Calculus or Analysis I: Sequences and Convergence

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Contents

Chapter 1

Ideas & Motivations

Welcome to Calculus or Analysis I (with *some*[1] theory) by me (Gudfit). The point of these notes is to cover everything I think is important as I build up to my current knowledge, while keeping it free and accessible for everyone from kids to adults.

I aim for each set of notes to be max 100 pages, as rigorous as possible, and far-reaching too. That means I'll cover the axioms and proofs of the most interesting stuff, plus I'll pull in other subjects we've already touched on to show how math builds on itself like funky Lego. These notes build on my existing informal logic, algebra I notes and geometry notes, and they're aimed at keeping the proofs, ideas, and build-up of calculus as informal as possible.

It'll be a mix of quick ideas and concepts, but in the appendix for each section, I'll go rigorous with the key axioms pulled from a bunch of books.

As you browse the contents, you may notice that these notes are somewhat dense in terms of theory. This is by design. Many standard Calculus concepts are simply extrapolations of Real Analysis; when we gloss over these roots, it becomes difficult to see the true beauty and underlying simplicity that Calculus offers.

Consequently, we are taking the rigorous "Analysis route." We will build the machinery from the ground up across a series of notes:

Part I (This Document): Limits and Convergence.
Part II: Topology, Functions, Differentiation, and Integration.
Part III: Multivariable Calculus.
Part IV: Calculus on Manifolds.
Part V?: Metrics Spaces.

[1] LOL

Chapter 2

Number System

This chapter will act as a sort of review of the algebra, set theory, and concrete abstraction notes with some other fun stuff sprinkled in, as the heart of analysis is the differentiation and integration of functions defined on the real line. We are (or should be) familiar with several systems of numbers:

- The natural numbers, $\mathbb{N} = \{0, 1, 2, 3, \dots\}$, used for counting.
- The integers, $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$, which extend the natural numbers to include zero and their negatives, allowing for subtraction.
- The rational numbers, $\mathbb{Q}$, the set of all numbers of the form a/b where $a, b \in \mathbb{Z}$ and $b \neq 0$. This set allows for division by any non-zero number.
- The real numbers, $\mathbb{R}$, which complete the rational numbers, filling in "gaps" with irrational numbers such as $\sqrt{2}$ and π.

Visually, the rational numbers form a dense collection of points on the number line, yet they do not cover it entirely. The real numbers represent the entire continuum of the number line, as suggested in Figure 2.1.

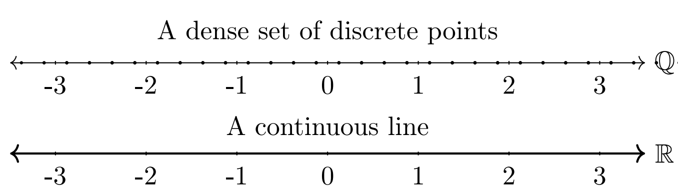


Figure 2.1: A conceptual representation of the rational and real number lines.

We will define the real numbers as any set that satisfies a specific list of axioms.

2.1 The Field Axioms

The real numbers, denoted $\mathbb{R}$, are a set containing two distinct special elements, 0 and 1, and equipped with two binary operations, addition $(+)$ and multiplication $(\cdot)$. This structure satisfies a collection of axioms, labelled P1 through P13. The first nine, known as the field axioms, define a mathematical structure called a field.

Axioms of Addition

The first four axioms govern the behaviour of addition.

Axiom 2.1.1. Additive Associativity. For all $a, b, c \in \mathbb{R}$, $a + (b + c) = (a + b) + c$.

***Axiom* 2.1.2. *Additive Identity*.** For all $a \in \mathbb{R}$, $a + 0 = 0 + a = a$.

***Axiom* 2.1.3. *Additive Inverse*.** For all $a \in \mathbb{R}$, there exists a number $-a \in \mathbb{R}$ such that $a + (-a) = (-a) + a = 0$.

***Axiom* 2.1.4. *Additive Commutativity*.** For all $a, b \in \mathbb{R}$, $a + b = b + a$.

Additive Associativity ensures that the grouping of terms in a sum is irrelevant, so an expression like $a+b+c$ is unambiguous. Similarly, Additive Commutativity allows us to reorder terms in a sum without changing the result.

***Theorem* 2.1.1. Uniqueness of Additive Identity.** If x and a are numbers satisfying $a + x = a$, then $x = 0$.

Remark. The proofs for elementary theorems derivable from these axioms were covered in the algebra notes and will be omitted here.

Definition 2.1.1. *Subtraction*. For any numbers $a, b \in \mathbb{R}$, we define $a - b$ to mean $a + (-b)$.

Axioms of Multiplication

Multiplication behaves similarly to addition.

***Axiom* 2.1.5. *Multiplicative Associativity*.** For all $a, b, c \in \mathbb{R}$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

***Axiom* 2.1.6. *Multiplicative Identity*.** For all $a \in \mathbb{R}$, $a \cdot 1 = 1 \cdot a = a$.

***Axiom* 2.1.7. *Multiplicative Inverse*.** For all $a \in \mathbb{R}$, if $a \neq 0$, there exists a number $a^{-1} \in \mathbb{R}$ such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$.

***Axiom* 2.1.8. *Multiplicative Commutativity*.** For all $a, b \in \mathbb{R}$, $a \cdot b = b \cdot a$.

Note. The crucial exception in Multiplicative Inverse is that 0 does not have a multiplicative inverse.

Definition 2.1.2. *Division*. For any numbers $a, b \in \mathbb{R}$ with $b \neq 0$, we define a/b to mean $a \cdot b^{-1}$.

The Distributive Axiom

The ninth axiom connects the operations of addition and multiplication.

***Axiom* 2.1.9. *Distributivity*.** For all $a, b, c \in \mathbb{R}$, $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

This axiom is fundamental to algebraic manipulation.

***Theorem* 2.1.2.** For any $a \in \mathbb{R}$, $a \cdot 0 = 0$.

***Theorem* 2.1.3.** If $a \cdot b = 0$, then either $a = 0$ or $b = 0$.

Proof. Suppose $a \cdot b = 0$. If $a = 0$, the statement holds. If $a \neq 0$, then a^{-1} exists by Multiplicative Inverse. Multiplying $a \cdot b = 0$ by a^{-1} gives $a^{-1} \cdot (a \cdot b) = a^{-1} \cdot 0$, which simplifies to $(a^{-1} \cdot a) \cdot b = 0$, so $1 \cdot b = 0$, and thus $b = 0$. ∎

2.1.1 The Axioms of Order

To define order, we first formally define an ordered field.

Definition 2.1.3. *Ordered Field*. An ordered field is a field $\mathbb{F}$ with a total order relation $<$ such that for all $x, y, z \in \mathbb{F}$:

1. if $x < y$, then $x + z < y + z$;
2. if $x < y$ and $z > 0$, then $xz < yz$.

To establish this structure for $\mathbb{R}$, we introduce a set of positive numbers, $P \subset \mathbb{R}$, and three axioms.

***Axiom* 2.1.10. *Trichotomy*.** For every $a \in \mathbb{R}$, exactly one of the following holds: $a = 0$, $a \in P$, or $-a \in P$.

***Axiom* 2.1.11. *Closure under Addition*.** If $a, b \in P$, then $a + b \in P$.

***Axiom* 2.1.12. *Closure under Multiplication*.** If $a, b \in P$, then $a \cdot b \in P$.

Numbers in P are called positive. Numbers a such that $-a \in P$ are called negative. We define order relations based on this set.

Definition 2.1.4. *Order Relations*.

- $a > b$ means $a - b \in P$.
- $a < b$ means $b > a$ (or $b - a \in P$).
- $a \geq b$ means $a > b$ or $a = b$.
- $a \leq b$ means $a < b$ or $a = b$.

The axioms P1–P12 define $\mathbb{R}$ as an ordered field. Both $\mathbb{Q}$ and $\mathbb{R}$ are ordered fields.

A Non-Ordered Field: The Complex Numbers

The axioms for an ordered field are not sufficient to uniquely characterise the real numbers, as the set of rational numbers, $\mathbb{Q}$, also satisfies all twelve. To illustrate the distinctions these axioms create, we consider a field that fails to be an ordered field: the complex numbers.

The set of complex numbers, $\mathbb{C}$, consists of elements of the form $a + bi$, where $a, b \in \mathbb{R}$ and i is a symbol defined by $i^2 = -1$. Arithmetic in $\mathbb{C}$ is defined as:

- Addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$
- Multiplication: $(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i$

With these operations, $\mathbb{C}$ satisfies the field axioms (axioms 2.1.1–2.1.9). A key result, the Fundamental Theorem of Algebra, states that any non-constant polynomial equation has a solution in $\mathbb{C}$. However, this algebraic completeness comes at a cost.

***Theorem* 2.1.4. The Complex Numbers are Not an Ordered Field.** It is not possible to define a subset of "positive" complex numbers $P \subset \mathbb{C}$ that satisfies the order axioms Trichotomy, Closure under Addition, and Closure under Multiplication.

Proof. Assume for contradiction that such a set P exists. Consider the element i. Since $i \neq 0$, by Trichotomy, either $i \in P$ or $-i \in P$. In any ordered field, the square of a non-zero element must be positive. If $i \in P$, then $i^2 = -1 \in P$ by Closure under Multiplication. If $-i \in P$, then $(-i)^2 = -1 \in P$ by Closure under Multiplication. In both cases, we must have $-1 \in P$. However, we also know that $1^2 = 1$, so $1 \in P$. By Trichotomy, if $1 \in P$, its additive inverse, -1, cannot be in P. This is a contradiction. Therefore, no such ordering is possible. ∎

***Theorem* 2.1.5. Properties of Inequalities.** For $a, b, c \in \mathbb{R}$:

1. If $a < b$, then $a + c < b + c$.
2. If $a < b$ and $c > 0$, then $ac < bc$.
3. If $a < b$ and $c < 0$, then $ac > bc$.
4. If $a < b$ and $b < c$, then $a < c$ (Transitivity).
5. $a^2 \geq 0$ for all $a \in \mathbb{R}$, and $a^2 = 0$ if and only if $a = 0$.

You should be able to prove this by now.

Absolute Value

If you've read my previous notes you should know that the absolute value measures a number's magnitude.

Definition 2.1.5. ***Absolute Value***. For a real number a, its absolute value, denoted $|a|$, is defined as $|a| = \sqrt{a^2}$. This is equivalent to:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

***Theorem* 2.1.6. Properties of Absolute Value**. Let $a, b \in \mathbb{R}$ and $\epsilon > 0$.

1. $|a| \geq 0$, and $|a| = 0 \iff a = 0$.
2. $|ab| = |a||b|$.
3. $|a| < \epsilon \iff -\epsilon < a < \epsilon$.
4. $|a| > \epsilon \iff a < -\epsilon$ or $a > \epsilon$.
5. **Triangle Inequality**: $|a + b| \leq |a| + |b|$.
6. **Reverse Triangle Inequality**: $||a| - |b|| \leq |a - b|$.

The triangle inequality generalises to any number of terms: $|a_1 + \cdots + a_n| \leq |a_1| + \cdots + |a_n|$.

Note. If you haven't already you can and should prove this using induction.

So why am i reintroducing the absolute value function? well this function appears so frequently in calculus, most notably in the definitions of limits and continuity. The standard approach to expressions involving absolute values is to break the problem into cases, such that within each case, the absolute value signs can be removed.

Example 2.1.1. Solving an Inequality with Absolute Values. Find all real x such that $|x - 1| + |x - 2| > 1$.

The expressions change sign at $x = 1$ and $x = 2$. We consider the intervals defined by these points.

Case 1: $x < 1$. Here $|x - 1| = -(x - 1) = 1 - x$ and $|x - 2| = -(x - 2) = 2 - x$. The inequality is $(1 - x) + (2 - x) > 1$, which simplifies to $3 - 2x > 1$, or $x < 1$. This holds for all x in this case, so the interval $(-\infty, 1)$ is part of the solution.

Case 2: $1 \leq x \leq 2$. Here $|x - 1| = x - 1$ and $|x - 2| = 2 - x$. The inequality is $(x - 1) + (2 - x) > 1$, which simplifies to $1 > 1$. This is false, so there are no solutions in this interval.

Case 3: $x > 2$. Here $|x - 1| = x - 1$ and $|x - 2| = x - 2$. The inequality is $(x - 1) + (x - 2) > 1$, which simplifies to $2x - 3 > 1$, or $x > 2$. This holds for all x in this case, so the interval $(2, \infty)$ is part of the solution.

Combining the results, the solution set is $(-\infty, 1) \cup (2, \infty)$.

Example 2.1.2. An Inequality for Large x. Prove that there exists a number $M > 0$ such that for all real numbers $x > M$, the following inequality holds:

$$\left| \frac{x^2}{x^2 + x - 2} - 1 \right| < \frac{1}{10}.$$

Proof. First, we simplify the expression within the absolute value:

$$\begin{aligned} \left| \frac{x^2}{x^2 + x - 2} - 1 \right| &= \left| \frac{x^2 - (x^2 + x - 2)}{x^2 + x - 2} \right| \\ &= \left| \frac{-x + 2}{x^2 + x - 2} \right| = \frac{|2 - x|}{|(x - 1)(x + 2)|}. \end{aligned}$$

The problem requires the property to hold for all $x > M$. We are free to choose M as large as needed to simplify the analysis. Let us search for an $M > 2$. For any $x > M$, we then have $x > 2$, which implies $x - 1 > 1$, $x + 2 > 4$, and $x - 2 > 0$. Consequently, $|2 - x| = x - 2$ and the denominator is positive, so the expression simplifies to:

$$\frac{x-2}{(x-1)(x+2)}.$$

To show this expression is less than $\frac{1}{10}$, we find a simpler upper bound. For $x > 2$, we know that $x-2 < x-1$. As the denominator is positive, we can write:

$$\frac{x-2}{(x-1)(x+2)} < \frac{x-1}{(x-1)(x+2)} = \frac{1}{x+2}.$$

Therefore, the original inequality is satisfied if we can find an M such that for all $x > M$,

$$\frac{1}{x+2} < \frac{1}{10}.$$

This is equivalent to $x + 2 > 10$, which simplifies to $x > 8$.

