Precalculus: Complex Numbers

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Chapter 1

Ideas & Motivations

Welcome to The Complex Numbers by me (Gudfit). The point of these notes is to cover everything I think is important as I build up to my current knowledge, while keeping it free and accessible for everyone from kids to adults.

I aim for each set of notes to be max 50 pages, as rigorous as possible, and far-reaching too. That means I'll cover the axioms and proofs of the most interesting stuff, plus I'll pull in other subjects we've already touched on to show how math builds on itself like funky Lego. These notes build on my existing informal logic and Set Theory notes (specifically the construction of the Real numbers), and they're aimed at demystifying "imaginary" numbers to show they are just as real as any other number.

It'll be a mix of quick ideas and concepts, but in the sections on Field definitions and Theorems, I'll go rigorous. Unlike the Geometry notes, we will be using Algebra heavily here (since Complex numbers are the marriage of Algebra and Geometry), but I will strictly avoid Analysis to keep the proofs self-contained based on what we've learned so far.

The original idea was a dry, purely algebraic construction of the field $\mathbb{C}$, but that felt too grindy and ignores the beautiful geometry. Why slog through abstract lists of rules when you can see how these numbers actually work in the plane? So this'll be more efficient, assuming you've got some mathematical rigor from the previous notes. Either way, let's dive in and enjoy!

(Also, credit where it's due: these notes have some basis in the Oxford Complex Numbers induction notes, just remixed to fit the Gudfit universe.)

Chapter 2

The Complex Numbers

In our construction of the number systems in the previous notes on set theory, we observed a recurring theme: the necessity to expand our mathematical universe to solve equations that are impossible within the current bounds. The natural numbers $\mathbb{N}$ allow for addition, but the equation $x + 7 = 3$ requires the integers $\mathbb{Z}$. The integers allow for multiplication, but $5x = 2$ necessitates the rationals $\mathbb{Q}$. The rationals, while dense, contain gaps where numbers like $\sqrt{2}$ reside, leading us to the completeness of the real numbers $\mathbb{R}$.

However, the polynomial equation $x^2 + 1 = 0$ exposes a fundamental deficiency in $\mathbb{R}$. Since the square of any real number is non-negative ($x^2 \geq 0$), this equation has no solution on the real line. To address this, we must extend our system once more. Unlike previous extensions which filled gaps on a line, this extension requires moving into a plane.

2.1 Construction of the Complex Field

Historically, numbers satisfying $x^2 = -1$ were termed "imaginary" and treated with suspicion. To place them on a rigorous foundation, comparable to our construction of $\mathbb{R}$ via Dedekind cuts, we define complex numbers not as mystical quantities, but as ordered pairs of real numbers with specific arithmetic rules.

Remark. (Historical Note). The notation $i = \sqrt{-1}$ was introduced by Leonhard Euler in 1777. Prior to this, and indeed for some time after, these numbers were treated with skepticism. It was not until the work of Caspar Wessel (1797) and Jean-Robert Argand (1806), who provided the geometric interpretation of complex numbers as points in a plane, that they were accepted as legitimate mathematical objects. Note that in engineering contexts, the symbol j is often used instead of i to avoid confusion with electric current.

Definition 2.1.1. ***Complex Numbers***. The set of complex numbers, denoted by $\mathbb{C}$, is defined as the Cartesian product $\mathbb{R} \times \mathbb{R}$. An element $z \in \mathbb{C}$ is an ordered pair of real numbers (a, b). Equality is defined component-wise: two complex numbers (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$.

We endow this set with two operations: addition and multiplication. While addition follows the intuitive vector addition in $\mathbb{R}^2$, multiplication is defined specifically to impart the desired algebraic structure.

Definition 2.1.2. ***Arithmetic on*** $\mathbb{C}$. Let $z = (a, b)$ and $w = (c, d)$ be elements of $\mathbb{C}$. We define:

1. **Addition:** $z + w := (a + c, b + d)$
2. **Multiplication:** $z \cdot w := (ac - bd, ad + bc)$

Note. The definition of multiplication may appear arbitrary at first glance. However, it is carefully constructed to ensure that specific elements of this set behave like roots of $x^2 + 1 = 0$, while preserving the distributive laws of algebra.

Field Properties

We must verify that $\mathbb{C}$ constitutes a field. A field is a set equipped with two operations that satisfy the commutative, associative, identity, inverse, and distributive laws (as detailed in Chapter 6 of the set theory notes).

***Theorem* 2.1.1.** The set $\mathbb{C}$, equipped with the addition and multiplication defined above, forms a field.

Proof. We verify the existence of identities and inverses. The verifications of associativity and distributivity are direct computational exercises relying on the field properties of $\mathbb{R}$.

- **Additive Identity:** Consider the element $\mathbf{0} = (0, 0)$. For any $(a, b) \in \mathbb{C}$,

$$(a, b) + (0, 0) = (a + 0, b + 0) = (a, b).$$

- **Additive Inverse:** For any $(a, b) \in \mathbb{C}$, the element $(-a, -b)$ satisfies $(a, b) + (-a, -b) = (0, 0)$.
- **Multiplicative Identity:** Consider the element $\mathbf{1} = (1, 0)$. For any $(a, b) \in \mathbb{C}$,

$$(a, b) \cdot (1, 0) = (a \cdot 1 - b \cdot 0, a \cdot 0 + b \cdot 1) = (a, b).$$

- **Multiplicative Inverse:** Let $z = (a, b)$ be non-zero (meaning at least one of a or b is not zero). We seek (x, y) such that $(a, b) \cdot (x, y) = (1, 0)$. This yields the system:

$$ax - by = 1 \quad \text{and} \quad bx + ay = 0$$

  Solving this system gives the unique solution:

$$x = \frac{a}{a^2 + b^2}, \quad y = \frac{-b}{a^2 + b^2}$$

  Since $a^2 + b^2 > 0$ for non-zero z, the inverse z^{-1} exists in $\mathbb{C}$. ■

Note. Unlike $\mathbb{R}$, the field $\mathbb{C}$ cannot be an *ordered field*. In an ordered field, squares are non-negative. In $\mathbb{C}$, we will see that there is an element whose square is -1, which would lead to a contradiction if a compatible order existed.

2.2 The Imaginary Unit

We can embed the real numbers into the complex numbers via the injection $x \mapsto (x, 0)$. This mapping preserves addition and multiplication, as $(a, 0) + (b, 0) = (a + b, 0)$ and $(a, 0) \cdot (b, 0) = (ab, 0)$. Thus, we identify the real number x with the complex number $(x, 0)$.

We now consider the element $(0, 1)$. Let us compute its square:

$$(0, 1) \cdot (0, 1) = (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 1 \cdot 0) = (-1, 0).$$

Under our identification of $(x, 0)$ with x, this result is simply -1. We denote this element by the symbol i.

Definition 2.2.1. ***Imaginary Unit*.** The element $i \in \mathbb{C}$ is defined as the pair $(0, 1)$. It satisfies the property $i^2 = -1$.

Using this notation, any complex number $z = (a, b)$ can be decomposed using the arithmetic rules:

$$(a, b) = (a, 0) + (0, 1) \cdot (b, 0).$$

Identifying $(a, 0)$ with a and $(b, 0)$ with b, we arrive at the standard algebraic form.

Example 2.2.1. A Cautionary Tale. Care must be taken when applying rules of surds valid for positive real numbers to complex numbers. Consider the following fallacious argument:

$$-1 = i \cdot i = \sqrt{-1} \cdot \sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1.$$

This contradiction arises because the rule $\sqrt{a}\sqrt{b} = \sqrt{ab}$ holds generally only when at least one of a or b is non-negative.

***Theorem* 2.2.1.** Standard Form Every complex number $z = (a, b)$ can be uniquely written as

$$z = a + bi$$

where $a, b \in \mathbb{R}$. The number a is called the *real part* of z, denoted $\Re(z)$, and b is called the *imaginary part* of z, denoted $\Im(z)$.

Calculations in this form follow the standard rules of algebra for binomials, replacing i^2 with -1 wherever it appears.

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i.$$

2.3 Conjugation and Modulus

The geometric interpretation of complex numbers as points in a plane (the Argand plane) suggests two fundamental operations: reflection across the real axis and the distance from the origin.

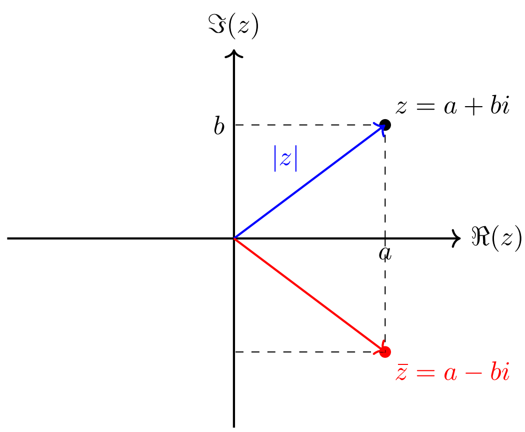


Figure 2.1: The complex plane. The modulus $|z|$ represents the distance from the origin, while the conjugate $\bar{z}$ is the reflection of z across the real axis.

Definition 2.3.1. ***Complex Conjugate***. Let $z = a+bi$. The complex conjugate of z, denoted $\bar{z}$, is defined as

$$\bar{z} := a - bi.$$

Geometrically, $\bar{z}$ is the reflection of z across the real axis. Conjugation respects the field operations:

$$\overline{z + w} = \bar{z} + \bar{w} \quad \text{and} \quad \overline{z \cdot w} = \bar{z} \cdot \bar{w}.$$

Definition 2.3.2. ***Modulus***. The modulus (or absolute value) of a complex number $z = a + bi$ is defined as the non-negative real number

$$|z| := \sqrt{a^2 + b^2}.$$

The modulus measures the magnitude of the vector (a, b). A crucial relationship connects the modulus and the conjugate.

