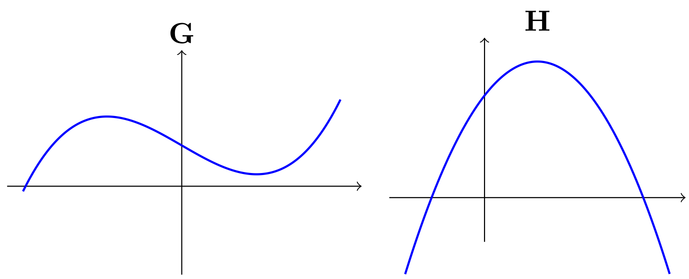
1 Exercises (Lesson 3)

1. Six Basic Shapes. The six graphs below represent functions defined for all real x.

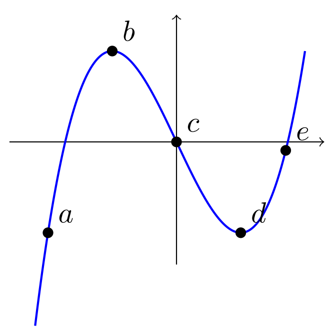


(a) Which functions are increasing for every real x?
(b) Which functions have the property that the slope always increases as x increases?
(c) Which functions have the property that the slope always decreases as x increases?

2. Describe the Graphs. For each graph below, give a short description using these six categories: where the function is positive or negative, where it is increasing or decreasing, relative extrema, absolute extrema on the displayed interval, concavity, and inflection points.



3. Reading Slope and Concavity at Labeled Points. The labelled points lie on the graph of $y = f(x)$.



(a) At which labelled points is the function increasing?
(b) At which labelled points is the graph concave up?
(c) Which labelled point has the most positive slope?

4. **Sketching from Verbal Information.** Draw a possible graph in each case.
   (a) The function decreases and the slope increases as x increases. In other words, the slope is negative but becomes less negative.
   (b) Both the function and the slope decrease as x increases. In other words, the slope is negative and becomes more negative.
   (c) Annual world consumption of oil rises each year, and the amount of the annual increase also rises each year. Sketch a graph that could represent annual world oil consumption as a function of time.

5. **A Patient's Temperature.** At noon, a child's temperature is 101°F and is rising at an increasing rate. At 1 PM the child is given medicine. After 2 PM the temperature is still increasing, but at a decreasing rate. The temperature reaches a peak of 103°F at 3 PM and decreases to 100°F by 5 PM. Draw a possible graph of $T(t)$, the child's temperature t hours after noon.

6. **First and Second Derivative Language.** Refer to the six graphs in Exercise 1.
   (a) Which functions have positive first derivative for every real x?
   (b) Which functions have negative first derivative for every real x?
   (c) Which functions have positive second derivative for every real x?
   (d) Which functions have negative second derivative for every real x?
   (e) Which graph could represent a function f for which $f(a) > 0$, $f'(a) = 0$, and $f''(a) > 0$ at some input a?

7. **Constructing a Curve from Local Data.** Sketch a function satisfying all of the following conditions:

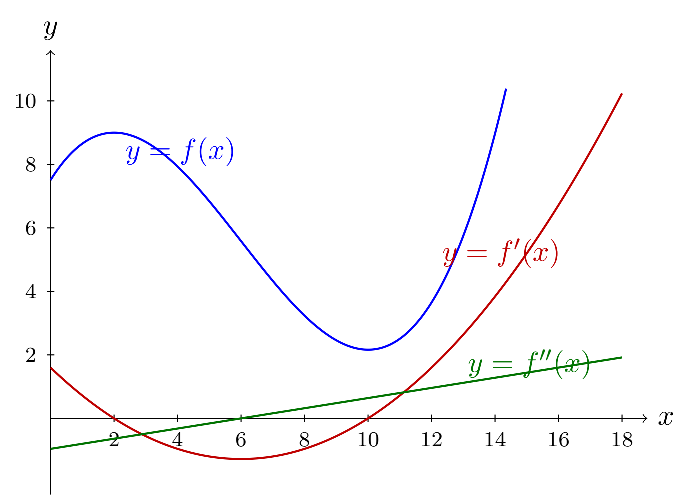
$$(0, 6), \quad (2, 3), \quad (4, 0)$$

are on the graph, $f'(0) = 0$, $f'(4) = 0$, the graph is decreasing on $0 < x < 4$, and the concavity changes at $x = 2$. Your sketch should make clear on which side of $x = 2$ the graph is concave down and on which side it is concave up.

8. **Reading f, f', and f'' Together.** In the figure below, the blue curve is $y = f(x)$, the red curve is $y = f'(x)$, and the green line is $y = f''(x)$. They are drawn from the formulas

$$f(x) = \frac{2}{75}x^3 - \frac{12}{25}x^2 + \frac{8}{5}x + \frac{563}{75},$$

$$f'(x) = \frac{2}{25}(x-2)(x-10), \qquad f''(x) = \frac{4}{25}(x-6).$$



(a) Looking at the graph of $f'(x)$, determine whether $f(x)$ is increasing or decreasing at $x = 9$. Confirm your answer from the graph of $f(x)$.
(b) Looking at the values of $f'(x)$ for $1 < x < 2$ and $2 < x < 3$, explain why $f(x)$ has a relative maximum at $x = 2$. What are the coordinates of this relative maximum point?
(c) Looking at the values of $f'(x)$ near $x = 10$, explain why $f(x)$ has a relative minimum at $x = 10$. What are the coordinates of this relative minimum point?
(d) Looking at $f''(x)$, determine whether $f(x)$ is concave up or concave down at $x = 2$. Confirm your answer from the graph of $f(x)$.
(e) Looking at $f''(x)$, determine where $f(x)$ has an inflection point. What are the coordinates of the inflection point?
(f) Find the x-coordinate of the point on $f(x)$ where the function is increasing at the rate of 6 units per unit change in x.

9. First-Derivative Test for Cubics. Each function below has one relative maximum point and one relative minimum point. Find both points using the first-derivative test.

(a) $f(x) = -x^3 + 6x^2 - 9x + 1$.
(b) $f(x) = -x^3 - 12x^2 + 1$.

10. Extrema, Concavity, and Sketching.

(a) The graph of $f(x) = 2x^2 - 8$ has one relative extreme point. Plot this point, check the concavity there, and sketch the graph using only this information and the symmetry of the formula.
(b) The graph of $f(x) = \frac{1}{2}x^2 + x - 4$ has one relative extreme point. Plot this point, check the concavity there, and sketch the graph.
(c) For $f(x) = x^3 + 6x^2 + 9x$, find all relative extrema and inflection points, then sketch the curve.
(d) For $f(x) = x^3 - 3x^2 - 9x + 1$, find all relative extrema and inflection points, then sketch the curve.
(e) Let a, b, c be fixed real numbers with $a \neq 0$, and let $f(x) = ax^2 + bx + c$. Is it possible for the graph of f to have an inflection point? Explain using $f''(x)$.