By choosing $M = 8$, we satisfy our initial assumption ($M > 2$) and the final condition. For any $x > 8$, all steps in the argument are valid, and we have

$$\left|\frac{x^2}{x^2+x-2} - 1\right| = \frac{x-2}{(x-1)(x+2)} < \frac{1}{x+2} < \frac{1}{10}.$$

This proves the existence of such an M. ■

Remark. This type of argument, finding a bound M for which a property holds for all $x > M$, is fundamental to the formal definition of a limit at infinity, a key concept in calculus.

2.2 The Completeness Axiom

The ordered field axioms are insufficient to distinguish the rational numbers $\mathbb{Q}$ from the real numbers $\mathbb{R}$, as $\mathbb{Q}$ also satisfies them. The deficiency of the rational numbers is that they contain "gaps", as demonstrated by the irrationality of numbers like $\sqrt{2}$.

Preliminaries for Irrationality

To prove rigorously that $\sqrt{2}$ is irrational, we first establish some elementary properties of integers.

***Proposition* 2.2.1.** An integer is either even or odd, but not both.

Proof. Suppose, for contradiction, that an integer a is both even and odd. Then there exist integers m and n such that $a = 2m$ and $a = 2n + 1$. Equating these gives $2m = 2n + 1$, which implies $2(m - n) = 1$, or $m - n = 1/2$. This is a contradiction, as the difference of two integers must be an integer. ■

***Theorem* 2.2.1. Parity of Squares**. An integer a is even if and only if its square a^2 is even.

Proof. ($\Rightarrow$) If a is even, then $a = 2k$ for some integer k. So $a^2 = (2k)^2 = 4k^2 = 2(2k^2)$, which is an even integer. ($\Leftarrow$) We prove the contrapositive: if a is odd, then a^2 is odd. If a is odd, then $a = 2k + 1$ for some integer k. So $a^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is an odd integer. Therefore, if a^2 is even, a must be even. ■

***Theorem* 2.2.2. Lowest Terms Representation**. Every rational number can be written as a fraction a/b where $a, b \in \mathbb{Z}$ and a and b are not both even.

Proof. Let a rational number be a_0/b_0. If both a_0 and b_0 are even, we can write $a_0 = 2a_1$ and $b_0 = 2b_1$, so $a_0/b_0 = a_1/b_1$. We can repeat this process if a_1 and b_1 are also both even. This generates a sequence of positive integers $|b_0| > |b_1| > |b_2| > \dots$. Such a strictly decreasing sequence of positive integers cannot continue indefinitely. Therefore, the process must terminate at a pair (a_n, b_n) where at least one of a_n or b_n is odd. ■

***Theorem* 2.2.3.** **$\sqrt{2}$ is irrational**. There is no rational number r such that $r^2 = 2$.

Proof. Assume, for contradiction, that $\sqrt{2}$ is rational. Then $\sqrt{2} = a/b$ for integers a, b. By the previous theorem, we can assume that not both a and b are even. Squaring both sides gives $2 = a^2/b^2$, which implies $2b^2 = a^2$. This shows that a^2 is an even number. By the theorem on the parity of squares, a must also be even. Let $a = 2k$ for some integer k. Substituting this into the equation gives $2b^2 = (2k)^2 = 4k^2$, which simplifies to $b^2 = 2k^2$. This implies b^2 is even, and thus b must also be even. We have shown that both a and b must be even, which contradicts our initial assumption that not both are even. Therefore, the initial assumption that $\sqrt{2}$ is rational must be false. ■

To address the "gaps" in $\mathbb{Q}$, we introduce the final axiom of the real numbers. First, we need some definitions.

Definition 2.2.1. ***Bounds***. Let $S \subseteq \mathbb{R}$ be a non-empty set.

1. A real number u is an **upper bound** for S if $s \leq u$ for all $s \in S$. The set S is said to be **bounded above** if it has an upper bound.
2. A real number l is a **lower bound** for S if $l \leq s$ for all $s \in S$. The set S is said to be **bounded below** if it has a lower bound.
3. The set S is said to be **bounded** if it is bounded both above and below.

Example 2.2.1.

- The interval $(-\infty, 1)$ is bounded above but not below. The interval $(0, \infty)$ is bounded below but not above. The interval $[0, 1)$ is bounded.
- Consider the set $S = \{x \in \mathbb{R} \mid x^2 < 2\}$. This set is non-empty, as $1 \in S$. We claim that 2 is an upper bound. Assume for contradiction that there is some $x \in S$ with $x > 2$. Then $x^2 > 4$, which contradicts the condition $x^2 < 2$. Thus, for all $x \in S$, we must have $x \leq 2$, meaning 2 is an upper bound for S.

Definition 2.2.2. ***Supremum and Infimum***. Let $S \subseteq \mathbb{R}$ be a non-empty set.

- The **least upper bound** (or supremum) of S, denoted $\sup S$, is an upper bound α for S such that for any other upper bound u of S, we have $\alpha \leq u$.
- The **greatest lower bound** (or infimum) of S, denoted $\inf S$, is a lower bound β for S such that for any other lower bound l of S, we have $l \leq \beta$.

***Proposition* 2.2.2.** *Uniqueness of Supremum.* A non-empty set $S \subseteq \mathbb{R}$ has at most one supremum.

Proof. Suppose α_1 and α_2 are both suprema of S. Since α_1 is a supremum and α_2 is an upper bound, the definition of supremum implies $\alpha_1 \leq \alpha_2$. Similarly, since α_2 is a supremum and α_1 is an upper bound, we have $\alpha_2 \leq \alpha_1$. Combining these inequalities gives $\alpha_1 = \alpha_2$. A similar argument holds for the uniqueness of the infimum. ■

***Axiom* 2.2.1. *The Completeness Axiom*.** Every non-empty set of real numbers that is bounded above has a least upper bound in $\mathbb{R}$.

The set of axioms for an ordered field, together with The Completeness Axiom, defines $\mathbb{R}$ as a complete ordered field.

Definition 2.2.3. ***Maximum and Minimum***. Let $S \subseteq \mathbb{R}$ be a non-empty set.

- If S is bounded above and $\sup S \in S$, then $\sup S$ is called the maximum of S, denoted $\max S$.
- If S is bounded below and $\inf S \in S$, then $\inf S$ is called the minimum of S, denoted $\min S$.

A maximum or minimum must belong to the set, whereas a supremum or infimum need not.

Example 2.2.2.

- For $S_1 = (-\infty, 1)$, the set of upper bounds is $[1, \infty)$. Thus, $\sup S_1 = 1$. The set has no maximum, as $1 \notin S_1$.
- For $S_2 = [0, 1)$, $\sup S_2 = 1$ and $\inf S_2 = 0$. The set has no maximum, but it has a minimum, $\min S_2 = 0$, since $0 \in S_2$.
- For $S_3 = \{1/n \mid n \in \mathbb{N}, n \geq 1\}$. We have $\sup S_3 = 1$, which is also the maximum. The infimum is $\inf S_3 = 0$, which is not a minimum as $0 \notin S_3$.

***Theorem* 2.2.4. $\mathbb{Q}$ is not complete**. The set of rational numbers does not satisfy the completeness axiom.

Proof. Consider the set $S = \{r \in \mathbb{Q} \mid r^2 \leq 2\}$. This set is non-empty (since $1 \in S$) and is bounded above (by 2, for instance). If $\mathbb{Q}$ were complete, S would have a least upper bound in $\mathbb{Q}$. Let this hypothetical supremum be $b \in \mathbb{Q}$. Since $\sqrt{2}$ is not rational, we know $b^2 \neq 2$. By the trichotomy property, either $b^2 < 2$ or $b^2 > 2$.

Case 1: $b^2 < 2$. We will show that b is not an upper bound by constructing a rational number $b + h \in S$ for some small rational $h > 0$. We require $(b + h)^2 < 2$, which is $b^2 + 2bh + h^2 < 2$. This is equivalent to $h(2b + h) < 2 - b^2$. If we choose a rational $h \in (0, 1)$, then $h < 1$, so $h(2b + h) < h(2b + 1)$. We can satisfy the inequality by choosing h such that $h(2b + 1) < 2 - b^2$. Such a rational h exists, for example $h = \min\left(\frac{1}{2}, \frac{2-b^2}{2(2b+1)}\right)$. For this h, $(b+h)^2 < 2$, so $b+h \in S$, which contradicts b being an upper bound.

Case 2: $b^2 > 2$. We will show b is not the least upper bound by finding a smaller rational upper bound, $b-h$, for some small rational $h > 0$. We require $(b-h)^2 > 2$ for any element in S. Consider $b^2 - 2bh + h^2 > 2$. This is equivalent to $b^2 - 2 > 2bh - h^2$. Since $h > 0$, $2bh - h^2 < 2bh$. We can satisfy the inequality by finding an h such that $b^2 - 2 > 2bh$. Such a rational h exists, for example $h = \frac{b^2-2}{2b}$. For this h, $b - h$ is an upper bound for S that is smaller than b, contradicting that b is the least upper bound.

Since both cases lead to a contradiction, no such least upper bound b can exist in $\mathbb{Q}$. ∎

The completeness axiom has a symmetric counterpart for lower bounds.

***Theorem* 2.2.5. Greatest Lower Bound Property**. Every non-empty set of real numbers that is bounded below has a greatest lower bound in $\mathbb{R}$.

Proof. Let S be a non-empty set that is bounded below. Consider the set $-S = \{-s \mid s \in S\}$. If l is a lower bound for S, then for any $s \in S$, $l \leq s$, which implies $-l \geq -s$. Thus, $-l$ is an upper bound for $-S$. Since $-S$ is non-empty and bounded above, by The Completeness Axiom it has a supremum, $\alpha = \sup(-S)$. We claim that $-\alpha$ is the infimum of S. First, we show $-\alpha$ is a lower bound for S. Since $\alpha = \sup(-S)$, for any $-s \in -S$, we have $-s \leq \alpha$, which means $s \geq -\alpha$. This holds for all $s \in S$, so $-\alpha$ is a lower bound. Next, we show it is the greatest lower bound. Let l be any lower bound of S. Then $-l$ is an upper bound of $-S$. Since α is the least upper bound of $-S$, we have $\alpha \leq -l$, which implies $-\alpha \geq l$. Thus, $-\alpha$ is greater than or equal to any other lower bound, making it the infimum of S. ∎

The definition of a supremum is often more conveniently applied in proofs using the following equivalent characterisation.

***Theorem* 2.2.6. Characterisation of Supremum**. Let $S \subseteq \mathbb{R}$ be a non-empty set bounded above. A number α is the supremum of S if and only if:

1. α is an upper bound for S.
2. For any $\varepsilon > 0$, there exists an element $s \in S$ such that $s > \alpha - \varepsilon$.

Proof. ($\Rightarrow$) Assume $\alpha = \sup S$. By definition, α is an upper bound, satisfying the first condition. For the second condition, let $\varepsilon > 0$ be given. Since $\alpha - \varepsilon < \alpha$, and α is the *least* upper bound, the number $\alpha - \varepsilon$ cannot be an upper bound for S. This implies the existence of some $s \in S$ such that $s > \alpha - \varepsilon$.

($\Leftarrow$) Assume conditions (1) and (2) hold. We must show that α is the least upper bound. Let u be any other upper bound for S. We need to show that $\alpha \leq u$. Assume for contradiction that $u < \alpha$. Let $\varepsilon = \alpha - u$. Since $u < \alpha$, we have $\varepsilon > 0$. By condition (2), there exists an $s \in S$ such that $s > \alpha - \varepsilon = \alpha - (\alpha - u) = u$. This result, $s > u$, contradicts the fact that u is an upper bound for S. Therefore, the assumption $u < \alpha$ must be false, so $\alpha \leq u$. This confirms that α is the least upper bound. ■

Consequences of Completeness

Alas the completeness axiom has several profound consequences that shape the structure of the real number line most notably:

***Theorem* 2.2.7. Archimedean Property**. The set of natural numbers $\mathbb{N}$ is not bounded above.

Proof. Assume for contradiction that $\mathbb{N}$ is bounded above. As $\mathbb{N}$ is non-empty, by the The Completeness Axiom axiom it must have a least upper bound, $\alpha = \sup \mathbb{N}$. By the characterisation of the supremum, for $h = 1$, there must be some $n \in \mathbb{N}$ such that $n > \alpha - 1$. This implies $n + 1 > \alpha$. Since $n + 1$ is also a natural number, this contradicts α being an upper bound for $\mathbb{N}$. ■

Closely related to this is the Archimedean property of the real numbers:

***Corollary* 2.2.1.** For any $x \in \mathbb{R}$ ("large" real number) and $\epsilon \in \mathbb{R}$ ("really small" real number) such that $0 < \epsilon < x$, there exists $n \in \mathbb{N}$ such that $r < n\epsilon$.