***Theorem* 2.3.1.** For any $z \in \mathbb{C}$,

$$z\bar{z} = |z|^2.$$

Proof. Let $z = a + bi$. Then,

$$z\bar{z} = (a+bi)(a-bi) = a^2 - abi + abi - b^2i^2 = a^2 - b^2(-1) = a^2 + b^2 = (\sqrt{a^2+b^2})^2 = |z|^2. \quad \blacksquare$$

This identity provides a practical method for computing the multiplicative inverse and performing division. To compute z^{-1}, we divide the conjugate by the squared modulus:

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}.$$

Remark. (Rationalising the Denominator). The process of dividing complex numbers is algebraically identical to "rationalising the denominator" for surds. Just as we multiply $\frac{1}{1+\sqrt{2}}$ by $\frac{1-\sqrt{2}}{1-\sqrt{2}}$ to remove the root from the denominator, we multiply a complex fraction by the conjugate of the denominator to remove the imaginary part.

2.4 Zero Divisors

In future study of structures like matrix algebra, we often encounter non-zero elements whose product is zero (zero divisors). It is important to establish that the field $\mathbb{C}$ maintains the integrity of the real numbers in this regard.

***Theorem* 2.4.1.** The complex numbers contain no zero divisors. That is, for $z, w \in \mathbb{C}$, if $zw = 0$, then $z = 0$ or $w = 0$.

Proof. Suppose $zw = 0$. By properties of the modulus (which maps to non-negative reals where standard rules apply):

$$|zw| = |0| = 0.$$

It can be shown that the modulus is multiplicative, $|zw| = |z||w|$. Thus $|z||w| = 0$. Since $|z|$ and $|w|$ are real numbers, this implies $|z| = 0$ or $|w| = 0$. By the definition of modulus, $|z| = 0$ if and only if $z = 0$. Thus, $z = 0$ or $w = 0$. $\blacksquare$

2.5 Exercises

Part I: Arithmetic and Elementary Properties

1. Put each of the following numbers into the form $a + bi$:

(a) $(3 + 2i) + (2 - 7i)$
(b) $(1 + 2i)(3 - i)$
(c) $\frac{1+2i}{3-i}$
(d) $(1 + i)^4$

2. Zero Divisors.

(a) Show directly (using algebraic expansion $z = x + iy$) that if $zw = 0$, where z, w are two complex numbers, then $z = 0$ or $w = 0$ (or both).

(b) Now re-prove this making use of properties of the modulus function.

3. Let $z = 3 - 2i$ and $w = 1 + 4i$. Compute the following and express the result in standard form $a + bi$:

(a) $z + w$

(b) $z \cdot w$

(c) $\frac{z}{w}$

(d) $|z|$

(e) $z^2 - 3z + (3 + i)$

4. We defined the complex numbers $\mathbb{C}$ as the set $\mathbb{R} \times \mathbb{R}$. Explain why the subset of pairs of the form $(a, 0)$ is isomorphic to the field of real numbers $\mathbb{R}$. That is, show that the bijection $\phi(a) = (a, 0)$ satisfies $\phi(a + b) = \phi(a) + \phi(b)$ and $\phi(ab) = \phi(a)\phi(b)$.

5. Prove the following identities regarding the real and imaginary parts.

(a) $z + \bar{z} = 2\Re(z)$.

(b) $z - \bar{z} = 2i\Im(z)$.

(c) Hence, show that z is a real number (i.e., $\Im(z) = 0$) if and only if $z = \bar{z}$.

6. Verify the multiplicative property of the modulus: $|zw| = |z||w|$ for all $z, w \in \mathbb{C}$.

Remark. Use the identity $|u|^2 = u\bar{u}$ and the properties of conjugation.

7. Solve the equation $z^2 = i$.

Remark. Let $z = x + yi$. Expand $(x + yi)^2$ and equate real and imaginary parts to the components of $i = 0 + 1i$. You will obtain a system of equations involving $x^2 - y^2$ and $2xy$.

8. Find all complex numbers z such that $\bar{z} = z^2$.

Part II: Geometry and Inequalities

7. The Triangle Inequality. Prove that for any $z, w \in \mathbb{C}$,

$$|z + w| \leq |z| + |w|.$$

Remark. Start by computing $|z + w|^2 = (z + w)(\overline{z + w})$. Expand this and use the fact that $\Re(u) \leq |u|$ for any complex number u.

8. The Parallelogram Law. Prove that

$$|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2).$$

Interpret this result geometrically in terms of the sides and diagonals of a parallelogram in the Argand plane.

9. Describe and sketch the locus of points z in the complex plane satisfying the following conditions:

(a) $|z - 2| = |z + 2|$

(b) $|z - i| = 2$

(c) $\Re(z^2) = 0$

(d) $|z - 1| + |z + 1| = 4$

10. General Square Roots. Let $w = a + bi$. We wish to find $z = x + yi$ such that $z^2 = w$.

(a) By equating real and imaginary parts, show that $x^2 = \frac{a+\sqrt{a^2+b^2}}{2}$ and $y^2 = \frac{-a+\sqrt{a^2+b^2}}{2}$.

(b) Explain how the signs of x and y must be chosen based on the sign of b.

Part III: Structure and Theory

11. $\star$ **Ordering the Complex Numbers.** In the text, it was noted that $\mathbb{C}$ cannot be an *ordered field*. An ordered field is a field equipped with a relation $>$ such that:

- (Trichotomy) For any x, exactly one of $x > 0$, $x = 0$, or $x < 0$ holds.
- If $x, y > 0$, then $x + y > 0$ and $xy > 0$.

While $\mathbb{C}$ cannot be an ordered field, one can define an ordering that satisfies *some* of the axioms. Consider the *dictionary order* defined by:

$$(a, b) > (c, d) \iff a > c \quad \text{or} \quad (a = c \text{ and } b > d).$$

(a) Prove that this relation satisfies the *Trichotomy Law* (for any z, exactly one of $z > 0, z = 0, z < 0$ holds).
(b) Prove that this relation is compatible with addition: if $z > 0$ and $w > 0$, then $z + w > 0$.
(c) Show, by counterexample, that this relation fails the compatibility with multiplication (find $z, w > 0$ such that $zw < 0$).

12. Gaussian Integers. The set of Gaussian integers is defined as $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$.

(a) Show that $\mathbb{Z}[i]$ is closed under addition and multiplication.

(b) An element $u \in \mathbb{Z}[i]$ is called a *unit* if there exists $v \in \mathbb{Z}[i]$ such that $uv = 1$. Prove that u is a unit if and only if $|u|^2 = 1$.

(c) Find all units in $\mathbb{Z}[i]$.

13. $\star$ **Roots of Unity.** A complex number ω is called an *n-th root of unity* if $\omega^n = 1$.

(a) Factorise $z^3 - 1$ into a linear and a quadratic factor using real coefficients.

(b) Solve $z^3 = 1$ to find the three cube roots of unity.

(c) Let $\omega = \frac{-1+i\sqrt{3}}{2}$. Show that the roots are $1, \omega, \omega^2$, and that $1 + \omega + \omega^2 = 0$.

14. $\star$ **Pythagorean Triples.** A Pythagorean triple is a set of three integers (a, b, c) such that $a^2 + b^2 = c^2$.

(a) Let $z = u + iv$ where $u, v \in \mathbb{Z}$ and $u > v > 0$. Calculate z^2.

(b) Let $z^2 = x + iy$. Show that $(x, y, |z|^2)$ forms a Pythagorean triple.

(c) Generate three distinct Pythagorean triples using this method.

Chapter 3

Roots and Order

In the previous chapter, we constructed the complex numbers $\mathbb{C}$ to provide a solution to the equation $x^2 + 1 = 0$. By defining $i^2 = -1$, we successfully extended the real numbers. A natural question arises: does this extension suffice? That is, does every polynomial equation have a solution within $\mathbb{C}$, or must we continue to invent new numbers to solve equations like $x^2 = i$ or $x^4 + x + 1 = 0$?

This chapter explores the algebraic completeness of the complex field, the properties of its ordering (or lack thereof), and the fundamental inequalities that govern the geometry of the complex plane.

3.1 Quadratic Equations

Consider the quadratic equation $az^2 + bz + c = 0$ with coefficients in $\mathbb{C}$. In the real case, we rely on the method of completing the square, which necessitates taking the square root of the discriminant $\Delta = b^2 - 4ac$. Since arithmetic operations in $\mathbb{C}$ mirror those in $\mathbb{R}$, the quadratic formula remains valid, provided we can interpret the square root of a complex number.

Example 3.1.1. Consider the equation $z^2 + (1+i)z - i = 0$. Completing the square yields:

$$\left(z + \frac{1+i}{2}\right)^2 - \left(\frac{1+i}{2}\right)^2 - i = 0$$

$$\left(z + \frac{1+i}{2}\right)^2 = \frac{1 + 2i - 1}{4} + i = \frac{2i}{4} + i = \frac{3i}{2}$$

To proceed, we require a number w such that $w^2 = \frac{3i}{2}$.

Square Roots in $\mathbb{C}$

We seek a general method to solve $z^2 = w$ for any $w \in \mathbb{C}$. Let $w = \alpha + i\beta$ and let the unknown root be $z = u + iv$, where $u, v, \alpha, \beta \in \mathbb{R}$. Equating the square of z to w:

$$(u + iv)^2 = (u^2 - v^2) + i(2uv) = \alpha + i\beta$$

This gives the system of real equations:

$$u^2 - v^2 = \alpha \quad \text{and} \quad 2uv = \beta \tag{3.1}$$

To solve for u and v, we utilise the modulus. Since $|z^2| = |w|$, we have $|z|^2 = |w|$, which implies $u^2 + v^2 = \sqrt{\alpha^2 + \beta^2}$. Combining this with the first equation in Equation 3.1:

$$(u^2 + v^2) + (u^2 - v^2) = \sqrt{\alpha^2 + \beta^2} + \alpha \implies 2u^2 = \sqrt{\alpha^2 + \beta^2} + \alpha$$

$$(u^2 + v^2) - (u^2 - v^2) = \sqrt{\alpha^2 + \beta^2} - \alpha \implies 2v^2 = \sqrt{\alpha^2 + \beta^2} - \alpha$$

Since $\sqrt{\alpha^2 + \beta^2} \geq |\alpha|$, the right-hand sides are non-negative, allowing us to solve for real u and v.

