1 Exercises (Lesson 2)

1. **Lines from Minimal Data.** For each item below, find an equation of the line in slope-intercept form $y = mx + b$.
   (a) The line has slope $m = \frac{7}{3}$ and passes through $\left(\frac{1}{4}, -\frac{2}{5}\right)$.
   (b) The line passes through $\left(\frac{1}{7}, 5\right)$ and $\left(-\frac{2}{5}, -4\right)$.
   (c) The line passes through $\left(\frac{1}{2}, 1\right)$ and $(1, 4)$.
   (d) The line passes through $(0, 0)$ and $(1, 0)$.
   (e) The line passes through $\left(-\frac{1}{2}, -\frac{1}{7}\right)$ and $\left(\frac{2}{3}, 1\right)$.

   In each case, verify your formula by substituting the given data back into the equation.

2. **Average Rate of Change and a Preview of Limits.** Let $f(x) = x^2$.
   (a) For $h \neq 0$, compute the slope of the line through the two points $\big(2, f(2)\big)$ and $\big(2+h, f(2+h)\big)$, and simplify your answer as far as possible.
   (b) Evaluate your slope formula at $h = 1,\ \frac{1}{2},\ \frac{1}{10},\ -\frac{1}{10},\ -\frac{1}{2}$.
   (c) As h gets closer and closer to 0, what number do your slopes seem to get closer to?

3. **A Corner and Two Different Slopes.** Let $f(x) = |x|$.
   (a) For $h > 0$, compute the slope of the line through $\big(0, f(0)\big)$ and $\big(h, f(h)\big)$.
   (b) For $h < 0$, compute the slope of the line through $\big(0, f(0)\big)$ and $\big(h, f(h)\big)$.
   (c) Explain, in one or two sentences, why these computations suggest that there is no single "best" slope for the graph at $x = 0$.

4. **Collinearity by Slopes.** Consider the three points $(-1, 2)$, $(2, 8)$, and $(5, 14)$.
   (a) Show that these three points lie on a single non-vertical line by comparing slopes.
   (b) Find an equation of the line in slope-intercept form $y = mx + b$.
   (c) Use your equation to find the y-intercept.

5. **An Unusual Slope Rule (Forces a Quadratic).** Let f be a function defined for all real x. Suppose that for every pair of distinct real numbers x_1 and x_2,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = x_1 + x_2 + 1.$$

   (a) Show that for every real $x \neq 0$,

$$\frac{f(x) - f(0)}{x} = x + 1.$$

   (b) Deduce that $f(x) = x^2 + x + b$ for some constant b.
   (c) Verify directly that every function of the form $f(x) = x^2 + x + b$ satisfies the given slope rule.

6. **Constant Difference Quotient Implies Linearity.** Let f be a function defined on an interval I. Suppose there is a constant m such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = m$$

   for all distinct real numbers x_1, x_2 in I.

(a) Prove that there is a constant b with $f(x) = mx + b$ for all real numbers x in I.

(b) Fix any function f with $f(3) = 2$. Let $h > 0$ and draw the straight line through $(3, f(3))$ and $(3 + h, f(3 + h))$. What is the slope of this line in terms of h?

7. Tangent Lines to $y = x^2$. Fix a real number a and consider the parabola $y = x^2$.

(a) For $h \neq 0$, compute the slope $m_h(a)$ of the line through the two points (a, a^2) and $(a + h, (a + h)^2)$, and simplify your answer.

(b) Show that $m_h(a) = 2a + h$. Explain why this makes $2a$ a natural candidate for the slope of the curve $y = x^2$ at (a, a^2).

(c) Let $L_a(x) = a^2 + 2a(x - a)$ be the line of slope $2a$ passing through (a, a^2). Show that for every real x,

$$x^2 - L_a(x) = (x - a)^2.$$

Deduce that $L_a(x) \leq x^2$ for all real x, with equality only at $x = a$. Interpret this as a precise algebraic meaning of "tangent line" for the parabola.

(d) Take $a = -\frac{1}{2}$. Write the equation of the tangent line L_a in slope-intercept form, and find its x-intercept.