Proof. Suppose for sake of contradiction this was false, then there is some $r > 0$ and $\epsilon > 0$ such that $n\epsilon \leq r$ for all $n \in \mathbb{N}$, so $n \leq (r/\epsilon)$ which implies $\mathbb{N}$ is bounded above which is a contradiction. ■

***Corollary* 2.2.2.** For any $x \in \mathbb{R}$, there exist integers m and n such that $m < x < n$.

Proof. Since $\mathbb{N}$ is not bounded above, x cannot be an upper bound. Thus, there exists an integer $n \in \mathbb{N}$ such that $n > x$. Similarly, $-x$ is not an upper bound for $\mathbb{N}$, so there exists $k \in \mathbb{N}$ such that $k > -x$, which implies $-k < x$. Let $m = -k$. Then m is an integer and $m < x < n$. ■

***Corollary* 2.2.3.** For any $\epsilon > 0$, there exists an $n \in \mathbb{N}$ such that $0 < 1/n < \epsilon$.

Proof. Consider $1/\epsilon$. By the Archimedean Property, there is an integer n such that $n > 1/\epsilon$. Since $\epsilon > 0$, we can rearrange this inequality to $1/n < \epsilon$. As $n \in \mathbb{N}$, $n > 0$, so $1/n > 0$. ■

These corollaries collectively show that integers are "spread out" enough to capture any real number. We can now formalize this into a more precise statement.

***Theorem* 2.2.8. Existence of Integer Part**. For any $x \in \mathbb{R}$, there is exactly one integer n such that $n \leq x < n + 1$.

Proof. Consider the set $S = \{k \in \mathbb{Z} \mid k \leq x\}$. By the Archimedean property, S is non-empty and it is bounded above by x. By the completeness axiom (extended to sets of integers), S must have a supremum which is also its maximum element, n. By definition, $n \in \mathbb{Z}$ and $n \leq x$. Since n is maximal in S, $n + 1 \notin S$, which means $n + 1 > x$. Thus, $n \leq x < n + 1$. Uniqueness follows because if m were another such integer, the distance between n and m would be less than 1, impossible for distinct integers. ■

One of the key implications of these properties is the distribution of rational and irrational numbers within the reals.

Density in $\mathbb{R}$

Definition 2.2.4. ***Dense Set.*** A set $S \subseteq \mathbb{R}$ is dense in $\mathbb{R}$ if for all $x < y$ in $\mathbb{R}$, there is an element of S in the interval (x, y). We also say that S is a dense subset of $\mathbb{R}$.

For example, the set of reals itself forms a dense subset of the reals, rather trivially, as does the set of reals minus one point. The set of positive numbers is not dense (there is no positive number between -2 and -1), and nor is the set of integers (there is no integer between 1.1 and 1.9).

***Theorem* 2.2.9. Approximation Property for Rationals.** For any real number x and any $\epsilon > 0$, there exists a rational number r such that $|x - r| < \epsilon$.

Proof. Let $x \in \mathbb{R}$ and $\epsilon > 0$ be given. By the Archimedean property, there exists an integer $n \in \mathbb{N}$ such that $n > 1/\epsilon$, or $1/n < \epsilon$. Consider the real number nx. By the Existence of Integer Part theorem, there is an integer m such that $m \leq nx < m + 1$. Dividing by n gives $m/n \leq x < (m + 1)/n$. Let $r = m/n$, which is a rational number. From the first inequality, $x - r \geq 0$. From the second, $x < m/n + 1/n = r + 1/n$, which means $x - r < 1/n$. Combining these, we have $0 \leq x - r < 1/n$. Since $1/n < \epsilon$, we have $|x - r| < \epsilon$. ∎

Remark. This theorem demonstrates that the rational numbers are dense in the real numbers. It implies that between any two distinct real numbers, there exists a rational number.

Note. For the sake of completion here is the second proof.

***Theorem* 2.2.10. $\mathbb{Q}$ is dense in $\mathbb{R}$.** If x, y are reals with $x < y$, then there is a rational number in the interval (x, y).

Proof. We first prove the result for $0 \leq x < y$. The general case follows: if $x < y \leq 0$, there exists a rational $r \in (-y, -x)$, so $-r$ is a rational in (x, y); if $x < 0 < y$, any rational in $(0, y)$ is also in (x, y).

So, let $0 \leq x < y$ be given. By the Consequences of Completeness, there is a natural number n with $1/n < y - x$. The informal idea behind the proof is that the gaps between consecutive elements in the sequence $\{\ldots, -2/n, -1/n, 0, 1/n, 2/n, \ldots\}$ are all smaller than the distance between x and y, so one of these rational numbers must fall between them.

Formally: As $\mathbb{N}$ is unbounded, there exists $m \in \mathbb{N}$ with $m \geq ny$. Let m_1 be the least such integer (an application of the well-ordering principle). Note that $m_1 > 1$, because $y - x > 1/n \implies y > 1/n \implies ny > 1$. Consider $(m_1 - 1)/n$. We have $m_1 - 1 < ny$ (else m_1 would not be the least integer $\geq ny$) and so $(m_1 - 1)/n < y$. If $(m_1 - 1)/n \leq x$, then together with $y \leq m_1/n$ this implies $y - x \leq 1/n$, a contradiction to our choice of n. Thus we must have $(m_1 - 1)/n > x$, and so $(m_1 - 1)/n \in (x, y)$. ∎

***Proposition* 2.2.3.** *Properties of Rational/Irrational Operations.* Let $r \in \mathbb{Q}$ and $x, y \in \mathbb{R} \setminus \mathbb{Q}$. Then:

1. $r + x \in \mathbb{R} \setminus \mathbb{Q}$.
2. $rx \in \mathbb{R} \setminus \mathbb{Q}$ for $r \neq 0$.
3. $x + y$ and xy may be rational or irrational. For example, $\sqrt{2} + (-\sqrt{2}) = 0 \in \mathbb{Q}$, but $\sqrt{2} + \sqrt{3} \in \mathbb{R} \setminus \mathbb{Q}$. Also, $\sqrt{2} \cdot \sqrt{2} = 2 \in \mathbb{Q}$, but $\sqrt{2} \cdot \sqrt{3} = \sqrt{6} \in \mathbb{R} \setminus \mathbb{Q}$.

Proof. For (1), suppose for contradiction that $r + x = q$ for some $q \in \mathbb{Q}$. Then $x = q - r$. Since $\mathbb{Q}$ is a field, $q - r \in \mathbb{Q}$, which contradicts that x is irrational. The proof for (2) is analogous. ∎

***Theorem* 2.2.11. $(\mathbb{R} \setminus \mathbb{Q})$ is dense in $\mathbb{R}$.** Between any two distinct real numbers, there exists an irrational number.

Proof. Let $a < b$ be real numbers. By the Approximation Property, there exists a rational number r in the interval $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$. So $\frac{a}{\sqrt{2}} < r < \frac{b}{\sqrt{2}}$. Multiplying by $\sqrt{2}$ gives $a < r\sqrt{2} < b$. Since r is rational ($r \neq 0$) and $\sqrt{2}$ is irrational, their product $r\sqrt{2}$ is irrational. Thus, we have found an irrational number between a and b. ∎

2.3 Basic Topology of the Real Line

We conclude with a brief introduction to the topological structure of $\mathbb{R}$, which provides the language needed to discuss concepts like continuity and limits with precision.

Intervals and Neighbourhoods

Definition 2.3.1. ***Intervals***. An interval is a subset of $\mathbb{R}$ with the property that any number that lies between two numbers in the set is also included in the set. Common forms for $a < b$ include:

- Open interval: $(a, b) = \{x \in \mathbb{F} \mid a < x < b\}$
- Closed interval: $[a, b] = \{x \in \mathbb{F} \mid a \leq x \leq b\}$
- Half-open intervals: $[a, b)$ or $(a, b]$
- Unbounded intervals (rays): $[a, \infty)$, (a, ∞), $(-\infty, b]$, $(-\infty, b)$

Definition 2.3.2. ***Neighbourhood***. An **open neighbourhood** of a point $a \in \mathbb{R}$ is any open interval containing a. The δ-neighbourhood of a is the set

$$B_\delta(a) = \{x \in \mathbb{F} \mid |x - a| < \delta\} = (a - \delta, a + \delta)$$

for some radius $\delta > 0$. A **deleted neighbourhood** of a excludes the point a itself:

$$B'_\delta(a) = \{x \in \mathbb{F} \mid 0 < |x - a| < \delta\} = (a - \delta, a) \cup (a, a + \delta)$$

These concepts are illustrated in Figure 2.2.

Example 2.3.1. Finding the Centre and Radius of a Neighbourhood. Find a and δ for which $B_\delta(a) = (3, 11)$. The centre a is the midpoint of the interval, $a = (11 + 3)/2 = 7$. The radius δ is half the length of the interval, $\delta = (11 - 3)/2 = 4$. Thus $B_4(7) = (3, 11)$.

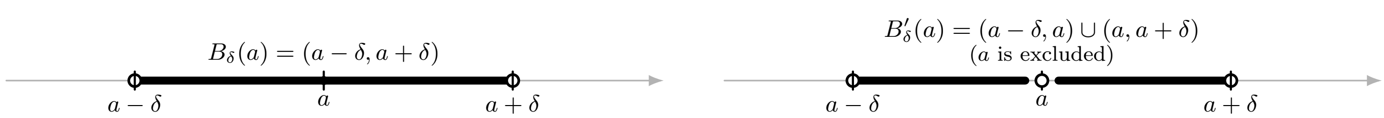


Figure 2.2: Open neighbourhood $B_\delta(a)$ (left) and *deleted* neighbourhood $B'_\delta(a)$ (right) on $\mathbb{R}$.

Topological Properties

Definition 2.3.3. ***Connected Subsets***. A set $U \subseteq \mathbb{R}$ is connected if for any two points in U, all points between them are also in U. The connected subsets of $\mathbb{R}$ are precisely the intervals.

Example 2.3.2. *Connected:* Let $U = [0, 5]$. For any two points $x, y \in U$, the entire interval $[x, y]$ remains inside U. *Disconnected:* Let $V = [0, 1] \cup [2, 3]$. There is a gap: while $0.5 \in V$ and $2.5 \in V$, the intermediate point $1.5 \notin V$.

Definition 2.3.4. ***Boundary Points***. A point $p \in \mathbb{R}$ is a boundary point of a set $U \subseteq \mathbb{R}$ if every neighbourhood of p contains at least one point in U and at least one point not in U. The set of all boundary points of U is denoted $\mathrm{bd}(U)$.

Remark. A boundary point of U need not be an element of U. For some sets, such as $\mathbb{N}$, every point is a boundary point, so $\mathrm{bd}(\mathbb{N}) = \mathbb{N}$.

Definition 2.3.5. ***Interior Points***. A point $p \in U$ is an interior point of $U \subseteq \mathbb{R}$ if there exists a neighbourhood of p that is entirely contained within U. That is, if there exists an $\epsilon > 0$ such that $B_\epsilon(p) \subseteq U$. The set of all interior points of U is denoted $\mathrm{int}(U)$.

Example 2.3.3. Boundary & Interior of an Interval. For the set $U = [0, 1) = \{x \in \mathbb{R} \mid 0 \le x < 1\}$:

- The boundary is $\mathrm{bd}(U) = \{0, 1\}$.
- The interior is $\mathrm{int}(U) = (0, 1)$.

Example 2.3.4. Boundary and Interior of a Union of Intervals. Suppose $U = \{x \in \mathbb{R} \mid x < -8 \text{ or } 10 \le x < 13\} = (-\infty, -8) \cup [10, 13)$.

- The boundary is $\mathrm{bd}(U) = \{-8, 10, 13\}$.
- The interior is $\mathrm{int}(U) = (-\infty, -8) \cup (10, 13)$.

Remark. Although $10 \in U$, it is not an interior point because any open neighbourhood of 10 contains points less than 10, which are not in U. Therefore, $10 \notin \mathrm{int}(U)$.

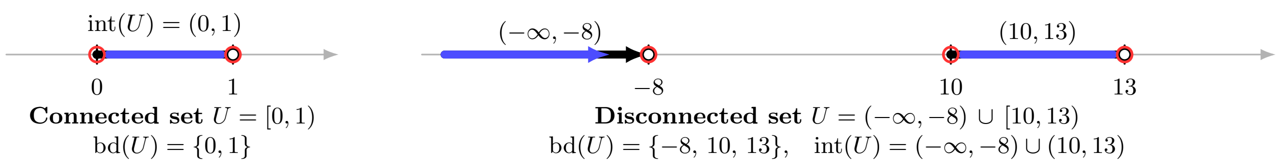


Figure 2.3: Left: a connected set (interval) with its interior and boundary. Right: a disconnected set with its interior and boundary.

2.4 Nested Intervals and Completeness

The Completeness Axiom (The Completeness Axiom) characterises the real numbers by asserting the existence of a supremum for every bounded non-empty set. However, an alternative characterisation exists, one that is constructive in nature and illuminates the topological structure of the real line: the Nested Interval Property.

Definition 2.4.1. ***Nested Interval Property***. An ordered field $\mathbb{F}$ is said to possess the *Nested Interval Property (NIP)* if for every sequence of closed intervals $I_n = [a_n, b_n]$ such that $I_{n+1} \subseteq I_n$ for all $n \in \mathbb{N}$, the intersection is non-empty:

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset$$

Geometrically, this property implies that a sequence of intervals shrinking inside one another must "trap" at least one point. While $\mathbb{Q}$ satisfies the Archimedean Property, it fails the NIP (one can construct nested rational intervals converging to $\sqrt{2}$ with empty intersection in $\mathbb{Q}$). Conversely, one can construct non-Archimedean fields that satisfy the NIP but are not complete. It is the conjunction of these two properties that implies completeness.

Example 2.4.1. Trapping a Single Point. Consider the sequence of intervals given by $I_n = [0, \frac{1}{n}]$ for all $n \in \mathbb{N}$.

- Observe that $I_1 = [0, 1]$, $I_2 = [0, \frac{1}{2}]$, etc.
- Since $\frac{1}{n+1} < \frac{1}{n}$, we have $[0, \frac{1}{n+1}] \subset [0, \frac{1}{n}]$, so the sequence is nested.

By the Nested Interval Property, the intersection is non-empty. In this specific case, the only element common to all intervals is 0 (by the Archimedean Property), so:

$$\bigcap_{n=1}^{\infty} \left[0, \frac{1}{n}\right] = \{0\}$$

***Theorem* 2.4.1. Equivalence of Completeness**. Let $\mathbb{F}$ be an ordered field. If $\mathbb{F}$ satisfies the Nested Interval Property and the Archimedean Property, then $\mathbb{F}$ is complete.