***Theorem* 3.1.1.** Square Roots of a Complex Number Let $w = \alpha + i\beta$ be a non-zero complex number. The equation $z^2 = w$ has exactly two solutions $z = \pm(u + iv)$, where:

$$u = \sqrt{\frac{\sqrt{\alpha^2 + \beta^2} + \alpha}{2}}, \quad v = \sigma(\beta)\sqrt{\frac{\sqrt{\alpha^2 + \beta^2} - \alpha}{2}}$$

Here, $\sigma(\beta)$ is the sign of β (defined as 1 if $\beta \geq 0$ and -1 if $\beta < 0$). This ensures the condition $2uv = \beta$ is satisfied.

This result confirms that we do not need to extend $\mathbb{C}$ further to solve quadratic equations; the complex numbers are sufficient.

***Theorem* 3.1.2. General Quadratic Solution.** The solutions to the equation $az^2 + bz + c = 0$ for $a, b, c \in \mathbb{C}$ and $a \neq 0$ are given by

$$z = \frac{-b \pm \xi}{2a}$$

where ξ is one of the square roots of the discriminant $\Delta = b^2 - 4ac$.

3.2 The Fundamental Theorem of Algebra

Having conquered quadratics, one might wonder about cubic or quartic equations. It is a profound fact of analysis that the complex numbers form an *algebraically closed* field. This means that every non-constant polynomial with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$.

***Theorem* 3.2.1. Fundamental Theorem of Algebra.** Every polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ with coefficients $a_k \in \mathbb{C}$, $n \geq 1$, and $a_n \neq 0$, has at least one root in $\mathbb{C}$.

The proof of this theorem is topological in nature and typically covered in a course on Complex Analysis. However, its algebraic consequences are immediate. By the Factor Theorem, if z_1 is a root, we can write $P(z) = (z - z_1)Q(z)$, where $Q(z)$ is a polynomial of degree $n - 1$. Applying the Fundamental Theorem recursively to $Q(z)$ yields the complete factorisation.

***Corollary* 3.2.1.** Every polynomial of degree $n \geq 1$ over $\mathbb{C}$ has exactly n roots in $\mathbb{C}$, provided roots are counted with their multiplicity.

***Corollary* 3.2.2.** *Conjugate Pairs.* Let $P(z)$ be a polynomial with *real* coefficients. If z_0 is a root of $P(z)$, then its conjugate $\bar{z}_0$ is also a root.

Proof. Let $P(z) = \sum a_k z^k$ where $a_k \in \mathbb{R}$. If $P(z_0) = 0$, then

$$\overline{P(z_0)} = \overline{\sum a_k z_0^k} = \sum \bar{a}_k \bar{z}_0^k = \sum a_k \bar{z}_0^k = P(\bar{z}_0).$$

Thus $\overline{0} = P(\bar{z}_0)$, implying $P(\bar{z}_0) = 0$. Consequently, non-real roots of real polynomials always occur in conjugate pairs. ■

Example 3.2.1. Solve $z^3 + i = 0$. We look for z such that $z^3 = -i$. Writing $-i$ in polar form is often the most efficient method (discussed in later sections), but we can also use algebraic manipulation. We know $i^3 = -i$, so $z^3 - i^3 = 0$. Factoring the difference of cubes:

$$(z - i)(z^2 + iz + i^2) = (z - i)(z^2 + iz - 1) = 0$$

One root is $z_1 = i$. The other two come from the quadratic $z^2 + iz - 1 = 0$. Using the quadratic formula:

$$z = \frac{-i \pm \sqrt{i^2 - 4(1)(-1)}}{2} = \frac{-i \pm \sqrt{-1 + 4}}{2} = \frac{-i \pm \sqrt{3}}{2}$$

Thus, the three roots are i, $\frac{\sqrt{3}}{2} - \frac{1}{2}i$, and $-\frac{\sqrt{3}}{2} - \frac{1}{2}i$.

3.3 Order in the Complex Field

The real numbers $\mathbb{R}$ constitute an *ordered field*. This means there exists a relation $>$ satisfying trichotomy (for any a, exactly one of $a > 0$, $a = 0$, $-a > 0$ holds) and compatibility with arithmetic ($a, b > 0 \implies a + b > 0$ and $ab > 0$). One might ask if it is possible to define such an order on $\mathbb{C}$.

***Theorem* 3.3.1.** No Order on $\mathbb{C}$ The field of complex numbers $\mathbb{C}$ cannot be made into an ordered field.

Proof. Suppose there exists an ordering $>$ on $\mathbb{C}$ satisfying the ordered field axioms. Consider the imaginary unit i. By trichotomy, $i \neq 0$, so we must have either $i > 0$ or $-i > 0$.

- **Case 1:** Assume $i > 0$. The product of positive elements must be positive, so $i \cdot i > 0$. This implies $-1 > 0$. Adding 1 to both sides (valid in an ordered field) gives $0 > 1$. However, squaring any non-zero element must yield a positive result: $1^2 = 1 > 0$. We have reached a contradiction ($0 > 1$ and $1 > 0$).
- **Case 2:** Assume $-i > 0$. Then $(-i) \cdot (-i) > 0$. This implies $i^2 > 0$, or $-1 > 0$, leading to the same contradiction.

Thus, no such ordering exists. ∎

Consequently, inequalities such as $z_1 < z_2$ have no meaning in $\mathbb{C}$ unless z_1 and z_2 are real. When we wish to compare complex numbers, we typically compare their moduli, $|z_1| < |z_2|$, which is a comparison of real numbers.

3.4 The Triangle Inequality

While we lack a linear order, the geometry of the complex plane is governed by a metric structure. The distance between points satisfies the triangle inequality, which states that the length of one side of a triangle is never greater than the sum of the other two.

***Theorem* 3.4.1. The Triangle Inequality.** For any $z_1, z_2 \in \mathbb{C}$:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Equality holds if and only if one of the numbers is a non-negative real multiple of the other (i.e., they lie on the same ray from the origin).

Proof. We analyse the square of the modulus to avoid square roots.

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)\overline{(z_1 + z_2)} \\ &= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) \\ &= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2 \\ &= |z_1|^2 + z_1\bar{z}_2 + \overline{z_1\bar{z}_2} + |z_2|^2 \end{aligned}$$

Recall that for any $w \in \mathbb{C}$, $w + \bar{w} = 2\Re(w)$. Setting $w = z_1\bar{z}_2$:

$$|z_1 + z_2|^2 = |z_1|^2 + 2\Re(z_1\bar{z}_2) + |z_2|^2$$

We know that for any complex number, the real part is less than or equal to the modulus: $\Re(w) \leq |w|$. Thus:

$$\begin{aligned} |z_1 + z_2|^2 &\leq |z_1|^2 + 2|z_1\bar{z}_2| + |z_2|^2 \\ &= |z_1|^2 + 2|z_1||z_2| + |z_2|^2 \\ &= (|z_1| + |z_2|)^2 \end{aligned}$$

Taking the square root of both sides (which is valid as moduli are non-negative) yields $|z_1 + z_2| \leq |z_1| + |z_2|$.

For equality to hold, we must have $\Re(z_1\bar{z}_2) = |z_1\bar{z}_2|$. This occurs only if $z_1\bar{z}_2$ is a non-negative real number. If $z_2 \neq 0$, this is equivalent to $z_1/z_2 \geq 0$. ∎

Note. This inequality extends to sums of n complex numbers by induction: $|\sum_{k=1}^{n} z_k| \leq \sum_{k=1}^{n} |z_k|$.

***Corollary* 3.4.1.** *Reverse Triangle Inequality.* For any $z_1, z_2 \in \mathbb{C}$:

$$\big||z_1| - |z_2|\big| \leq |z_1 - z_2|$$

Proof. We write $z_1 = (z_1 - z_2) + z_2$. Applying the triangle inequality:

$$|z_1| = |(z_1 - z_2) + z_2| \leq |z_1 - z_2| + |z_2|$$

Rearranging gives $|z_1| - |z_2| \leq |z_1 - z_2|$. Swapping z_1 and z_2 yields $|z_2| - |z_1| \leq |z_2 - z_1| = |z_1 - z_2|$. Combining these gives the result. ∎

3.5 Exercises

Part I: Solving Equations

1. Which of the following quadratic equations require the use of complex numbers to solve them?

(a) $3x^2 + 2x - 1 = 0$
(b) $2x^2 - 6x + 9 = 0$
(c) $-4x^2 + 7x - 9 = 0$
(d) Find the two square roots of $-5 - 12i$.
(e) Hence solve the quadratic equation $z^2 - (4 + i)z + (5 + 5i) = 0$.

2. Compute the square roots of the following complex numbers in standard form $a + bi$.

(a) $z = 3 - 4i$
(b) $z = -15 + 8i$
(c) $z = -2i$

3. Solve the quadratic equation $z^2 + (2i - 3)z + (5 - i) = 0$.

4. Solve the simultaneous equations for $z, w \in \mathbb{C}$:

$$z + w = 3 - i \quad \text{and} \quad zw = 4 + 2i.$$

Remark. Recall that z and w are roots of the quadratic $t^2 - (z + w)t + zw = 0$.

5. Let $P(z) = z^4 - 6z^3 + 14z^2 - 24z + 40$. Given that $2i$ is a root of $P(z)$, find the remaining roots and factorise $P(z)$ completely over $\mathbb{C}$.

6. Complex Coefficients.

(a) Prove that if the coefficients a_k of a polynomial $P(z)$ are all real, and if z_0 is a root, then its conjugate $\bar{z}_0$ is also a root.
(b) Construct a quadratic equation with complex coefficients that has i as a root, but does *not* have $-i$ (the conjugate) as a root. This demonstrates that the result in (a) relies strictly on the reality of the coefficients.

7. Find the complex number z that satisfies the equation $z^2 + |z| = 0$.

Part II: Order and Inequalities

7. Interpret the triangle inequality geometrically by drawing the vectors z_1, z_2, and $z_1 + z_2$ in the Argand plane. Under what precise geometric condition does $|z_1 + z_2| = |z_1| + |z_2|$?

8. Prove the generalised triangle inequality by induction.

9. A Minimum Value Problem. Let z be a complex number with $|z| = 1$.

(a) Show that $|z - 1|^2 = 2 - 2\Re(z)$.
(b) Hence, or otherwise, find the maximum and minimum values of $|z^n - 1|$ for $n \in \mathbb{N}$.

10. Let $z, w \in \mathbb{C}$ such that $|z| = |w| = 1$ and $1 + z + w = 0$. Compute $z^3 + w^3$.

Part III: Polynomial Theory

8. Solving Cubics.

(a) Let p, q be complex numbers. Show that Vieta's substitution $z = w - \frac{p}{3w}$ turns the equation $z^3 + pz = q$ into a quadratic equation in w^3.

(b) Use Vieta's substitution to solve the cubic $z^3 - 12z + 8 = 0$.

9. Roots of the Derivative.

(a) Let $P(z)$ be a polynomial with (possibly repeated) roots $\alpha_1, \alpha_2, \ldots, \alpha_k$. Show that:

$$\frac{P'(z)}{P(z)} = \frac{1}{z - \alpha_1} + \frac{1}{z - \alpha_2} + \cdots + \frac{1}{z - \alpha_k}.$$

(b) Deduce that if $\Im(\alpha_i) > 0$ for each i, then $\Im(\beta) > 0$ for any root β of $P'(z)$.

(c) Deduce further that if all the roots α of a polynomial $P(z)$ satisfy $|\alpha| < R$, then all the roots β of $P'(z)$ satisfy $|\beta| < R$.

Chapter 4

The Complex Plane

As established in chapter 2, the set of complex numbers $\mathbb{C}$ is defined as the Cartesian product $\mathbb{R} \times \mathbb{R}$. This algebraic definition naturally induces a geometric interpretation. By identifying the complex number $z = x + iy$ with the ordered pair (x, y), we map $\mathbb{C}$ to the Euclidean plane $\mathbb{R}^2$. This plane, equipped with the real and imaginary axes, is referred to as the *complex plane* or the *Gaussian plane*.

4.1 Vector Representation

The correspondence between $z = x+iy$ and the point (x, y) allows us to interpret complex numbers as position vectors. The complex number z represents the vector $\boldsymbol{Oz}$ originating from the origin and terminating at (x, y). Consequently, the arithmetic operations defined in chapter 2 possess direct geometric analogues.

Addition The sum $z + w$ corresponds to vector addition. If z and w are non-collinear vectors starting from the origin, $z + w$ is the diagonal of the parallelogram determined by z and w.

Scalar Multiplication For a real scalar $c \in \mathbb{R}$, the product cz scales the vector z by a factor of c. If $c > 0$, the direction is preserved; if $c < 0$, the direction is reversed.

Subtraction The difference $z - w$ represents the vector displacement from w to z. Geometrically, if vectors z and w originate at the origin, $z - w$ is the vector connecting the terminal point of w to the terminal point of z.

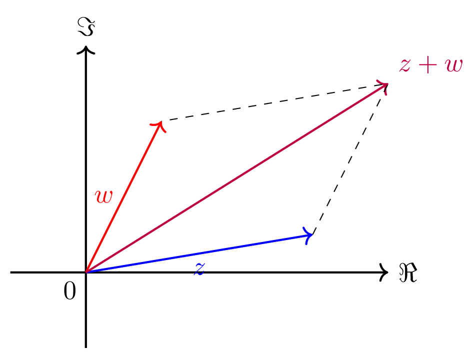


Figure 4.1: Addition in the complex plane follows the parallelogram law.

The modulus $|z|$, defined as $\sqrt{x^2 + y^2}$, is precisely the Euclidean length of the vector z. The triangle inequality proved in previous sections, $|z + w| \leq |z| + |w|$, is thus a restatement of the geometric fact that the length of a triangle's side $(z + w)$ cannot exceed the sum of the lengths of the other two sides (z and w).

4.2 Applications in Plane Geometry

Complex numbers can simplify many proofs in plane geometry by treating points as algebraic objects.

Example 4.2.1. Midpoint Formula. Let z_1 and z_2 be two points. The midpoint M of the segment connecting them is given by the average of their coordinates:

$$m = \frac{z_1 + z_2}{2}.$$

Example 4.2.2. Section Formula. A point z dividing the segment connecting z_1 and z_2 in the ratio $m : n$ is given by:

$$z = \frac{nz_1 + mz_2}{m + n}.$$

Alternatively, using a parameter $t \in [0, 1]$, any point on the segment can be written as:

$$z = (1 - t)z_1 + tz_2.$$

When $t = 0$, $z = z_1$; when $t = 1$, $z = z_2$; and when $t = 1/2$, z is the midpoint.

Example 4.2.3. Centroid of a Triangle. Consider a triangle with vertices z_1, z_2, z_3. The median from z_1 connects to the midpoint of the opposite side, $m_{23} = \frac{z_2+z_3}{2}$. The centroid divides this median in a $2 : 1$ ratio. Using the section formula with $t = 2/3$:

$$g = (1 - \tfrac{2}{3})z_1 + \tfrac{2}{3}\left(\frac{z_2 + z_3}{2}\right) = \frac{1}{3}z_1 + \frac{1}{3}(z_2 + z_3) = \frac{z_1 + z_2 + z_3}{3}.$$

Since the expression is symmetric in z_1, z_2, z_3, the centroid lies on all three medians, proving they are concurrent.

Example 4.2.4. Midpoint Theorem. Let A, B, C be vertices of a triangle represented by complex numbers z_A, z_B, z_C. Let D and E be the midpoints of sides AB and AC. Then $z_D = \frac{z_A+z_B}{2}$ and $z_E = \frac{z_A+z_C}{2}$. The vector representing the segment DE is:

$$z_E - z_D = \frac{z_A + z_C}{2} - \frac{z_A + z_B}{2} = \frac{z_C - z_B}{2} = \frac{1}{2}(z_C - z_B).$$

The vector $z_C - z_B$ represents the side BC. Thus, the segment DE is parallel to BC (as they are scalar multiples) and its length is exactly half that of BC.

4.3 Lines and Circles in the Complex Plane

Geometric shapes can be described by equations involving z.

Definition 4.3.1. ***Circle***. The set of points at a fixed distance $r > 0$ from a centre z_0 is defined by:

$$|z - z_0| = r.$$

Squaring this gives the algebraic form:

$$|z - z_0|^2 = r^2 \iff (z - z_0)(\bar{z} - \bar{z}_0) = r^2 \iff z\bar{z} - z\bar{z}_0 - \bar{z}z_0 + |z_0|^2 - r^2 = 0.$$

Definition 4.3.2. ***Line***. The perpendicular bisector of the segment connecting two points z_1 and z_2 is the set of points equidistant from both:

$$|z - z_1| = |z - z_2|.$$

Squaring both sides and simplifying yields a linear equation in z and $\bar{z}$:

$$(\bar{z}_2 - \bar{z}_1)z + (z_2 - z_1)\bar{z} + |z_1|^2 - |z_2|^2 = 0.$$

This is the general equation of a line in the complex plane, often written as $\bar{\beta}z + \beta\bar{z} + \gamma = 0$ where $\gamma \in \mathbb{R}$.

4.4 Exercises

Part I: Geometry and Construction

1. Let $z_1 = 2 + i$ and $z_2 = -1 + 3i$.
   (a) Plot z_1, z_2, and the vector $z_1 - z_2$ on the complex plane.
   (b) Calculate the distance between z_1 and z_2.
   (c) Find the midpoint of the segment connecting them.

2. Show that the points 1, $\frac{-1+i\sqrt{3}}{2}$, and $\frac{-1-i\sqrt{3}}{2}$ form an equilateral triangle in the complex plane.

3. Loci Sketches. On separate Argand diagrams sketch the following sets:
   (i) $|z| < 1$
   (ii) $\Re(z) = 3$
   (iii) $|z - 1| = |z + i|$
   (iv) $\arg(z - i) = \pi/4$
   (v) $-\pi/4 < \arg(z) < \pi/4$
   (vi) $\Re(z + 1) = |z - 1|$
   (vii) $|z - 3 - 4i| = 5$
   (viii) $\Re((1 + i)z) = 1$
   (ix) $\Im(z^3) > 0$

4. Let $ABCD$ be a parallelogram with vertices in counterclockwise order represented by complex numbers z_A, z_B, z_C, z_D.
   (a) Express the midpoints of the diagonals AC and BD in terms of the vertices.
   (b) Hence, prove that $z_A + z_C = z_B + z_D$.
   (c) Interpret this result geometrically regarding the intersection of the diagonals.

5. Rotation by i. Let $z = x + iy$.
   (a) Express iz in terms of x and y.
   (b) Show that the vector representing iz is perpendicular to the vector z and has the same length.
   (c) Conclude that multiplication by i corresponds to a counterclockwise rotation of $90°$ about the origin.

Part II: Lines, and Circles

6. Collinearity.
   (a) Prove that three distinct points z_1, z_2, z_3 are collinear if and only if the ratio $\frac{z_2-z_1}{z_3-z_1}$ is a real number.
   (b) Use this property to show that the points $1 + 2i$, $3 - i$, and $5 - 4i$ lie on a straight line.

7. Equation of a Line. Using the collinearity condition derived above, show that the equation of the line passing through two distinct points z_1 and z_2 can be written as:

$$\frac{z - z_1}{z_2 - z_1} = \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1}.$$

Rearrange this to show it fits the general form $\bar{\beta}z + \beta\bar{z} + \gamma = 0$.