Proof. Let $S \subseteq \mathbb{F}$ be a non-empty set bounded above. We wish to show that $\sup S$ exists in $\mathbb{F}$. We employ the method of successive bisection to approximate the supremum. Let b_0 be an upper bound of S. Since S is non-empty, we choose an element $a_0 \in S$. Note that $a_0 \leq b_0$. We define a sequence of intervals $I_n = [a_n, b_n]$ recursively.

(i) Let $I_0 = [a_0, b_0]$.
(ii) Given $I_n = [a_n, b_n]$, let $m_n = \frac{a_n+b_n}{2}$ be the midpoint.
(iii) If m_n is an upper bound for S, set $I_{n+1} = [a_n, m_n]$ (i.e., $a_{n+1} = a_n, b_{n+1} = m_n$).
(iv) If m_n is not an upper bound for S, set $I_{n+1} = [m_n, b_n]$ (i.e., $a_{n+1} = m_n, b_{n+1} = b_n$).

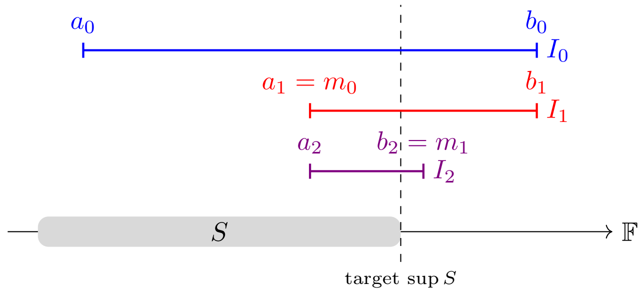


Figure 2.4: The construction of nested intervals narrowing down to the supremum. If the midpoint is an upper bound, we discard the right half; otherwise, we discard the left half.

By construction, $I_{n+1} \subseteq I_n$. Furthermore, for every n, b_n is an upper bound for S, and a_n is not an upper bound (or technically, there exists $x \in S$ such that $x \geq a_n$, specifically since $a_0 \in S$ and the sequence a_n is non-decreasing). By the Nested Interval Property, there exists a unique element $\xi \in \bigcap_{n=0}^{\infty} I_n$. We claim $\xi = \sup S$.

The length of the interval I_n is $(b_0 - a_0)/2^n$. By the Archimedean Property, for any $\epsilon > 0$, there exists n large enough such that the length of I_n is less than ϵ. This implies that ξ is the *unique* point in the intersection.

Step 1: ξ is an upper bound. Suppose ξ is not an upper bound. Then there exists $s \in S$ such that $s > \xi$. Let $\epsilon = s - \xi > 0$. By the Archimedean Property, we can choose n such that $b_n - a_n < \epsilon$. Since $\xi \in [a_n, b_n]$, we have $b_n - \xi < \epsilon$.

$$b_n < \xi + \epsilon = \xi + (s - \xi) = s$$

This implies $b_n < s$, which contradicts the fact that b_n is an upper bound for S. Thus, ξ is an upper bound.

Step 2: ξ is the least upper bound. Suppose there exists an upper bound u such that $u < \xi$. Let $\epsilon = \xi - u > 0$. Again, using the Archimedean Property, choose n such that the length of I_n is less than ϵ. Since $\xi \in [a_n, b_n]$, we have $\xi - a_n < \epsilon$.

$$a_n > \xi - \epsilon = \xi - (\xi - u) = u$$

Thus $a_n > u$. However, by our construction, a_n is not an upper bound (specifically, if $a_n > a_0$, a_n was a midpoint that was *not* an upper bound). This contradicts u being an upper bound (since $u < a_n$ and a_n is not an upper bound, u cannot be an upper bound). Therefore, $\xi = \sup S$. ∎

Note. The proof relies heavily on the Archimedean Property to force the interval lengths to zero. Without it, the "hole" might be infinitesimally small but non-zero, allowing the intersection to contain points that are not the supremum. Thus, NIP + Archimedean $\Leftrightarrow$ Completeness.

Uniqueness of the Real Numbers

We have established that $\mathbb{R}$ is a complete ordered field. A natural question arises: are there other systems that satisfy these axioms? The answer is effectively no.

***Theorem* 2.4.2.** There is, up to isomorphism, exactly one complete ordered field.

This theorem (the proof of which is beyond the scope of this review) assures us that any construction of the real numbers (whether via Dedekind cuts, Cauchy sequences, or axiomatic definition), results in the same mathematical structure. Whether we view the real numbers as algebraic objects or geometric points on a line (reconciled via the axioms of Euclidean geometry), we are dealing with a unique system that bridges the discrete and the continuous. This structure forms the bedrock upon which the analysis of functions, limits, and continuity is built in the subsequent chapters.

2.5 Uncountability of the Real Numbers

A fundamental consequence of the Nested Interval Property is the distinction regarding the size of infinity between the rational numbers and the real numbers. While $\mathbb{Q}$ is countable (i.e., can be placed in a one-to-one correspondence with $\mathbb{N}$), $\mathbb{R}$ is uncountable. The standard proof typically employs Cantor's diagonalisation argument on decimal expansions (for the rigorous set-theoretic construction, see the Set Theory notes, especially Section 7.4). However, establishing the properties of decimal representations rigorously is tedious. Instead, we utilise the Nested Interval Property to provide a direct topological proof.

***Theorem* 2.5.1. Uncountability of $\mathbb{R}$.** The set of real numbers $\mathbb{R}$ is uncountable.

Proof. Assume, for the sake of contradiction, that $\mathbb{R}$ is countable. Then, the elements of $\mathbb{R}$ can be enumerated in a sequence:

$$S = \{x_1, x_2, x_3, \dots\} = \mathbb{R}$$

We construct a sequence of nested closed intervals $\{I_n\}_{n=1}^{\infty}$ such that the n-th interval excludes the n-th real number in our list.

(i) Let I_1 be any closed interval such that $x_1 \notin I_1$. (For instance, if $x_1 = 0$, we could choose $I_1 = [1, 2]$).

(ii) Suppose we have constructed nested closed intervals $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_k$ such that $x_j \notin I_j$ for all $1 \leq j \leq k$.

(iii) To construct I_{k+1}, consider the interval I_k. We must choose a sub-interval $I_{k+1} \subseteq I_k$ such that $x_{k+1} \notin I_{k+1}$.
   - If $x_{k+1} \notin I_k$, then any closed sub-interval of I_k suffices.
   - If $x_{k+1} \in I_k$, since I_k contains infinitely many points, we can select a closed sub-interval that does not contain x_{k+1}. (For example, divide I_k into disjoint sub-intervals and select one that excludes x_{k+1}).

This recursive process yields a sequence of non-empty closed intervals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ such that for every $n \in \mathbb{N}$, $x_n \notin I_n$.

By the Nested Interval Property, the intersection contains at least one point. Let ξ be an element of this intersection. Since $\xi \in I_n$ for all n, and specifically $x_n \notin I_n$, it follows that $\xi \neq x_n$ for any $n \in \mathbb{N}$. Thus, ξ is a real number that is not included in the enumeration S. This contradicts the assumption that S contains all real numbers. Therefore, $\mathbb{R}$ is uncountable. ■

Note. This result highlights a strict hierarchy of infinities; the cardinality of the continuum strictly exceeds that of the rationals.

2.6 Decimal Representations and the Number Line

While the axiomatic construction of $\mathbb{R}$ as a complete ordered field provides a rigorous foundation for analysis, it is computationally convenient to represent real numbers using the decimal system. This representation links the algebraic properties of $\mathbb{R}$ with the intuitive notion of magnitude and provides a concrete method for approximation.

Decimal Expansions

Any real number x can be represented as an integer part plus an infinite series of fractions with denominators that are powers of 10.

Definition 2.6.1. ***Decimal Expansion***. Let $x \in \mathbb{R}$ with $x \geq 0$. A decimal expansion of x is a series of the form:

$$x = a_0 + \sum_{n=1}^{\infty} \frac{d_n}{10^n}$$

where $a_0 \in \mathbb{N} \cup \{0\}$ is the non-negative integer part, and each digit $d_n \in \{0, 1, \dots, 9\}$.

The existence of such an expansion is a consequence of the Archimedean Property. We define $a_0 = \lfloor x \rfloor$. For $n \geq 1$, the digits are defined recursively by $d_n = \lfloor 10^n(x - a_{n-1}) - \dots \rfloor$, essentially extracting the tenths, hundredths, and so forth.

Remark. This representation is not unique for all real numbers. Specifically, numbers with a terminating decimal expansion (where $d_n = 0$ for all $n > N$) have an alternative representation ending in infinitely many nines. For instance:

$$1.25 = 1.2500 \cdots = 1.2499 \dots$$

To ensure uniqueness in the mapping between $\mathbb{R}$ and decimal sequences, one conventionally forbids expansions ending in an infinite sequence of nines, or treats them as equivalent classes. In this way, real numbers are simply all infinite decimals.

Rationals and Periodic Expansions

The distinction between rational and irrational numbers manifests clearly in their decimal expansions.

***Theorem* 2.6.1. Characterisation of Rational Decimals**. A real number x is rational if and only if its decimal expansion is eventually periodic. That is, there exist integers N and $k > 0$ such that $d_{n+k} = d_n$ for all $n > N$.

Proof.

($\Leftarrow$) Suppose x has a periodic expansion. It can be written as the sum of a terminating part and a geometric series. Since the sum of rationals is rational, and the limit of a rational geometric series is rational, $x \in \mathbb{Q}$.

($\Rightarrow$) Let $x = p/q$ with $p, q \in \mathbb{N}$. The digits of the decimal expansion are determining by performing long division of p by q. At each step, the remainder r_n must satisfy $0 \leq r_n < q$. Since there are only q possible values for the remainder, by the Pigeonhole Principle, a remainder must eventually repeat. Once a remainder repeats, the sequence of dividends and thus the sequence of digits d_n enters a cycle.

■

Example 2.6.1. Converting Decimals to Fractions Terminating decimals correspond to fractions with denominators of the form $2^a 5^b$. For example:

$$3.25 = \frac{325}{100} = \frac{13}{4}$$

Infinite repeating decimals require algebraic manipulation to identify the underlying rational. Consider the number $3.\dot{1}4285\dot{7}$. Let $3.\dot{1}4285\dot{7} = 3 + \alpha$, where $\alpha = 0.\dot{1}4285\dot{7}$. Since the repeating block has length 6, we multiply by 10^6:

$$10^6\alpha = 142857.\dot{1}4285\dot{7} = 142857 + \alpha$$

Solving for α:

$$\alpha = \frac{142857}{10^6 - 1} = \frac{142857}{999999} = \frac{1}{7}$$

Therefore:

$$3.\dot{1}4285\dot{7} = 3 + \frac{1}{7} = \frac{22}{7}$$

Consequently, the irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ correspond precisely to the infinite non-repeating decimals. This aligns with our earlier proof of the uncountability of $\mathbb{R}$, as the set of all infinite sequences of digits is uncountable.

Approximation and the Number Line

We previously established that $\mathbb{Q}$ is dense in $\mathbb{R}$. The decimal representation offers a specific sequence of rational approximations: for any $x \in \mathbb{R}$, the sequence of truncated decimals $r_n = a_0.d_1 \dots d_n$ is a sequence of rationals converging to x, satisfying $|x - r_n| \leq 10^{-n}$.

More generally, we can approximate real numbers by rationals with arbitrary denominators, a concept fundamental to Diophantine approximation.

***Theorem* 2.6.2. Basic Approximation Theorem**. For any $x \in \mathbb{R}$ and any integer $q \geq 1$, there exists an integer p such that:

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q}$$

Proof. Consider the number axis partitioned into intervals of length $1/q$. These intervals are of the form $[k/q, (k+1)/q)$ for $k \in \mathbb{Z}$. Since the union of these intervals covers $\mathbb{R}$, the number x must lie in one such interval. Thus, there exists an integer p (where $p = k$ or $p = k + 1$) such that:

$$\frac{p}{q} \leq x < \frac{p+1}{q}$$

The distance between x and the closer endpoint (or simply p/q) is bounded by the length of the interval. Specifically:

$$0 \leq x - \frac{p}{q} < \frac{1}{q} \implies \left| x - \frac{p}{q} \right| < \frac{1}{q}$$

By choosing q sufficiently large, we can approximate x with a rational number p/q to any desired degree of precision. ■

The Geometric Continuum

Throughout these notes, we have treated $\mathbb{R}$ primarily as an algebraic object — a complete ordered field. However, intuition often relies on the geometric representation of $\mathbb{R}$ as the unique coordinate system for a straight line, the *number axis*. We assume the existence of a one-to-one correspondence between the set $\mathbb{R}$ and the points on a geometric line.

- An arbitrary point O is chosen as the origin, corresponding to 0.
- A unit length is established to locate 1.
- Every point P to the right of O corresponds to a positive real number x representing the length of the segment OP.
- Every point Q to the left of O corresponds to a negative real number.

The Completeness Axiom ensures that this correspondence is perfect: there are no "holes" in the line. Every point on the line corresponds to a real number, and every real number corresponds to a point. This geometric completeness allows us to graph functions, define derivatives as slopes, and interpret integrals as areas, bridging the gap between discrete arithmetic and continuous analysis. The rigorous justification of this correspondence requires the axioms of Euclidean geometry (specifically the Cantor-Dedekind axiom), which we assume for the remainder of this course.

2.7 Exercises

Part I: Inequalities and Field Properties

1. **Bernoulli's Inequality.** A fundamental tool in analysis is Bernoulli's Inequality, which estimates powers of numbers close to 1.
   (a) Prove that for any real number $x > -1$ and any natural number $n \in \mathbb{N}$, $(1+x)^n \geq 1 + nx$.
   Remark. Use mathematical induction. Be careful to justify where the condition $x > -1$ is used.
   (b) Show that for $n \geq 2$ and $x \neq 0$, the inequality is strict.