8. Perpendicularity. Prove that the line connecting z_1 and z_2 is perpendicular to the line connecting z_3 and z_4 if and only if the ratio $\frac{z_1-z_2}{z_3-z_4}$ is purely imaginary.

Remark. Consider the Pythagorean theorem on the triangle formed by vectors representing these segments, or use the concept of rotation by i.

9. Cassini Ovals. Let $k > 0$.
   (a) Sketch the curve C_k with equation $|z + 1/z| = k$.

(b) What are the extreme values of $|z|$ on C_k?

(c) Show that the curve C_2 consists of precisely two circles.

10. Rhombus Diagonals. Use complex numbers to prove that the diagonals of a rhombus are perpendicular.

Remark. Place the centre of the rhombus at the origin. Let the sides be represented by vectors z and w with $|z| = |w|$. The diagonals are $z + w$ and $z - w$. Use the result from the previous exercise.

11. Apollonius Circle. Let $k > 0$ be a real constant such that $k \neq 1$. Consider the locus of points z such that the distance from z to -1 is k times the distance from z to 1:

$$|z + 1| = k|z - 1|.$$

(a) Square both sides and use the identity $|w|^2 = w\bar{w}$ to expand the expression.

(b) Rearrange the terms to show that the equation takes the form $|z - c|^2 = r^2$.

(c) Find the centre c and radius r in terms of k.

12. General Circle Equation.

(a) Show that the equation $z\bar{z} + \bar{\beta}z + \beta\bar{z} + \gamma = 0$ represents a circle if and only if γ is a real number and $|\beta|^2 > \gamma$.

(b) Find the centre and radius of the circle described by $z\bar{z} - (2 + i)z - (2 - i)\bar{z} - 3 = 0$.

Part III: Advanced Geometric Properties

12. ⋆ The Real Inner Product. We can define a "dot product" for complex numbers z, w by $\langle z, w\rangle = \Re(z\bar{w})$.

(a) If $z = x_1 + iy_1$ and $w = x_2 + iy_2$, show that $\langle z, w\rangle = x_1x_2 + y_1y_2$, which is the standard Euclidean dot product in $\mathbb{R}^2$.

(b) Prove that two vectors z and w are perpendicular if and only if $\Re(z\bar{w}) = 0$.

(c) Prove the Law of Cosines identity: $|z - w|^2 = |z|^2 + |w|^2 - 2\Re(z\bar{w})$.

13. ⋆ The Euler Line. Consider a triangle with vertices z_1, z_2, z_3 inscribed in the unit circle (so $|z_1| = |z_2| = |z_3| = 1$). The circumcentre O is therefore at the origin 0.

(a) The centroid G is given by $g = \frac{z_1+z_2+z_3}{3}$.

(b) The orthocentre H (intersection of altitudes) for such a triangle is given by $h = z_1 + z_2 + z_3$. Verify this by showing that the vector $h - z_1$ is perpendicular to the side $z_2 - z_3$.

Remark. Use the fact that for unit modulus numbers, $\bar{z} = 1/z$, and the perpendicularity condition derived in problem 12(b).

(c) Show that O, G, and H are collinear and determine the ratio in which G divides the segment OH.

14. ⋆ Ptolemy's Inequality. For any four points z_1, z_2, z_3, z_4 in the complex plane:

$$|z_1 - z_3||z_2 - z_4| \leq |z_1 - z_2||z_3 - z_4| + |z_2 - z_3||z_4 - z_1|.$$

(The product of the diagonals of a quadrilateral is less than or equal to the sum of the products of opposite sides).

Remark. Use the algebraic identity $(z_1 - z_3)(z_2 - z_4) = (z_1 - z_2)(z_3 - z_4) + (z_2 - z_3)(z_4 - z_1)$ and the Triangle Inequality.

Chapter 5

Polar Representation of Complex Numbers

While the vector interpretation of complex numbers provides a robust geometric intuition for addition and subtraction, it offers little insight into the nature of multiplication. The dot product of two vectors yields a scalar, and the cross product yields a vector orthogonal to the plane; neither operation remains within the two-dimensional complex plane in a way that preserves the field structure. To fully appreciate the power of complex numbers, particularly regarding multiplication and powers, we turn to the polar coordinate system.

5.1 Polar Coordinates and the Argument

Any point (x, y) in the plane (excluding the origin) can be uniquely identified by its distance r from the origin and the angle θ subtended by the positive real axis and the position vector.

Definition 5.1.1. ***Polar Form***. Let $z = x + iy$ be a non-zero complex number. We can express x and y as:

$$x = r\cos\theta, \quad y = r\sin\theta$$

where $r = \sqrt{x^2 + y^2} = |z|$ is the modulus, and θ is the angle measured counterclockwise from the positive real axis. Thus,

$$z = r(\cos\theta + i\sin\theta).$$

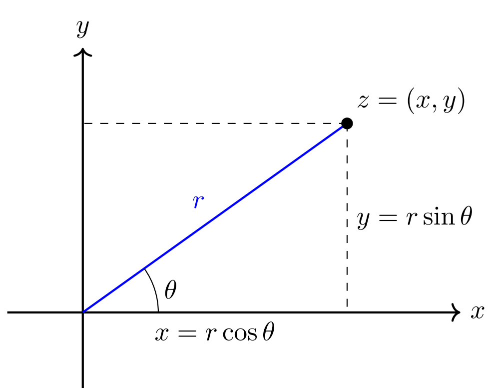


Figure 5.1: Polar representation. The complex number z corresponds to the point (x, y), where x and y are the geometric projections satisfying $x = r\cos\theta$ and $y = r\sin\theta$.

The angle θ is called the *argument* of z, denoted by $\arg(z)$. Unlike the modulus, the argument is not unique;

adding any integer multiple of 2π to θ yields the same position.

$$\arg(z) = \theta + 2n\pi, \quad n \in \mathbb{Z}.$$

Usually, we identify $\arg(z)$ as a set of values or work modulo 2π. When a specific value is required, we often choose the *principal value* in the interval $(-\pi, \pi]$ or $[0, 2\pi)$.

Example 5.1.1.

- **Real Numbers:** If z is a positive real number, $\arg(z) = 2n\pi$. If negative, $\arg(z) = \pi + 2n\pi$.
- **Imaginary Numbers:** If $z = i$, $\arg(z) = \frac{\pi}{2} + 2n\pi$. If $z = -i$, $\arg(z) = -\frac{\pi}{2} + 2n\pi$.
- **Example:** Let $z = 1 - i$. Then $|z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$. The point lies in the fourth quadrant, so $\arg(z) = -\frac{\pi}{4}$. Thus, $z = \sqrt{2}(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4}))$.

5.2 Multiplication and Geometry

The polar form reveals the geometric essence of complex multiplication: it is a combination of scaling and rotation.

***Theorem* 5.2.1. Multiplication in Polar Form**. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then:

$$z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)).$$

Consequently:

$$|z_1 z_2| = |z_1||z_2| \quad \text{and} \quad \arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \pmod{2\pi}.$$

Proof. Using the distributive law and the trigonometric addition formulas:

$$\begin{aligned} z_1 z_2 &= [r_1(\cos\theta_1 + i\sin\theta_1)] \cdot [r_2(\cos\theta_2 + i\sin\theta_2)] \\ &= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \\ &= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]. \end{aligned}$$

■

Geometrically, multiplying a complex number w by $z = r(\cos\theta + i\sin\theta)$ scales the vector w by a factor of r and rotates it counterclockwise by an angle θ.

***Corollary* 5.2.1.** Multiplying by i corresponds to a rotation by $\frac{\pi}{2}$ (90 degrees) counterclockwise, as $|i| = 1$ and $\arg(i) = \frac{\pi}{2}$.

Geometric Construction of the Product

We can construct the product $z_1 z_2$ using similar triangles. Consider the triangle formed by the origin O, the point 1, and z_1. We wish to construct a triangle similar to $\triangle O1z_1$ on the base Oz_2. Let z_3 be the point such that $\triangle O1z_1 \sim \triangle Oz_2z_3$. Then the ratio of sides implies $|z_3|/|z_2| = |z_1|/1$, so $|z_3| = |z_1||z_2|$. The angles add, so $\arg(z_3) = \arg(z_2) + \arg(z_1)$. Thus, $z_3 = z_1 z_2$.

De Moivre's Theorem

The multiplicative property naturally extends to powers, leading to one of the most useful theorems in complex analysis.

***Theorem* 5.2.2. De Moivre's Theorem**. For any integer $n \in \mathbb{Z}$ and any complex number $z = r(\cos\theta + i\sin\theta)$:

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta)).$$

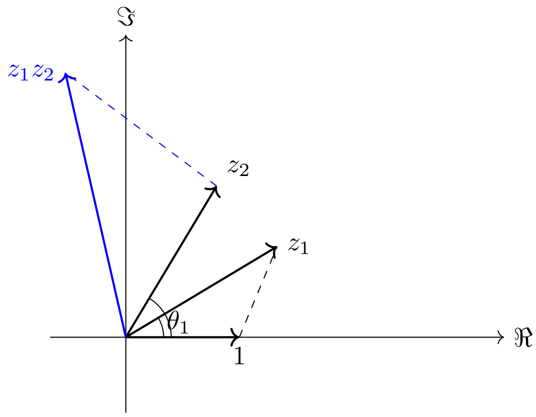


Figure 5.2: Geometric construction of the product. The triangle with vertices $0, 1, z_1$ is similar to the triangle with vertices $0, z_2, z_1 z_2$.

Proof. For $n \geq 0$, this follows from induction on the multiplication theorem. For $n = -1$:

$$z^{-1} = \frac{1}{r(\cos\theta + i\sin\theta)} = \frac{1}{r}(\cos\theta - i\sin\theta) = r^{-1}(\cos(-\theta) + i\sin(-\theta)).$$

The general negative case follows from $(z^{-1})^{-n}$. ∎

***Corollary* 5.2.2.** *Division.*

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)).$$

In terms of argument: $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2)$.

Example 5.2.1. Roots of Unity Consider the number $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. In polar form:

$$|\omega| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1.$$

$$\cos\theta = -1/2, \sin\theta = \sqrt{3}/2 \implies \theta = \frac{2\pi}{3}.$$

Using De Moivre's Theorem:

$$\omega^2 = 1^2(\cos\tfrac{4\pi}{3} + i\sin\tfrac{4\pi}{3}) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

$$\omega^3 = 1^3(\cos\tfrac{6\pi}{3} + i\sin\tfrac{6\pi}{3}) = \cos(2\pi) + i\sin(2\pi) = 1.$$

The points $1, \omega, \omega^2$ form the vertices of an equilateral triangle inscribed in the unit circle.

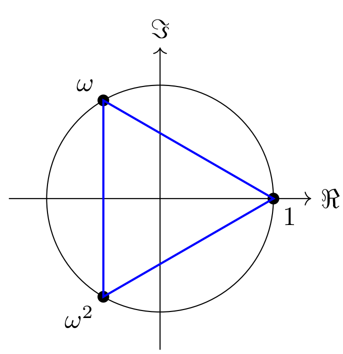


Figure 5.3: The cube roots of unity form an equilateral triangle.

Applications of De Moivre's Theorem

This theorem is a powerful tool for deriving trigonometric identities. For instance, expanding $(\cos\theta + i\sin\theta)^3$ via the binomial theorem and comparing it to $\cos(3\theta) + i\sin(3\theta)$ yields:

$$\cos(3\theta) + i\sin(3\theta) = (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta).$$

Equating real and imaginary parts:

$$\begin{aligned}\cos(3\theta) &= 4\cos^3\theta - 3\cos\theta\\ \sin(3\theta) &= 3\sin\theta - 4\sin^3\theta\end{aligned}$$

Revisiting the Triangle Inequality

In earlier chapter, we established the triangle inequality $|z_1 + z_2| \le |z_1| + |z_2|$. Using polar coordinates, we can precisely characterise the equality case. Recall that $|z_1 + z_2| = |z_1| + |z_2|$ if and only if $z_1\bar{z}_2 \ge 0$. In polar form, let $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$.

$$z_1\bar{z}_2 = r_1r_2e^{i(\theta_1-\theta_2)}.$$

For this product to be a non-negative real number, the angle $\theta_1 - \theta_2$ must be an integer multiple of 2π. Thus, equality holds if and only if $\arg(z_1) = \arg(z_2)$, meaning z_1 and z_2 lie on the same ray from the origin.

Geometric Applications of the Argument

The polar representation provides a rigorous link between algebraic operations and geometric angles. We can utilise this to prove classical Euclidean theorems with remarkable efficiency.

Example 5.2.2. The Cosine Rule. Let A, B, C be the vertices of a triangle. We place A at the origin ($z_A = 0$). Let u and v be the complex numbers representing vectors AB and AC. The side BC corresponds to the vector $v - u$. We compute the squared length of BC:

$$\begin{aligned}|v-u|^2 &= (v-u)(\bar{v}-\bar{u})\\ &= v\bar{v} - v\bar{u} - u\bar{v} + u\bar{u}\\ &= |v|^2 + |u|^2 - (v\bar{u} + \overline{v\bar{u}})\\ &= |v|^2 + |u|^2 - 2\Re(v\bar{u}).\end{aligned}$$

In polar form, let $u = r_1e^{i\theta_1}$ and $v = r_2e^{i\theta_2}$. Then:

$$v\bar{u} = r_2e^{i\theta_2} \cdot r_1e^{-i\theta_1} = r_1r_2e^{i(\theta_2-\theta_1)}..$$

The real part is $\Re(v\bar{u}) = r_1r_2\cos(\theta_2 - \theta_1)$. Since $\theta_2 - \theta_1$ is the angle A between the vectors, and r_1, r_2 are the lengths $|AB|, |AC|$:

$$|BC|^2 = |AC|^2 + |AB|^2 - 2|AC||AB|\cos A.$$

Example 5.2.3. Thales' Theorem. Let A and B be points on a circle with diameter AB, and let P be any other point on the circle. We may set the origin at the centre of the circle and scale so the radius is 1. Thus A corresponds to -1, B to 1, and P to some z with $|z| = 1$. The angle $\angle APB$ is the difference in arguments between the vector $P \to A$ and $P \to B$. This is the argument of the quotient:

$$w = \frac{z-1}{z+1}.$$

We check if w is purely imaginary (which would imply $\arg(w) = \pm\pi/2$) by computing $w + \bar{w}$:

$$\frac{z-1}{z+1} + \frac{\bar{z}-1}{\bar{z}+1} = \frac{(z-1)(\bar{z}+1) + (\bar{z}-1)(z+1)}{(z+1)(\bar{z}+1)}.$$

Expanding the numerator:

$$(z\bar{z} + z - \bar{z} - 1) + (z\bar{z} + \bar{z} - z - 1) = 2z\bar{z} - 2 = 2|z|^2 - 2.$$

Since P lies on the unit circle, $|z|^2 = 1$, so the numerator is 0. Thus $w + \bar{w} = 0$, implying w is purely imaginary. Consequently, $\arg(w) = \pm\frac{\pi}{2}$, proving that the angle at P is a right angle.

Quadratic Equations with Complex Coefficients

The familiar quadratic formula holds even when the coefficients are complex. However, one must be careful when computing the square root of the discriminant.

Example 5.2.4. Complex Coefficients. Solve $z^2 - (3+i)z + (2+i) = 0$. Using the quadratic formula with $a = 1, b = -(3+i), c = (2+i)$:

$$\Delta = b^2 - 4ac = (3+i)^2 - 4(2+i) = (9 + 6i - 1) - 8 - 4i = 2i.$$

We need the square roots of $2i$. Since $2i = 2e^{i\pi/2}$, the roots are $\sqrt{2}e^{i\pi/4}$ and $\sqrt{2}e^{i5\pi/4}$.

$$\sqrt{2}e^{i\pi/4} = \sqrt{2}(\tfrac{1}{\sqrt{2}} + \tfrac{i}{\sqrt{2}}) = 1 + i.$$

Thus $\sqrt{\Delta} = \pm(1+i)$. The solutions are:

$$z = \frac{(3+i) \pm (1+i)}{2} \implies z_1 = \frac{4+2i}{2} = 2 + i, \quad z_2 = \frac{2}{2} = 1.$$

Note that z_1 and z_2 are **not** conjugates. The Conjugate Pairs corollary applies only when coefficients are real.

5.3 Exercises

Part I: Polar Arithmetic and De Moivre's Theorem

1. Convert the following complex numbers to polar form $r(\cos\theta + i\sin\theta)$ using the principal argument in $(-\pi, \pi]$:
   (a) $z = -2 + 2i$
   (b) $z = -\sqrt{3} - i$
   (c) $z = -4i$

2. Find the modulus and argument of each of the following numbers:
   (a) $1 + i\sqrt{3}$
   (b) $(2+i)(3-i)$
   (c) $(1+i)^5$
   (d) $\frac{(1+2i)^3}{(2-i)^3}$

3. De Moivre Applications. Let $z = \operatorname{cis}\theta = e^{i\theta}$ and let n be an integer.
   (a) Show that $2\cos\theta = z + z^{-1}$ and that $2i\sin\theta = z - z^{-1}$.
   (b) Using De Moivre's theorem, show further that $2\cos n\theta = z^n + z^{-n}$ and that $2i\sin n\theta = z^n - z^{-n}$.
   (c) Deduce that $16\cos^5\theta = \cos 5\theta + 5\cos 3\theta + 10\cos\theta$ and evaluate $\int_0^{\pi/2}\cos^5\theta\, d\theta$.

4. Use De Moivre's Theorem to compute $(1+i)^{10}$. Express the result in the standard form $a + bi$.

5. Trigonometric Identities.
   (a) By expanding $(\cos\theta + i\sin\theta)^4$, express $\cos(4\theta)$ in terms of powers of $\cos\theta$ and $\sin\theta$.
   (b) Express $\sin(5\theta)$ purely in terms of $\sin\theta$.

6. Let $z_1 = 3(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$ and $z_2 = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$. Compute the product z_1z_2 and the quotient z_1/z_2 in polar form. Sketch the vectors z_1, z_2, and z_1z_2 to verify the geometric interpretation of multiplication.

7. Geometric Transformations.

(a) Interpret the function $f(z) = (1+i)z$ geometrically. What are the scaling factor and the angle of rotation?

(b) Find a complex number w such that multiplication by w results in a rotation of $45°$ clockwise and a scaling by a factor of 2.

(c) A square $ABCD$ in the complex plane has its centre at the origin. If the vertex A is at $3+4i$, find the complex numbers representing B, C, and D.

8. General Roots. To solve the equation $z^n = w$, we write w in polar form $R(\cos\phi + i\sin\phi)$ and assume $z = r(\cos\theta + i\sin\theta)$.

(a) Show that $r = \sqrt[n]{R}$ and $\theta = \frac{\phi+2k\pi}{n}$ for $k = 0, 1, \ldots, n-1$.

(b) Find all fourth roots of -16.

(c) Solve $z^3 = -2 + 2i$.

Part II: Geometry and Series

7. Let $\omega = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$.

(a) Prove that $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$.

Remark. Use the geometric series formula or vector symmetry.

(b) Hence, show that $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$.

8. Summation of Trigonometric Series. Let $S_n = 1 + z + z^2 + \cdots + z^n$ where $z = \cos\theta + i\sin\theta$ and $0 < \theta < 2\pi$.

(a) Show that $S_n = \frac{1-z^{n+1}}{1-z}$.