2. **The Cauchy-Schwarz Inequality.** Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be real numbers. Prove that:
$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$
   Remark. Consider the quadratic function $f(x) = \sum_{i=1}^{n}(a_i x + b_i)^2$. Observe that $f(x) \geq 0$ for all $x \in \mathbb{R}$. What does this imply about the discriminant of the corresponding quadratic equation?

3. Using the result from the previous exercise, prove the following:
   (a) If $x, y, z > 0$, then $(x+y+z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \geq 9$.
   (b) $\left(\sum_{i=1}^{n} a_i\right)^2 \leq n \sum_{i=1}^{n} a_i^2$.

4. **Complex Ordering.** We stated that $\mathbb{C}$ cannot be an ordered field. However, one can define an order on $\mathbb{C}$. Let us define the *lexicographical order*: for $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$, we say $z_1 < z_2$ if $a_1 < a_2$, or if $a_1 = a_2$ and $b_1 < b_2$.
   (a) Verify that this relation satisfies the Trichotomy law (Trichotomy).
   (b) Verify that this relation satisfies Closure under Addition (Closure under Addition).
   (c) Show that this relation fails Closure under Multiplication (Closure under Multiplication).

Part II: Supremum and Completeness

5. **Supremum of a Sum.** Let A and B be non-empty bounded sets of real numbers. Define the set sum $A + B = \{a + b \mid a \in A, b \in B\}$.
   (a) Prove that $\sup(A+B) = \sup A + \sup B$.
   Remark. To prove equality, prove two inequalities: $\sup(A+B) \leq \sup A + \sup B$ and $\sup(A+B) \geq \sup A + \sup B$. For the latter, use the ϵ characterisation of the supremum.
   (b) Formulate and prove a similar statement for $\inf(A+B)$.
   (c) Give an example where A and B are bounded, but $\sup(AB) \neq \sup A \cdot \sup B$.
   Remark. Consider sets containing negative numbers.

6. **Nested Open Intervals.** The Nested Interval Property (fig 2.4) specifically requires the intervals to be *closed*.
   (a) Construct a sequence of nested *open* intervals $J_n = (a_n, b_n)$ such that $J_{n+1} \subseteq J_n$ for all n, but $\bigcap_{n=1}^{\infty} J_n = \emptyset$.
   (b) Construct a sequence of nested *unbounded* closed intervals $K_n = [a_n, \infty)$ such that $\bigcap_{n=1}^{\infty} K_n = \emptyset$.

7. **Existence of Roots.** Let $n \in \mathbb{N}$ and $a > 0$. We can rigorously prove that $y = \sqrt[n]{a}$ exists using the Completeness Axiom, without relying on geometric intuition. Let $S = \{x \in \mathbb{R} \mid x \geq 0 \text{ and } x^n < a\}$.
   (a) Show that S is non-empty and bounded above. Let $y = \sup S$.
   (b) Prove that $y^n = a$.
   Remark. Use a contradiction argument (Trichotomy). If $y^n < a$, use Bernoulli's inequality to find a small $h > 0$ such that $(y+h)^n < a$, implying y is not an upper bound. If $y^n > a$, find $h > 0$ such that $(y-h)^n > a$, implying $y - h$ is an upper bound smaller than the least upper bound.

Part III: Topology and Structure

8. The Hausdorff Property. Prove that the real line is a Hausdorff space. That is, if $x, y \in \mathbb{R}$ and $x \neq y$, there exist $\epsilon > 0$ and $\delta > 0$ such that the neighbourhoods $B_\epsilon(x)$ and $B_\delta(y)$ are disjoint ($B_\epsilon(x) \cap B_\delta(y) = \emptyset$).

Remark. Let $\Delta = |x - y|$. Construct radii based on Δ.

9. Topology of the Rationals. Consider the set of rational numbers $\mathbb{Q}$ as a subset of $\mathbb{R}$.

(a) Determine the interior of $\mathbb{Q}$, $\text{int}(\mathbb{Q})$.
(b) Determine the boundary of $\mathbb{Q}$, $\text{bd}(\mathbb{Q})$.
(c) Determine the closure of $\mathbb{Q}$, $\overline{\mathbb{Q}} = \mathbb{Q} \cup \text{bd}(\mathbb{Q})$.

Remark. Recall that every interval contains both rational and irrational numbers.

10. Derived Sets. Let $S \subseteq \mathbb{R}$. A point x is an *accumulation point* of S if every deleted neighbourhood of x contains a point of S. The set of all accumulation points is denoted S'.

(a) Find S' for $S = \{1/n \mid n \in \mathbb{N}\}$.
(b) Find S' for $S = \mathbb{Z}$.
(c) Prove that if $x \in S'$, then every neighbourhood of x contains infinitely many points of S.

11. $\star$ **Irrational Exponents.** We have defined a^b for rational b and $a > 0$. For irrational x, we define $a^x = \sup\{a^q \mid q \in \mathbb{Q}, q < x\}$ (assuming $a > 1$).

(a) Prove that if a, b are irrational numbers, it is possible for a^b to be rational.

Remark. Consider the number $\sqrt{2}^{\sqrt{2}}$. If it is rational, you are done. If it is irrational, consider $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$.

(b) Can a rational number to an irrational power be rational?

12. $\star$ **Uniqueness of Decimal Expansions.** Let $x \in (0, 1)$. We stated that decimal expansions are unique barring the case of trailing nines/zeros. Prove that if x has two distinct decimal expansions $0.a_1a_2a_3\ldots$ and $0.b_1b_2b_3\ldots$, then one expansion must end in infinite 9s and the other in infinite 0s.

Remark. Let k be the first index where the digits differ, say $a_k < b_k$. Express x as a sum and estimate the magnitude of the tail of the series $\sum_{i=k+1}^{\infty} 9 \cdot 10^{-i}$. Recall from algebra that $\sum_{n=1}^{\infty} ar^n = \frac{ar}{1-r}$ for $|r| < 1$. Use this to sum the tail of nines.

13. $\star\star$ **Open Covers and Compactness.** A set $K \subseteq \mathbb{R}$ is called *compact* if every collection of open sets that covers K (i.e., $K \subseteq \bigcup_\alpha U_\alpha$) has a finite sub-collection that also covers K.

(a) Show that the interval $(0, 1)$ is **not** compact.

Remark. Consider the open sets $U_n = (1/n, 1)$ for $n \geq 2$.

(b) Show that the set $\mathbb{N}$ is **not** compact.
(c) (Heine-Borel Theorem Preview) The interval $[0, 1]$ *is* compact. While proving this is difficult, try to explain why the open cover used in part (a) fails to cover the closed interval $[0, 1]$ without using infinitely many sets.

Chapter 3

Sequences

In previous notes regarding discrete structures, we encountered the concept of a finite sequence — an ordered list of elements $\{a_k\}_{k=1}^n$. As we transition into the realm of analysis, we extend this concept to the infinite. The study of infinite sequences is the bedrock of calculus; concepts such as limits, continuity, derivatives, and integrals are ultimately founded upon the behaviour of sequences.

3.1 Definition and Notation

Intuitively, a sequence is an ordered list of real numbers. Formally, it is defined as a function whose domain is the set of natural numbers.

Definition 3.1.1. ***Sequence.*** A sequence of real numbers is a function $x : \mathbb{N} \to \mathbb{R}$. We typically describe a sequence as a list $(x_n)_{n\in\mathbb{N}}$ consisting of one real number for each natural number n:

$$x_1, \quad x_2, \quad x_3, \quad \ldots, \quad x_n, \quad \ldots$$

Although a sequence is a function, the standard functional notation $x(n)$ is rarely used. Instead, we utilise subscript notation to emphasise the ordinal nature of the terms. We denote a sequence variously as:

$$(x_n), \quad \{x_n\}, \quad \text{or} \quad \{x_n\}_{n=1}^\infty$$

Remark. As noted in my concrete abstractions notes $(x_n) \neq \{x_n\}$. The former is the ordered sequence, the latter is the set of values contained in the sequence.

The subscript n in x_n indicates the position of this term in the sequence. It is important to note that n does not have substantive meaning beyond being a placeholder; such a symbol is called a *dummy index*. Therefore, a sequence denoted (x_n) could equally be denoted (x_m), where the subscript m sequentially takes values from $\mathbb{N}$. Also just as with finite sequences, the index need not invariably commence at 1. We may define a sequence starting from any integer k, denoted $\{x_n\}_{n=k}^\infty$, provided the domain is a countably infinite set of consecutive integers (e.g., $n \geq 0$). If we use our intuition of a real number as corresponding to a point on a line, we can think of a sequence $(x_n)_{n\geq 1}$ as describing the motion of an object along the line, where x_n describes the position of that object at time n.

Basic Examples

To develop intuition, we examine several sequences with distinct behaviours.

Example 3.1.1. The Natural Sequence. Let $x_n = n$. The terms are $1, 2, 3, 4, \ldots$. This sequence is strictly increasing. It does not approach any specific real number but grows without bound.

Example 3.1.2. Decimal Approximations. We can artificially create a sequence that constructs a real number digit by digit:

$$a_1 = 0.3, \quad a_2 = 0.33, \quad a_3 = 0.333, \quad \ldots$$

The general term is $a_n = 0.\underbrace{333\ldots3}_{n}$. Obviously, as n gets larger and larger, the term a_n gets closer and closer to the real number $1/3$. This is a prototypical example of a sequence converging to a limit.

Example 3.1.3. Arithmetic Progression. The arithmetic progression with initial term $a \in \mathbb{R}$ and ratio $r \in \mathbb{R}$ is the sequence $a, a+r, a+2r, a+3r, \ldots$. For example, the sequence $3, 7, 11, 15, 19, \ldots$ is an arithmetic progression with initial term 3 and ratio 4. The constant sequence $a, a, a, \ldots$ is an arithmetic progression with initial term a and ratio 0.

Example 3.1.4. Geometric Progression. The geometric progression with initial term $a \in \mathbb{R}$ and ratio $r \in \mathbb{R}$ is the sequence $a, ar, ar^2, ar^3, \ldots$. For example, the sequence $1, -1, 1, -1, \ldots$ is the geometric progression with initial term 1 and ratio -1.

Example 3.1.5. The Harmonic Sequence. Let $x_n = 1/n$. The terms are:

$$1, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}, \quad \ldots$$

This sequence is strictly decreasing. As n increases, the terms become arbitrarily close to 0. We intuitively say this sequence approaches 0.

Example 3.1.6. Bounded Increasing Sequence. Let $x_n = 1 - (1/n)$. The terms are:

$$0, \quad \frac{1}{2}, \quad \frac{2}{3}, \quad \frac{3}{4}, \quad \ldots$$

This sequence is strictly increasing, yet it never exceeds 1. The terms "accumulate" near 1, suggesting the sequence approaches 1.

A common misconception is that a sequence approaching a value must eventually become strictly increasing or decreasing. This is false, as demonstrated by oscillating sequences.

Example 3.1.7. The Alternating Harmonic Sequence. Consider the sequence defined by $x_n = (-1)^n/n$. The terms are:

$$-1, \quad \frac{1}{2}, \quad -\frac{1}{3}, \quad \frac{1}{4}, \quad -\frac{1}{5}, \quad \ldots$$

The sign of the terms alternates between negative and positive. However, the magnitude $|x_n| = \frac{1}{n}$ decreases towards 0. Consequently, we can see the sequence approaches 0 from both sides (Fig 3.1), illustrating that approaching a specific number does not require the terms to be strictly increasing or decreasing.

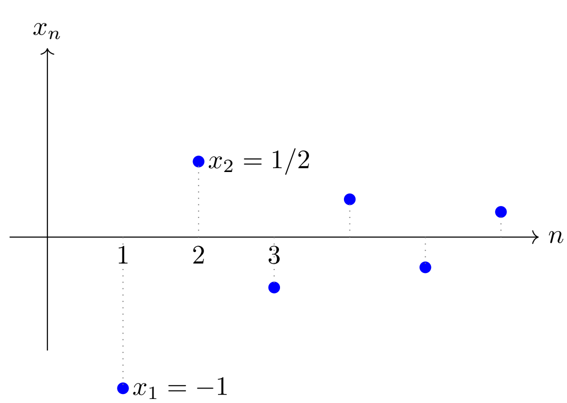


Figure 3.1: Graphical representation of the sequence $x_n = (-1)^n/n$. Unlike continuous functions, the graph of a sequence consists of discrete points.

We can easily construct sequences that behave erratically.

Example 3.1.8. Piecewise Definition. Sequences, being functions, can be defined piecewise.

$$x_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

This generates the sequence $1, 0, 1, 0, 1, \ldots$. This sequence is bounded but does not settle at any single value; it oscillates perpetually.

Example 3.1.9. The "Spoiler" Sequence. Consider the sequence:

$$x_n = \begin{cases} 1 & \text{if } 10 \mid n \quad (n \text{ is divisible by } 10) \\ 1 - \frac{1}{n} & \text{otherwise} \end{cases}$$

The terms generally follow the pattern $1 - 1/n$, approaching 1. However, at indices $10, 20, 30, \ldots$, the term is exactly 1. While intuitively this clusters around 1, rigorous analysis requires careful handling of such "interrupted" patterns.

Monotonicity and Boundedness

To precisely describe the behaviour of sequences, we introduce specific terminology regarding their direction and magnitude.

Definition 3.1.2. ***Monotonicity and Boundedness***. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of real numbers.