(b) By considering the real part of S_n, prove the *Lagrange Identity*:

$$1 + \cos\theta + \cos(2\theta) + \cdots + \cos(n\theta) = \frac{1}{2} + \frac{\sin((n+\frac{1}{2})\theta)}{2\sin(\frac{\theta}{2})}.$$

Remark. Multiply the numerator and denominator of the fraction in (a) by $1 - \bar{z}$ or use the half-angle substitution $1 - e^{i\theta} = -2i\sin(\theta/2)e^{i\theta/2}$.

9. Locus of an Arc. Recall that $\arg(\frac{z-a}{z-b})$ represents the angle between the vector $(z-b)$ and $(z-a)$.

(a) Identify and sketch the locus of points z satisfying $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$.

(b) Prove geometrically (using circle theorems) that this locus is an arc of a circle passing through 1 and -1.

(c) Find the Cartesian equation of this circle.

10. ⋆ **Product of Chord Lengths.** Consider the polynomial $P(z) = z^n - 1$. Its roots are the n-th roots of unity, denoted $1, \omega, \omega^2, \ldots, \omega^{n-1}$.

(a) Explain why $z^n - 1 = (z-1)(z-\omega)(z-\omega^2)\cdots(z-\omega^{n-1})$.

(b) By dividing both sides by $z-1$ and taking the limit as $z \to 1$, prove that:

$$(1-\omega)(1-\omega^2)\cdots(1-\omega^{n-1}) = n.$$

(c) Interpret this result geometrically regarding the product of the lengths of the chords connecting the vertex 1 of a regular n-gon to all other vertices.

11. ⋆ **Rotation about a General Point.** Multiplying by $e^{i\alpha}$ rotates a vector about the origin. To rotate a point z by an angle α about a centre c:

(a) Justify the formula $z' = c + e^{i\alpha}(z-c)$.

(b) Let $z_1 = 1$ and $z_2 = i$. Find the point z_3 such that triangle $z_1z_2z_3$ is equilateral and vertices are labelled counterclockwise.

Remark. Rotate z_2 about z_1 by $\pi/3$?

Chapter 6

Powers, Roots, and Exponentials

Having established the polar form and De Moivre's Theorem, we now possess the tools to explore two powerful applications: finding roots of complex numbers and defining the complex exponential function. These concepts not only complete our algebraic understanding but also bridge the gap between algebra, geometry, and analysis.

6.1 Roots of Complex Numbers

In chapter 3, we solved quadratic equations using algebraic manipulation. The polar form allows us to find the n-th roots of any complex number for any integer $n \geq 1$ with elegance and geometric clarity.

We seek all solutions z to the equation $z^n = w$, where w is a given non-zero complex number. Let $w = R(\cos\phi + i\sin\phi)$ and $z = r(\cos\theta + i\sin\theta)$. By De Moivre's Theorem, $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$. Equating the modulus and argument:

$$r^n = R \implies r = \sqrt[n]{R} \quad \text{(real, positive } n\text{-th root)}$$

$$n\theta = \phi + 2k\pi \implies \theta_k = \frac{\phi + 2k\pi}{n}, \quad k \in \mathbb{Z}.$$

Although k can be any integer, the values of θ_k repeat modulo 2π every n steps. Specifically, $\theta_n = \frac{\phi+2n\pi}{n} = \frac{\phi}{n} + 2\pi \equiv \theta_0 \pmod{2\pi}$. Thus, there are exactly n distinct roots.

***Theorem* 6.1.1.** n-th Roots Theorem The equation $z^n = w$, where $w = R(\cos\phi + i\sin\phi) \neq 0$, has exactly n distinct solutions given by:

$$z_k = \sqrt[n]{R}\left(\cos\left(\frac{\phi + 2k\pi}{n}\right) + i\sin\left(\frac{\phi + 2k\pi}{n}\right)\right)$$

for $k = 0, 1, 2, \ldots, n-1$. Geometrically, these roots lie on a circle of radius $\sqrt[n]{R}$ and form the vertices of a regular n-sided polygon.

Roots of Unity

The special case $w = 1$ yields the n-th roots of unity. Since $1 = 1(\cos 0 + i\sin 0)$, the roots are:

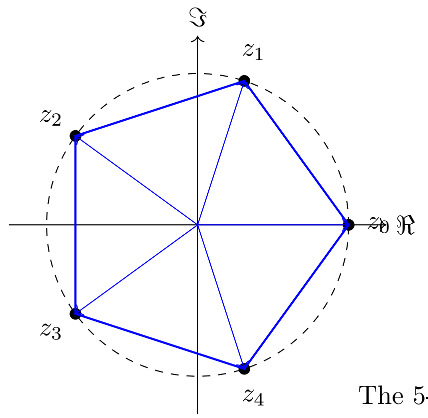
$$\omega_k = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right), \quad k = 0, 1, \ldots, n-1.$$

The root for $k = 1$, denoted $\omega = \cos(2\pi/n) + i\sin(2\pi/n)$, is called a *primitive* n-th root of unity because all other roots can be generated as powers of ω: the roots are $1, \omega, \omega^2, \ldots, \omega^{n-1}$.

Example 6.1.1. Cube Roots of Unity For $n = 3$, the roots are 1, $\omega = e^{i2\pi/3}$, and $\omega^2 = e^{i4\pi/3}$. In Cartesian form:

$$1 = 1$$
$$\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$
$$\omega^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

Note that $1 + \omega + \omega^2 = 0$, a property that holds for the sum of n-th roots of unity for any $n > 1$.



The 5-th roots of unity form a regular pentagon.

Figure 6.1: Geometric representation of the 5-th roots of unity.

6.2 The Complex Exponential

The Taylor series expansion for the real exponential function suggests a natural definition for complex arguments. Recall that for $x \in \mathbb{R}$, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. If we substitute a purely imaginary number ix into this series:

$$\begin{aligned} e^{ix} &= 1 + \frac{ix}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \dots \\ &= 1 + i\frac{x}{1!} - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + \dots \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) \end{aligned}$$

Recognising the series for cosine and sine, we arrive at one of the most famous formulas in mathematics.

Remark. (Motivation via Rotation). Before formally defining the complex exponential, consider the geometric behaviour of rotation. Rotating a point (x, y) by an angle θ and then by an angle β results in a total rotation of $\theta + \beta$. In complex polar arithmetic, a rotation corresponds to multiplication by a complex number on the unit circle. If we denote a rotation by θ as a function $f(\theta)$, the property that consecutive rotations add angles suggests the rule:

$$f(\theta) \cdot f(\beta) = f(\theta + \beta).$$

This functional equation is characteristic of the exponential function (where $e^a \cdot e^b = e^{a+b}$). Thus, it is natural to define the complex exponential such that it maps the additive group of angles to the multiplicative group of rotations.

Definition 6.2.1. ***Euler's Formula***. For any real number θ:

$$e^{i\theta} = \cos\theta + i \sin\theta.$$

This definition allows us to write the polar form compactly as $z = re^{i\theta}$. A specific instance of this formula, when $\theta = \pi$, gives Euler's Identity:

$$e^{i\pi} + 1 = 0.$$

This equation remarkably links five fundamental constants: $0, 1, e, i, \pi$.

Properties of the Complex Exponential

The complex exponential e^z for $z = x + iy$ is defined as:

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y).$$

This function retains the key property of the real exponential:

$$e^{z_1+z_2} = e^{z_1} e^{z_2}.$$

However, unlike the real exponential, the complex exponential is periodic with period $2\pi i$:

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z(\cos 2\pi + i \sin 2\pi) = e^z(1) = e^z.$$

Applications to Trigonometry

Euler's formula allows us to express trigonometric functions in terms of exponentials:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

These identities simplify the derivation of trigonometric integrals and sums.

Example 6.2.1. Power Reduction. To derive a formula for $\cos^2\theta$, substitute the exponential form and use the laws of exponents:

$$\begin{aligned}
\cos^2\theta &= \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^2 \\
&= \frac{1}{4}\left(e^{2i\theta} + 2e^{i\theta}e^{-i\theta} + e^{-2i\theta}\right) \\
&= \frac{1}{4}\left((e^{2i\theta} + e^{-2i\theta}) + 2\right) \\
&= \frac{1}{2}\left(\frac{e^{2i\theta} + e^{-2i\theta}}{2}\right) + \frac{1}{2} \\
&= \frac{1}{2}\cos(2\theta) + \frac{1}{2}.
\end{aligned}$$

Similarly, for $\sin^2\theta$:

$$\sin^2\theta = \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^2 = \frac{e^{2i\theta} - 2 + e^{-2i\theta}}{-4} = \frac{1}{2} - \frac{1}{2}\cos(2\theta).$$

Example 6.2.2. Product-to-Sum Identities. Consider the product $\cos(x)\sin(4x)$. Converting to exponentials:

$$\begin{aligned}
\cos(x)\sin(4x) &= \left(\frac{e^{ix} + e^{-ix}}{2}\right)\left(\frac{e^{i4x} - e^{-i4x}}{2i}\right) \\
&= \frac{1}{4i}\left(e^{i5x} - e^{-i3x} + e^{i3x} - e^{-i5x}\right) \\
&= \frac{1}{2}\left(\frac{e^{i5x} - e^{-i5x}}{2i}\right) - \frac{1}{2}\left(\frac{e^{-i3x} - e^{i3x}}{2i}\right) \\
&= \frac{1}{2}\sin(5x) + \frac{1}{2}\left(\frac{e^{i3x} - e^{-i3x}}{2i}\right) \\
&= \frac{1}{2}\sin(5x) + \frac{1}{2}\sin(3x).
\end{aligned}$$

6.3 Applications in Wave Mechanics

In physics and engineering, particularly in the study of electromagnetism and signal processing, sinusoidal oscillations are fundamental. Complex numbers provide a powerful method, known as the *phasor method*, to analyse these oscillations algebraically rather than trigonometrically.

The Phasor Representation

A sinusoidal wave with amplitude A, angular frequency ω, and phase angle ϕ can be written as:

$$y(t) = A\cos(\omega t + \phi).$$

Using Euler's formula, we observe that:

$$A\cos(\omega t + \phi) = Re\left(Ae^{i(\omega t+\phi)}\right) = Re\left(Ae^{i\phi} \cdot e^{i\omega t}\right).$$

The complex number $\tilde{A} = Ae^{i\phi}$ is called the *phasor* of the wave. It encodes the amplitude and phase shift in a single static complex number, separating them from the time-dependent component $e^{i\omega t}$.

Remark. (Engineering Notation). In Electrical Engineering, the imaginary unit is often denoted by j to avoid confusion with electric current i. Furthermore, impedance, capacitance, and inductance are modelled as complex quantities, allowing alternating current (AC) circuit problems to be solved using algebraic rules similar to direct current (DC) circuits.

Superposition and Interference

When two waves of the same frequency interact, they satisfy the principle of superposition. If $y_1 = A_1\sin(kx + \phi_1)$ and $y_2 = A_2\sin(kx + \phi_2)$, their sum is difficult to compute using standard trigonometric identities. However, using complex numbers, this becomes vector addition.

Let us represent the waves as the imaginary parts of complex exponentials (since $\Im(e^{i\theta}) = \sin\theta$):

$$\tilde{y}_1 = A_1 e^{i(kx+\phi_1)}, \quad \tilde{y} * 2 = A_2 e^{i(kx+\phi_2)}.$$

The sum is:

$$\tilde{y} * \mathrm{sum} = \tilde{y}_1 + \tilde{y} * 2 = e^{ikx}\left(A_1 e^{i\phi_1} + A_2 e^{i\phi_2}\right).$$

The term in the brackets is simply the sum of two constant complex numbers (vectors). If we let $Ae^{i\gamma} = A_1 e^{i\phi_1} + A_2 e^{i\phi_2}$, then:

$$y * \mathrm{sum} = \Im(Ae^{i\gamma}e^{ikx}) = A\sin(kx + \gamma).$$

The amplitude A and phase γ of the resulting wave are determined by the vector sum of the individual phasors.

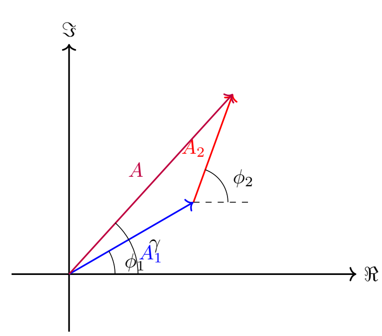


Figure 6.2: Phasor addition (Tip-to-Tail method). The superposition of two waves corresponds to the vector addition of their complex amplitudes.

Example 6.3.1. Standing Waves. Consider two waves of equal amplitude A travelling in opposite directions:

$$y_1 = Ae^{i(kx-\omega t)}, \quad y_2 = Ae^{i(kx+\omega t)}..$$

Summing them:

$$y = Ae^{ikx}(e^{-i\omega t} + e^{i\omega t}) = Ae^{ikx}(2\cos(\omega t)).$$

The real part represents a physical wave $2A\cos(kx)\cos(\omega t)$. This describes a *standing wave*, where the spatial dependence $\cos(kx)$ and temporal dependence $\cos(\omega t)$ are separated. The "complex" arithmetic elegantly reveals that the nodes (where amplitude is zero) occur fixed in space where $\cos(kx) = 0$.

6.4 Exercises

Part I: Roots and Polynomials

1. Find all solutions to the following equations in the form $re^{i\theta}$ where $-\pi < \theta \leq \pi$. Sketch the roots in the complex plane and identify the regular polygon they form.
   (a) $z^3 = -8i$
   (b) $z^4 = -1$
   (c) $z^6 - 64 = 0$

2. The Heptagon. Write down the seven roots of $z^7 + 1 = 0$. By considering the coefficient of z^6 in the factorization of $z^7 + 1$, show that $\cos(\pi/7) + \cos(3\pi/7) + \cos(5\pi/7) = \frac{1}{2}$.

3. Heptagon Identities.
   (a) Let $\zeta = e^{i\pi/7}$. Simplify the expression $(\zeta - \zeta^6)(\zeta^3 - \zeta^4)(\zeta^5 - \zeta^2)$. Hence show that:

$$\cos(\pi/7)\cos(3\pi/7)\cos(5\pi/7) = -1/8.$$

   (b) Given that $\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$, rederive the result of (a).

4. The Pentagon. Let $\zeta = e^{2\pi i/5}$.
   (a) Show that $1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 = 0$.
   (b) Show further that $(z - \zeta - \zeta^4)(z - \zeta^2 - \zeta^3) = z^2 + z - 1$.
   (c) Deduce that $\cos(2\pi/5) = \frac{\sqrt{5}-1}{4}$.

5. Inequalities.
   (a) Let a, b, c be positive real numbers. By expanding $(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$ and considering the second factor as a quadratic in a, show that $a^3 + b^3 + c^3 > 3abc$.
   (b) Let ω be a complex cube root of unity ($\omega \neq 1$). Show that $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c)$.

6. Solve the equation $(z + 1)^5 + z^5 = 0$.

   Remark. Rearrange the equation to the form $\left(\frac{z+1}{z}\right)^5 = -1$. The solutions will involve cotangents.

7. Exact Trigonometric Values. Consider the equation $z^5 = 1$.
   (a) Factorise $z^5 - 1$ into linear factors over $\mathbb{C}$.
   (b) By grouping conjugate roots, factorise $z^5 - 1$ into a linear factor and two quadratic factors with *real* coefficients.
   (c) By comparing the coefficients of the quadratic factors with the algebraic expansion $(z - 1)(z^4 + z^3 + z^2 + z + 1)$, derive the value of $\cos(72°)$ and $\cos(144°)$.

8. Primitive Roots. A root of unity ω is called a *primitive n-th root of unity* if $\omega^n = 1$ and $\omega^k \neq 1$ for all $1 \leq k < n$.
   (a) List all primitive 4th, 6th, and 8th roots of unity.
   (b) Show that if ω is a primitive n-th root of unity, then the set $\{1, \omega, \omega^2, \ldots, \omega^{n-1}\}$ contains all distinct n-th roots of unity.
   (c) Let $\Phi_n(z)$ be the polynomial whose roots are exactly the primitive n-th roots of unity (the Cyclotomic Polynomial). Find $\Phi_3(z)$ and $\Phi_4(z)$.

Part II: The Complex Exponential and Trigonometry

5. Use Euler's formula to prove the following trigonometric identities:
(a) $\cos^3\theta = \frac{3}{4}\cos\theta + \frac{1}{4}\cos 3\theta$
(b) $\sin^4\theta = \frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta$

6. Binomial Sums. Let $C = \sum_{k=0}^{n}\binom{n}{k}\cos(k\theta)$ and $S = \sum_{k=0}^{n}\binom{n}{k}\sin(k\theta)$.
(a) By considering the sum $C + iS$, show that it equals $(1+e^{i\theta})^n$.
(b) Convert $1+e^{i\theta}$ into polar form (using half-angles, e.g., $e^{i\theta/2}(e^{-i\theta/2}+e^{i\theta/2})$).
(c) Hence, derive closed-form expressions for C and S in terms of θ and n.

7. Product Expansions.
(a) Show, for any complex number z and positive integer n, that:

$$z^{2n} - 1 = (z^2-1)\prod_{k=1}^{n-1}\left(z^2 - 2z\cos\left(\frac{k\pi}{n}\right) + 1\right).$$

(b) Deduce that for any real θ:

$$\sin n\theta = 2^{n-1}\sin\theta\prod_{k=1}^{n-1}\left(\cos\theta - \cos\left(\frac{k\pi}{n}\right)\right).$$

(c) Determine the value of $\prod_{k=1}^{2n}\cos\left(\frac{k\pi}{2n+1}\right)$.

8. Dirichlet Kernel. The function $D_n(x) = \frac{1}{2} + \sum_{k=1}^{n}\cos(kx)$ is fundamental in the study of series. Prove that for $x \neq 2m\pi$:

$$D_n(x) = \frac{\sin((n+\frac{1}{2})x)}{2\sin(\frac{x}{2})}.$$

Remark. Write the cosine terms as exponentials ($2\cos(kx) = e^{ikx} + e^{-ikx}$) and sum the resulting geometric series.

9. Chebyshev Polynomials. De Moivre's Theorem implies that $\cos(n\theta)$ can be written as a polynomial in $\cos\theta$.
(a) Expand $(\cos\theta + i\sin\theta)^n$ and take the real part to express $\cos(n\theta)$ as a sum involving powers of $\cos\theta$ and $\sin^2\theta$.
(b) Substitute $\sin^2\theta = 1-\cos^2\theta$ to show that $\cos(n\theta) = T_n(\cos\theta)$ for some polynomial T_n of degree n.
(c) Find the explicit polynomials $T_2(x), T_3(x)$, and $T_4(x)$.

Part III: Phasors and Applications

9. Superposition. Two voltage signals are given by $V_1(t) = 10\cos(\omega t)$ and $V_2(t) = 10\cos(\omega t + \pi/3)$.
(a) Express V_1 and V_2 as the real parts of complex phasors.
(b) Find the amplitude and phase of the resulting signal $V(t) = V_1(t) + V_2(t)$ by adding the phasors as vectors.
(c) Verify your result using the trigonometric identity $\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$.

10. Interference Patterns. Consider N sources of light arranged in a line, each emitting a wave $Ae^{i(\omega t + k\delta)}$ where δ is the phase difference between adjacent sources. The total amplitude is the modulus of the sum $S = \sum_{k=0}^{N-1} Ae^{ik\delta}$.
(a) Sum the geometric series to show that $S = A\frac{1-e^{iN\delta}}{1-e^{i\delta}}$.
(b) Show that the intensity $I = |S|^2$ is given by:

$$I = A^2\frac{\sin^2(N\delta/2)}{\sin^2(\delta/2)}.$$

(c) Sketch the intensity I as a function of δ for $N = 3$. Where do the principal maxima occur?