1. The sequence is called **increasing** if $x_n < x_{n+1}$ for all $n \in \mathbb{N}$.
2. The sequence is called **decreasing** if $x_n > x_{n+1}$ for all $n \in \mathbb{N}$.
3. The sequence is called **nonincreasing** if $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$.
4. The sequence is called **nondecreasing** if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$.
5. A sequence is called **monotone** if it is either nondecreasing or nonincreasing. It is called **strictly monotone** if it is either increasing or decreasing.
6. The sequence is called **bounded** if there exist real numbers m, M such that:

$$m \leq x_n \leq M \quad \text{for all } n \in \mathbb{N}.$$

We can apply these definitions to our previous examples:

- An arithmetic progression is increasing if and only if its ratio is positive ($r > 0$).
- A geometric progression with positive initial term and positive ratio is monotone: it is increasing if the ratio is > 1, decreasing if the ratio is < 1, and constant if the ratio is $= 1$.
- A geometric progression is bounded if and only if its ratio r satisfies $|r| \leq 1$.

3.2 Subsequences

We can construct new sequences by selecting infinitely many terms from an existing sequence, maintaining their original relative order.

Definition 3.2.1. ***Subsequence***. Let (x_n) be a sequence. A subsequence is a restriction of x to an infinite subset $S \subseteq \mathbb{N}$. This subset S can be viewed as an increasing sequence of natural numbers $n_1 < n_2 < n_3 < \ldots$, where:

$$n_1 := \min S, \quad n_2 := \min(S \setminus \{n_1\}), \quad \ldots$$

Thus, a subsequence is denoted $(x_{n_k})_{k \in \mathbb{N}}$, where (n_k) is a strictly increasing function $\sigma : \mathbb{N} \to \mathbb{N}$.

Essentially, we choose indices $n_1 < n_2 < n_3 < \ldots$ and consider the restricted list:

$$x_{n_1}, \quad x_{n_2}, \quad x_{n_3}, \quad \ldots$$

Example 3.2.1. Subsequences of the Alternating Sequence. Recall $x_n = (-1)^n/n$.

- Selecting odd indices ($n_k = 2k - 1$) yields the subsequence of negative terms:

$$-1, \quad -\frac{1}{3}, \quad -\frac{1}{5}, \quad \dots$$

- Selecting even indices ($n_k = 2k$) yields the subsequence of positive terms:

$$\frac{1}{2}, \quad \frac{1}{4}, \quad \frac{1}{6}, \quad \dots$$

Both subsequences appear to approach 0, consistent with the behaviour of the parent sequence.

Conflicting Subsequences

Subsequences provide a powerful tool for detecting when a sequence fails to approach a single value. If a sequence has two subsequences approaching distinct values, the sequence itself cannot be settling on one specific number.

Example 3.2.2. Oscillating Binary Sequence. Define x_n by:

$$x_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

The sequence is $1, 0, 1, 0, 1, 0, \dots$. We can extract two constant subsequences:

- Odd indices yield $(1, 1, 1, \dots)$, which constantly stays at 1.
- Even indices yield $(0, 0, 0, \dots)$, which constantly stays at 0.

Since the sequence oscillates between 0 and 1 without settling on a single value, it does not have a single value its approaching.

Example 3.2.3. The "Pesky" Term. Consider a sequence that generally decreases but is periodically interrupted.

$$x_n = \begin{cases} 1 & \text{if } n \text{ is divisible by 10} \\ \frac{1}{n} & \text{otherwise} \end{cases}$$

The terms look like $1, \frac{1}{2}, \dots, \frac{1}{9}, \mathbf{1}, \frac{1}{11}, \dots, \frac{1}{19}, \mathbf{1}, \frac{1}{21}, \dots$. While the majority of terms (the subsequence where n is not a multiple of 10) approach 0, the subsequence $x_{10k} = 1$ remains constant at 1. The persistence of these "pesky" terms prevents the sequence from settling at 0.

Recursive Sequences

Sequences need not be defined by an explicit formula for the n-th term. They are often defined recursively, where a term depends on preceding terms.

Example 3.2.4. The Fibonacci Sequence. The Fibonacci sequence (F_n) is defined by the recurrence relation:

$$F_{n+2} = F_{n+1} + F_n \quad \text{for } n \geq 0$$

with initial conditions $F_0 = 1$ and $F_1 = 1$. The sequence begins:

$$1, \quad 1, \quad 2, \quad 3, \quad 5, \quad 8, \quad 13, \quad \dots$$

This sequence grows without bound. However, derived sequences often exhibit interesting behaviours. For instance, the sequence of reciprocals $(1/F_n)$:

$$1, \quad 1, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{5}, \quad \dots$$

approaches 0.

Throughout these examples, we have relied on intuitive language to describe sequence behaviour. We observed that the alternating subsequence "approaches" 0, the oscillating binary sequence "fails to settle," and the pesky term sequence is interrupted by outliers. To move from these qualitative observations to rigorous proof, we must formalize exactly what it means for terms to accumulate around a specific value.

3.3 Convergence

In this section, we define convergence, the central concept of real analysis. Having seen how subsequences and recurrence relations generate various patterns, we now aim to rigorously describe what it means for a sequence (x_n) to "approach" or "tend to" a specific value x_0, which we call the limit.

Intuition and Motivation

Consider a sequence of points $x_1, x_2, x_3, \ldots$ on the real line. Intuitively, we say this sequence approaches a point x_0 if the terms eventually get closer and closer to x_0.

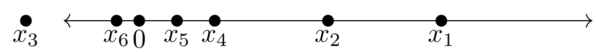


Figure 3.2: Terms of a sequence clustering around 0.

But what does "close" mean? Distance is subjective.

- On a standard ruler with millimetre markings, two points are indistinguishable if their distance is less than 1 mm.
- Under a microscope, we might require the distance to be less than 0.001 mm.
- With an electron microscope, the threshold might be nanometres.

For a sequence to truly converge to x_0, the terms must eventually become indistinguishable from x_0 *regardless of the magnification.* No matter how small a tolerance (let's call it ϵ) one specifies, the sequence must eventually stay within distance ϵ of x_0.

This leads to the formal definition. We are not saying every term is close; the first few terms $x_1, x_2, \ldots$ might be far away. We are saying that eventually, for all sufficiently large indices n, the terms x_n are close to x_0.

Definition of Convergence

We present two equivalent formulations of the definition. The first explicitly uses a function to determine the "threshold" index.

Definition 3.3.1. ***Functional Definition of Convergence*.** A sequence of real numbers (x_n) converges to a point $x_0 \in \mathbb{R}$ if there exists a function $N : \mathbb{R}^+ \to \mathbb{N}$ such that for any $\epsilon > 0$:

$$|x_n - x_0| < \epsilon \quad \text{whenever} \quad n > N(\epsilon).$$

We denote this by $\lim_{n\to\infty} x_n = x_0$ or $x_n \to x_0$.

Here, ϵ represents the desired level of approximation (the "least count" of our measuring device), and $N(\epsilon)$ represents the point in the sequence beyond which this approximation is guaranteed.

Standard textbooks often state this definition using quantifiers rather than an explicit function.

Definition 3.3.2. ***Standard ($\epsilon - N$) Definition of Convergence***. A sequence (x_n) converges to x_0 if for every $\epsilon > 0$, there exists a natural number N (which may depend on ϵ) such that:

$$|x_n - x_0| < \epsilon \quad \text{for all } n > N.$$

Remark. Observe that the definition of convergence implies a direct relationship between a limit and a null sequence (a sequence converging to 0):

$$\lim_{n\to\infty} x_n = x \iff \lim_{n\to\infty} |x_n - x| = 0.$$

This allows us to translate problems of converging to a general limit x into problems of converging to 0.

Both definitions assert the same condition: for any error tolerance ϵ, the sequence eventually enters and stays within the interval $(x_0 - \epsilon, x_0 + \epsilon)$.

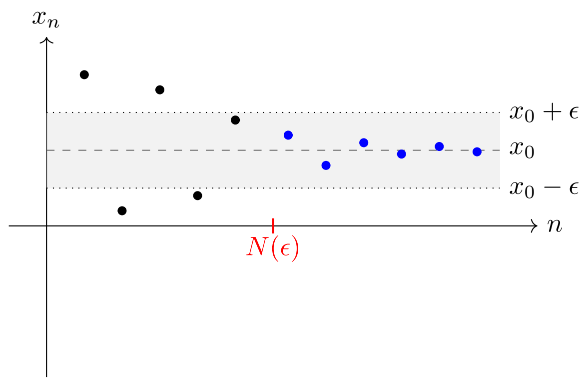


Figure 3.3: Visualising convergence. Once $n > N(\epsilon)$, all terms x_n (blue) must lie within the shaded ϵ-strip around x_0.

Remark. It suffices to verify the condition for small ϵ. If we can find a threshold function $N(\epsilon)$ for all $\epsilon \in (0, \epsilon_0)$, we can extend it to all $\mathbb{R}^+$ by simply defining $N(\epsilon) = N(\epsilon_0)$ for $\epsilon \geq \epsilon_0$. If terms are within distance ϵ_0, they are certainly within any larger distance ϵ.

Example: Convergence of $1/n$

Let us apply the definition to a concrete example. We claim that the sequence $x_n = 1/n$ converges to 0.

Example 3.3.1. Convergence of Harmonic Sequence. Show that $\lim_{n\to\infty} \frac{1}{n} = 0$.

Analysis: We are given the sequence $x_n = 1/n$ and the candidate limit $x_0 = 0$. We must show that for any $\epsilon > 0$, there exists an N such that $n > N$ implies $|x_n - x_0| < \epsilon$. Substituting the specific terms:

$$\left|\frac{1}{n} - 0\right| < \epsilon \iff \frac{1}{n} < \epsilon \iff n > \frac{1}{\epsilon}$$

Formal Proof: Let $\epsilon > 0$ be given. By the Archimedean Property, there exists a natural number N such that $N > 1/\epsilon$. If we choose $n > N$, then:

$$n > N > \frac{1}{\epsilon} \implies \frac{1}{n} < \epsilon$$

Therefore, $|x_n - 0| = 1/n < \epsilon$ for all $n > N$. This proves that $x_n \to 0$.

In the functional notation, we have explicitly constructed the function $N(\epsilon)$ by choosing any integer greater than $1/\epsilon$ (e.g., $N(\epsilon) = \lfloor 1/\epsilon \rfloor + 1$).

3.4 Special Limits I

With just the formal definition, combined with basic algebraic manipulation, we can establish convergence for most examples rigorously. The following examples illustrate standard techniques for determining the threshold $N(\epsilon)$ in non-trivial scenarios.

Example 3.4.1. Rational Functions Involving Roots. Prove that $\lim_{n\to\infty} \frac{3\sqrt{n}+1}{2\sqrt{n}-1} = \frac{3}{2}$.

Proof. We examine the difference between the general term and the limit:

$$\left|\frac{3\sqrt{n}+1}{2\sqrt{n}-1} - \frac{3}{2}\right| = \left|\frac{2(3\sqrt{n}+1) - 3(2\sqrt{n}-1)}{2(2\sqrt{n}-1)}\right| = \frac{5}{4\sqrt{n}-2}$$

To simplify finding N, we want to bound this term by something simpler. Note that $4\sqrt{n} - 2 > 2\sqrt{n}$ is true whenever $2\sqrt{n} > 2$, i.e., $n > 1$. Using this inequality:

$$\frac{5}{4\sqrt{n}-2} < \frac{5}{2\sqrt{n}} < \frac{3}{\sqrt{n}}$$

(We use $3/\sqrt{n}$ for simplicity, as $2.5 < 3$). For any given $\epsilon > 0$, we require $3/\sqrt{n} < \epsilon$, which implies $\sqrt{n} > 3/\epsilon$, or $n > 9/\epsilon^2$. Thus, take $N = \lfloor 9/\epsilon^2 \rfloor + 1$. For any $n > N$, the inequality holds. ■

Note. The strategy used above applies generally. For any two polynomials $P(n) = a_k n^k + \cdots + a_0$ and $Q(n) = b_k n^k + \cdots + b_0$ (where $b_k \neq 0$), the limit depends solely on the ratio of the leading coefficients:

$$\lim_{n\to\infty} \frac{a_k n^k + a_{k-1} n^{k-1} + \dots}{b_k n^k + b_{k-1} n^{k-1} + \dots} = \frac{a_k}{b_k}$$

If the degree of the numerator is strictly less than the denominator, the limit is 0; if strictly greater, it diverges to $\pm\infty$.

Example 3.4.2. Power Functions. For any $\alpha > 0$, prove that $\lim_{n\to\infty} (1/n^\alpha) = 0$.

Proof.

Case 1: $\alpha \geq 1$. Then $n^\alpha \geq n$, so $0 < (1/n^\alpha) \leq (1/n)$. Taking $N = \lfloor 1/\epsilon \rfloor$, if $n > N$, then $1/n^\alpha \leq 1/n < \epsilon$.

Case 2: $0 < \alpha < 1$. Since $\alpha > 0$, the number $1/\alpha$ exists. By the Archimedean Property, there exists a natural number m such that $m > 1/\alpha$, which implies $m\alpha > 1$.

Since $m\alpha > 1$, we know from Case 1 that $\lim_{n\to\infty} 1/(n^{m\alpha}) = 0$. Thus, for any $\epsilon > 0$, there exists N such that for $n > N$, $1/(n^{m\alpha}) < \epsilon^m$. Taking the m-th root of both sides yields $1/(n^\alpha) < \epsilon$. ■

Example 3.4.3. Geometric Sequence Decay. When $|q| < 1$, prove that $\lim_{n\to\infty} q^n = 0$.

Proof. If $q = 0$, the result is trivial. Let $0 < |q| < 1$. We can write $|q| = \frac{1}{1+\alpha}$ where $\alpha = \frac{1}{|q|} - 1 > 0$. By the Binomial Expansion, $(1+\alpha)^n = 1 + n\alpha + \frac{n(n-1)}{2}\alpha^2 + \cdots > n\alpha$. Thus:

$$|q^n - 0| = |q|^n = \frac{1}{(1+\alpha)^n} < \frac{1}{n\alpha}$$

For any $\epsilon > 0$, we require $1/n\alpha < \epsilon \implies n > 1/\alpha\epsilon$. Take $N = \lfloor 1/\alpha\epsilon \rfloor + 1$. When $n > N$, $|q^n| < \epsilon$. ■

Example 3.4.4. The n-th Root of n. Prove that $\lim_{n\to\infty} n^{1/n} = 1$.

Proof. We use the Arithmetic Mean-Geometric Mean (AM-GM) inequality. Consider the n numbers: $\underbrace{1, 1, \ldots, 1}_{n-2 \text{ times}}, \sqrt{n}, \sqrt{n}$. The geometric mean of these numbers is:

$$(1 \cdot \dots \cdot 1 \cdot \sqrt{n} \cdot \sqrt{n})^{1/n} = (n)^{1/n} = n^{1/n}$$

The arithmetic mean is:

$$\frac{(n-2) \cdot 1 + 2\sqrt{n}}{n} = \frac{n - 2 + 2\sqrt{n}}{n} = 1 - \frac{2}{n} + \frac{2}{\sqrt{n}} < 1 + \frac{2}{\sqrt{n}}$$

By AM-GM, Geometric Mean $\leq$ Arithmetic Mean, so:

$$1 \leq n^{1/n} < 1 + \frac{2}{\sqrt{n}} \implies 0 \leq n^{1/n} - 1 < \frac{2}{\sqrt{n}}$$

For any $\epsilon > 0$, take $N = \lfloor 4/\epsilon^2 \rfloor + 1$. When $n > N$:

$$|n^{1/n} - 1| < \frac{2}{\sqrt{n}} < \frac{2}{\sqrt{4/\epsilon^2}} = \epsilon$$

■

Remark. (**AM-GM Inequality:**). For any set of non-negative real numbers $x_1, x_2, \ldots, x_n$, the arithmetic mean is greater than or equal to the geometric mean:

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdot x_2 \cdot \cdots \cdot x_n}$$

Proof. Let $A = \frac{1}{n} \sum x_k$ be the arithmetic mean. We assume $A > 0$ (otherwise the result is trivial). Consider the quantities x_k/A. Using the inequality $e^{t-1} \geq t$, we set $t = x_k/A$:

$$e^{\frac{x_k}{A} - 1} \geq \frac{x_k}{A}$$

Multiplying this inequality for all $k = 1 \ldots n$:

$$\prod_{k=1}^{n} e^{\frac{x_k}{A} - 1} \geq \prod_{k=1}^{n} \frac{x_k}{A}$$

The exponent on the left becomes $\sum(\frac{x_k}{A} - 1) = \frac{\sum x_k}{A} - n = n - n = 0$. Thus, $e^0 \geq \frac{\prod x_k}{A^n}$, which simplifies to $1 \geq \frac{G^n}{A^n}$, or $A \geq G$. ■

Uniqueness of Limits

Before we delve deeper into examples and properties of limits, we must establish a fundamental fact: a sequence cannot converge to two different values simultaneously. If a sequence settles down, the place it settles is unique.

***Theorem* 3.4.1. Uniqueness of Limits**. If a sequence (x_n) converges to a limit a, then it cannot converge to a different limit b. That is, the limit of a convergent sequence is unique.

Proof. We proceed by contradiction. Assume that the sequence (x_n) converges to two distinct limits, a and b, with $a \neq b$. Without loss of generality, assume $a < b$. The intuition is that if x_n gets arbitrarily close to a and arbitrarily close to b, the terms must eventually be in two disjoint neighbourhoods simultaneously, which is impossible. Let $\epsilon = (b-a)/2$. This distance represents half the separation between the two proposed limits. Note that since $a < b$, $\epsilon > 0$. Since $x_n \to a$, there exists a threshold N_a such that for all $n > N_a$, $|x_n - a| < \epsilon$.

Since $x_n \to b$, there exists a threshold N_b such that for all $n > N_b$, $|x_n - b| < \epsilon$. Let $N = \max(N_a, N_b)$. For any $n > N$, both conditions must hold. Consider the distance between a and b. By the Triangle Inequality:

$$|a - b| = |(a - x_n) + (x_n - b)| \leq |a - x_n| + |x_n - b|$$

Substituting the bounds from our convergence definitions:

$$|a - b| < \epsilon + \epsilon = 2\epsilon$$

However, we defined $\epsilon = (b - a)/2$, which implies $2\epsilon = b - a = |a - b|$. This leads to the contradiction:

$$|a - b| < |a - b|$$

This is impossible. Therefore, the assumption that $a \neq b$ must be false. The limit is unique. ■

The "Game" of Convergence

The definition of convergence can be daunting. It is often helpful to view it as a game between two players: a Challenger (the sceptic) and a Defender (the prover).

The Setup: The Defender claims that the sequence (x_n) converges to L.

The Challenge: The Challenger picks a number $\epsilon > 0$. This represents a margin of error. The Challenger is essentially asking, "Can you guarantee the sequence stays within distance ϵ of L?"

The Response: The Defender must find a threshold index N. This is the Defender's way of saying, "Wait until the N-th term. After that, I guarantee the terms are within your margin."

The Verification: If $|x_n - L| < \epsilon$ for all $n > N$, the Defender wins this round.

For the sequence to be convergent, the Defender must have a winning strategy for *every possible* ϵ the Challenger might throw, no matter how small.

Example 3.4.5. Achilles and the Tortoise.

Challenger: I bet the sequence $x_n = 1/n$ doesn't converge to 0. What if I set the tolerance to $\epsilon = 0.0001$?

Defender: That is small, but eventually the sequence gets smaller. I choose $N = 10,000$. For any $n > 10,000$, we have $1/n < 0.0001$.

Challenger: Okay, but what if I choose ϵ to be $1/K$, where K is the number of sand particles on Earth? It's tiny!

Defender: It doesn't matter how huge K is. If $\epsilon = 1/K$, I simply choose $N = K$. Then for any $n > K$, we have $1/n < 1/K = \epsilon$.

Defender's Strategy: For any ϵ you give me, I define my response function as $N(\epsilon) = \lfloor 1/\epsilon \rfloor + 1$. Since I have a function that works for any input, I win. The sequence converges.

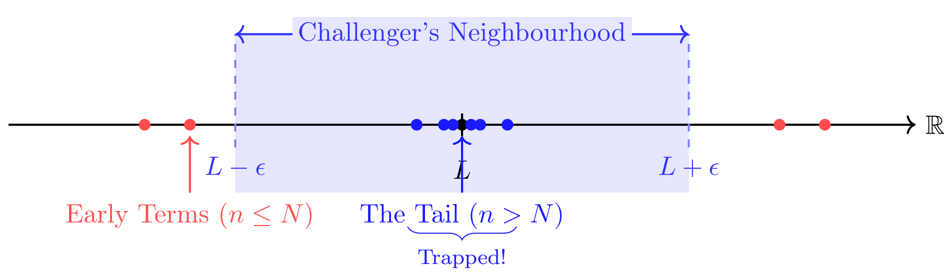


Figure 3.4: The Convergence Game: The blue shaded region represents the Challenger's ϵ-neighborhood. The Defender wins if eventually all future points (the blue dots) stay strictly inside this region, regardless of the red early points.

Consider the counter-example of a sequence that gets close but does not converge to the proposed limit.

Example 3.4.6. The Constant Disappointment. Does the constant sequence $y_n = 0.0001$ converge to 0?

Challenger: I choose $\epsilon = 0.00001$ (which is 10^{-5}).
Defender: I need to find N such that $|0.0001 - 0| < 0.00001$.

Analysis: The condition is $0.0001 < 0.00001$, which is false. No matter what N the Defender picks, the terms never get closer than 0.0001. The Defender has no strategy for this specific ϵ. Thus, y_n does *not* converge to 0 (it converges to 0.0001).

3.5 Deep Dive into the Definition

Let us revisit the standard definition of convergence to appreciate its logical structure.

$$\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N} \text{ such that } \forall n > N_\epsilon, |x_n - x_0| < \epsilon$$

The order of quantifiers is crucial.

For each $\epsilon > 0$: We start by fixing an arbitrary error tolerance.
There exists $N_\epsilon \in \mathbb{N}$: Based on that specific ϵ, we find a threshold.
Such that for all $n > N_\epsilon$: Beyond that threshold, the condition holds.

The notation N_ϵ emphasises that the threshold depends on ϵ. Generally, as ϵ becomes smaller (a tighter tolerance), the required N_ϵ becomes larger (we must go further into the sequence).

What if we swap quantifiers?

Consider the statement with the first two quantifiers swapped:

$$\exists N \in \mathbb{N} \text{ such that } \forall \epsilon > 0, \forall n > N, |x_n - x_0| < \epsilon$$

This says there is a "universal" threshold N that works for *every* possible ϵ. If such an N existed, then for $n > N$, $|x_n - x_0|$ would be less than every positive number. The only non-negative number smaller than every positive number is 0. Thus, this modified definition would imply $x_n = x_0$ for all $n > N$. In other words, swapping the quantifiers restricts the definition to include only sequences that are eventually constant. This is far too restrictive for analysis.

Divergence

To fully understand convergence, we must understand its negation. What does it mean for a sequence to *not* converge to x_0?

***Theorem* 3.5.1. Negation of Convergence**. A sequence (x_n) does not converge to x_0 if:

$$\exists \epsilon > 0 \text{ such that } \forall N \in \mathbb{N}, \exists n > N \text{ such that } |x_n - x_0| \geq \epsilon$$

In plain English: There is a "bad" tolerance ϵ such that no matter how far out we go (no matter what N we pick), we can always find a term x_n further out that violates the tolerance condition.

Definition 3.5.1. *Divergence*. A sequence (x_n) is said to *diverge* if it does not converge to any $x_0 \in \mathbb{R}$. That is, for every $L \in \mathbb{R}$, the sequence fails to converge to L.

Divergence to Infinity

A specific and important type of divergence occurs when a sequence grows without bound.

Definition 3.5.2. ***Divergence to Infinity***. A sequence (x_n) diverges to $+\infty$ (written $x_n \to \infty$) if for every $M \in \mathbb{R}$, there exists an $N \in \mathbb{N}$ such that:

$$x_n > M \quad \text{for all } n > N.$$

Similarly, $x_n \to -\infty$ if for every M, eventually $x_n < M$.

Here, M acts as a "barrier" rather than a tolerance. The condition says the sequence eventually stays above any barrier we set.

Topological Viewpoint

The definition of convergence can be elegantly rephrased using the language of topology. Recall that an open neighbourhood $B_\epsilon(x)$ (sometimes called open ball $B(x, \epsilon)$) is the set of points within distance ϵ of x, as defined in the section on the basic topology of the real line (see dfn 2.3.2).

Theorem **3.5.2. Topological Convergence**. A sequence (x_n) converges to x_0 if and only if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n > N$,

$$x_n \in B(x_0, \epsilon).$$

This phrasing shifts the focus from arithmetic inequalities to set membership: the "tail" of the sequence must eventually lie entirely within any open neighbourhood centred at the limit.

Definition 3.5.3. ***Neighbourhoods of Infinity***. While a neighborhood of a real number x is an open interval containing x, we extend this concept to the extended real line:

- A neighborhood of $+\infty$ is any interval of the form $(M, +\infty)$ for some $M \in \mathbb{R}$.
- A neighborhood of $-\infty$ is any interval of the form $(-\infty, M)$ for some $M \in \mathbb{R}$.

3.6 A Descriptive Language for Convergence

The formal definition of convergence, involving nested quantifiers and explicit functional dependencies, can be cumbersome in practice. To streamline our arguments without sacrificing rigour, we introduce descriptive terminology that captures the essence of "eventual" behaviour.

Sufficiently Large n

The core idea of convergence is that a property holds for the "tail" of a sequence.

Definition 3.6.1. ***Tail of a Sequence***. Let (x_n) be a sequence and let $N \in \mathbb{N}$. The N**-tail** (or simply the tail) of the sequence is the subsequence $(x_n)_{n>N}$. It corresponds to the set of values:

$$\{x_{N+1}, \quad x_{N+2}, \quad x_{N+3}, \quad \ldots\}$$

Definition 3.6.2. ***Eventually True***. Let $P(n)$ be a property defined on the natural numbers. We say $P(n)$ is *true for sufficiently large* n (or *true eventually*, or *true for all but finitely many* n) if there exists some $N \in \mathbb{N}$ such that $P(n)$ holds for all $n > N$.

This definition does not specify *how* large n must be — whether $N = 10$ or $N = 10^{10}$ is irrelevant to the qualitative fact that the property eventually holds.

Proposition **3.6.1.** *Principle of Finite Modification.* The convergence or divergence of a sequence is not affected if we modify only finitely many of its terms. Specifically, if (x_n) and (y_n) are two sequences such that there exists an $N_0 \in \mathbb{N}$ where $x_n = y_n$ for all $n > N_0$, then:

$$\lim_{n\to\infty} x_n = L \iff \lim_{n\to\infty} y_n = L.$$

Proof. Assume $\lim_{n\to\infty} x_n = L$. Let $\epsilon > 0$ be given. By definition, there exists N_x such that for all $n > N_x$, $|x_n - L| < \epsilon$. We must find a threshold for the sequence (y_n). Let $N_y = \max(N_x, N_0)$. For any $n > N_y$, we know $n > N_0$, which implies $y_n = x_n$. Since we also know $n > N_x$, it follows that $|y_n - L| = |x_n - L| < \epsilon$. Thus, $\lim_{n\to\infty} y_n = L$. The converse holds by symmetry. ■

Using this terminology, the definition of convergence can be restated succinctly:

***Theorem* 3.6.1. Descriptive Definition of Convergence**. A sequence (x_n) converges to x if and only if for each $\epsilon > 0$, the inequality $|x_n - x| < \epsilon$ holds for sufficiently large n.

Proof. This follows directly from the definition of "sufficiently large."

($\Rightarrow$) If $x_n \to x$, then for any $\epsilon > 0$, there exists N such that for all $n > N$, $|x_n - x| < \epsilon$. The existence of such an N is exactly the definition of the property holding for sufficiently large n.

($\Leftarrow$) Conversely, if for every $\epsilon > 0$ the inequality holds for sufficiently large n, then by definition there exists an N such that the inequality holds for all $n > N$. This is precisely the $\epsilon - N$ definition of convergence.

■

Topologically: (x_n) converges to x if for each $\epsilon > 0$, $x_n \in B(x, \epsilon)$ eventually.

The Role of ϵ and the $K - \epsilon$ Principle

In convergence proofs, ϵ represents an arbitrary, fixed error tolerance. Our goal is to show that the distance $|x_n - x_0|$ can be made smaller than this tolerance by choosing n large enough. Frequently, algebraic manipulations yield an upper bound of the form $K\epsilon$ rather than exactly ϵ, where $K > 0$ is a constant independent of n and ϵ.

***Lemma* 3.6.1.** *The $K - \epsilon$ Principle.* Let (E_n) be a sequence of non-negative real numbers (typically error terms $|x_n - x_0|$). Suppose there exists a constant $K > 0$ such that for every $\epsilon > 0$,

$$E_n < K\epsilon \quad \text{for sufficiently large } n.$$

Then for every $\epsilon > 0$,

$$E_n < \epsilon \quad \text{for sufficiently large } n.$$

In other words, proving the bound $K\epsilon$ is sufficient to prove convergence.

Proof. Let an arbitrary target $\eta > 0$ be given. We wish to show $E_n < \eta$ eventually. In the hypothesis, the condition holds for *every* positive number. Specifically, let us choose the value $\epsilon = \eta/K$. Since $\eta > 0$ and $K > 0$, this ϵ is positive. By the hypothesis, there exists a threshold N such that for all $n > N$:

$$E_n < K\left(\frac{\eta}{K}\right) = \eta$$

Thus, $E_n < \eta$ for sufficiently large n. Since η was arbitrary, the conclusion follows. ■

This principle allows us to be less pedantic with constants during proofs. If we arrive at $|x_n - x| < 2\epsilon$ or 100ϵ, we can simply conclude convergence without redefining $\epsilon' = \epsilon/100$ in every step.

3.7 Exercises

Part I: The Definitions of Convergence and Divergence

1. **Basic $\epsilon - N$ Proofs.** Use the formal definition of convergence to prove the following limits. Explicitly define $N(\epsilon)$ in each case.

(a) $\lim_{n\to\infty} \frac{1}{1+\sqrt{n}} = 0$

(b) $\lim_{n\to\infty} \frac{2n+3}{5n-10} = \frac{2}{5}$

(c) $\lim_{n\to\infty} 0.\underbrace{99\ldots9}_{n} = 1$

2. **Understanding Quantifiers.** Consider the modified definition of convergence where the order of quantifiers is swapped:

$$\exists N \in \mathbb{N} \text{ such that } \forall \epsilon > 0, \forall n > N, |x_n - x_0| < \epsilon.$$

Prove formally that a sequence satisfies this definition if and only if it is eventually constant (i.e., there exists N such that $x_n = x_0$ for all $n > N$).

Remark. To prove the "only if" direction, assume the sequence is not eventually constant and construct a specific ϵ that yields a contradiction.

3. **Alternative Definitions?** Determine whether the following statements can serve as a valid definition for $\lim_{n\to\infty} a_n = a$. If yes, prove equivalence to the standard definition. If no, provide a counter-example.

(a) For infinitely many $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $|a_n - a| < \epsilon$.

(b) For any $\epsilon > 1$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $|a_n - a| < \epsilon$.

(c) For any $\epsilon \in (0, 1)$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $|a_n - a| < \epsilon$.

(d) For every positive integer k, there exists $N_k \in \mathbb{N}$ such that for all $n > N_k$, $|a_n - a| < 1/k$.

(e) For any $\epsilon, \delta > 0$, the interval $(a - \epsilon, a + \delta)$ contains all but finitely many terms of the sequence.

4. **Discrete Convergence.** Let (a_n) be a sequence of integers. Prove that (a_n) converges if and only if the sequence is eventually constant (i.e., there exists N such that $a_n = a_{n+1}$ for all $n > N$).

Remark. To prove the forward direction, consider the definition of convergence with a specific choice of $\epsilon < 1$.

5. **Formal Negation.** Write out the formal definition for the statement "The sequence (x_n) diverges."

Remark. This requires combining the definition of divergence (for all $L \in \mathbb{R}$, (x_n) does not converge to L) with the logical negation of the convergence definition.

6. **Divergence to Infinity.**

(a) Prove that if a sequence of positive terms (x_n) satisfies $x_n \to \infty$, then $1/x_n \to 0$.

(b) Rewrite the definition of $x_n \to \infty$ using functional notation (i.e., explicitly defining a function $N : \mathbb{R} \to \mathbb{N}$).

7. **Topological Proof of Uniqueness.** Use the topological definition of convergence (using open neighbourhoods) to prove the Uniqueness of Limits theorem.

Remark. Assume $x_n \to a$ and $x_n \to b$ with $a \neq b$. Use the Hausdorff property of the real line to choose disjoint neighbourhoods $B_\epsilon(a)$ and $B_\epsilon(b)$.

Part II: Properties of Convergent Sequences

8. **Boundedness of Convergent Sequences.** Prove that every convergent sequence is bounded.

Remark. Let $x_n \to L$. Choose a specific value for ϵ (e.g., $\epsilon = 1$) to bound the tail of the sequence. Then consider the finite set of terms preceding the tail.

9. **Preservation of Inequalities.** Let (x_n) be a convergent sequence with limit L.

(a) Prove that if $x_n \geq 0$ for all n, then $L \geq 0$.

(b) Give a counter-example to show that $x_n > 0$ for all n does *not* imply $L > 0$.

(c) Suppose (y_n) converges to M and $x_n \leq y_n$ for all n. Prove that $L \leq M$.

10. Subsequence Inheritance. Prove that if a sequence (x_n) converges to L, then every subsequence (x_{n_k}) of (x_n) also converges to L.

Remark. Recall that for a subsequence, the index function satisfies $n_k \geq k$ for all k.

11. Absolute Values. Let $\lim_{n\to\infty} a_n = a$. Prove that $\lim_{n\to\infty} |a_n| = |a|$. Give an example to show that the converse is not generally true. Under what specific condition is the converse true?

12. The Maximum Mean. Let (a_n) be a sequence such that $\lim_{n\to\infty} \frac{a_n}{n} = 0$. Prove that:

$$\lim_{n\to\infty} \frac{\max(|a_1|, |a_2|, \ldots, |a_n|)}{n} = 0.$$

Remark. Fix ϵ. Since $a_n/n \to 0$, for large n, $|a_n| < \epsilon n$. What about the terms before that threshold? They are fixed finite numbers.

13. Limits of Sums. Use the identity $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ to prove:

$$\lim_{n\to\infty} \frac{1 + 2 + \cdots + n}{n^2} = \frac{1}{2}.$$

Part III: Advanced Topics

14. ⋆ **Recursive Sequences and Monotonicity.** Let a sequence be defined recursively by $x_1 = 1$ and $x_{n+1} = \sqrt{2 + x_n}$.

(a) Show by induction that the sequence is bounded above by 2.

(b) Show by induction that the sequence is increasing ($x_{n+1} > x_n$).

15. ⋆ **Characterisation of Supremum.** Let $S \subset \mathbb{R}$ be a non-empty set bounded above, and let $\alpha = \sup S$. Prove that there exists a sequence (x_n) such that $x_n \in S$ for all n, and $\lim_{n\to\infty} x_n = \alpha$.

Remark. Use the characterisation of the supremum: for every $\epsilon = 1/n$, there exists an element in S close to α.

16. ⋆ **Approximation of Irrationals.** Let (x_n) be a sequence of rational numbers, where each term is written in lowest terms as $x_n = p_n/q_n$ (with $p_n \in \mathbb{Z}, q_n \in \mathbb{N}$). Prove that if (x_n) converges to an irrational number ξ, then $\lim_{n\to\infty} q_n = \infty$.

17. ⋆ **The Ratio Lemma for Sequences.** Let (x_n) be a sequence of positive real numbers. Suppose that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L < 1$. Prove that $\lim_{n\to\infty} x_n = 0$.

18. ⋆ **Coupled Recursive System.** Let a, b, c be real numbers. Define sequences recursively by $a_0 = a, b_0 = b, c_0 = c$ and for $n \geq 1$:

$$a_n = \frac{b_{n-1} + c_{n-1}}{2}, \quad b_n = \frac{a_{n-1} + c_{n-1}}{2}, \quad c_n = \frac{a_{n-1} + b_{n-1}}{2}.$$

Prove that all three sequences converge to the same limit $\frac{a+b+c}{3}$.

Remark. Hint 1: Show that $a_n + b_n + c_n$ is constant for all n.
Hint 2: Let $d_n = \max(a_n, b_n, c_n) - \min(a_n, b_n, c_n)$ be the "diameter" of the set. Show that d_n decreases by a factor of 2 at each step ($d_n = d_{n-1}/2$), which implies the diameter goes to 0.

Chapter 4

Limits Laws

The definition of convergence, involving the precise manipulation of ϵ and N, is the rigorous foundation of analysis. However, determining the limit of a sequence directly from the definition is often arduous. It requires one to have a candidate for the limit *a priori*, and the resulting inequality manipulations can be cumbersome for complex expressions.

To streamline the analysis of sequences, we establish the Limit Laws. These theorems allow us to decompose complex sequences into simpler components, compute their limits individually, and reassemble them algebraically. Rather than playing the "Challenger-Defender" game for every new sequence, we prove that the game is won for basic arithmetic operations, allowing us to compute limits mechanically.

4.1 Algebraic Limit Laws

We begin with the arithmetic properties of limits. The central philosophy is that the limit operation respects the standard algebraic structures of the real numbers.

***Theorem* 4.1.1. Algebraic Limit Laws**. Let (a_n) and (b_n) be convergent sequences such that $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. Let $c \in \mathbb{R}$ be a constant. Then:

(i) **Sum Law:** The sequence $(a_n + b_n)$ converges to $A + B$.
(ii) **Difference Law:** The sequence $(a_n - b_n)$ converges to $A - B$.
(iii) **Constant Multiple Law:** The sequence (ca_n) converges to cA.
(iv) **Product Law:** The sequence $(a_n b_n)$ converges to AB.
(v) **Quotient Law:** If $B \neq 0$ and $b_n \neq 0$ for all n, the sequence (a_n/b_n) converges to A/B.

Remark. As noted in the previous chapter, sequences may not be defined for the first few terms (e.g., division by zero). The Quotient Law implies that if $B \neq 0$, then $b_n \neq 0$ for all *sufficiently large* n, so the quotient sequence is eventually well-defined.

Proofs of Linear Properties

The proofs for addition and scalar multiplication rely on the Triangle Inequality and the judicious choice of a maximum threshold index.

Proof of the Sum Law. We wish to show that $|(a_n + b_n) - (A + B)| < \epsilon$ for sufficiently large n. By the Triangle Inequality:

$$|(a_n + b_n) - (A + B)| = |(a_n - A) + (b_n - B)| \leq |a_n - A| + |b_n - B|$$

Let $\epsilon > 0$. Since $a_n \to A$ and $b_n \to B$, there exist thresholds N_1 and N_2 such that:

$$|a_n - A| < \epsilon/2 \quad \text{for } n > N_1,$$
$$|b_n - B| < \epsilon/2 \quad \text{for } n > N_2.$$

Let $N = \max\{N_1, N_2\}$. For any $n > N$, both inequalities hold simultaneously. Thus:

$$|(a_n + b_n) - (A + B)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

By the definition of convergence, $a_n + b_n \to A + B$. ■

Proof of the Constant Multiple Law. We examine $|ca_n - cA| = |c||a_n - A|$. If $c = 0$, the sequence is constantly 0, which trivially converges to 0. If $c \neq 0$, let $\epsilon > 0$. We require $|a_n - A| < \epsilon/|c|$. Since $a_n \to A$, there exists N such that this holds for all $n > N$. Thus, for $n > N$:

$$|ca_n - cA| = |c||a_n - A| < |c|\left(\frac{\epsilon}{|c|}\right) = \epsilon$$

■

Combining these two results proves the Difference Law, as $a_n - b_n = a_n + (-1)b_n$.

Proof of the Product Law

The limit of a product is the product of the limits. The proof utilizes a standard analytical technique: adding and subtracting a "cross-term" (either a_nB or Ab_n) to apply the Triangle Inequality.

Proof. We analyze the difference $|a_nb_n - AB|$. By adding and subtracting the term a_nB inside the absolute value:

$$\begin{aligned} |a_nb_n - AB| &= |a_nb_n - a_nB + a_nB - AB| \\ &= |a_n(b_n - B) + B(a_n - A)| \\ &\leq |a_n||b_n - B| + |B||a_n - A| \quad \text{(Triangle Inequality)} \end{aligned}$$

We must bound the term $|a_n|$. Since (a_n) is a convergent sequence, it is bounded (see Properties of Convergent Sequences). Thus, there exists a constant $M > 0$ such that $|a_n| \leq M$ for all n. Consequently:

$$|a_nb_n - AB| \leq M|b_n - B| + |B||a_n - A|$$

Let $\epsilon > 0$. We apply the convergence of a_n and b_n:

- Since $b_n \to B$, there exists N_1 such that $|b_n - B| < \epsilon$ for $n > N_1$.
- Since $a_n \to A$, there exists N_2 such that $|a_n - A| < \epsilon$ for $n > N_2$.

Let $N = \max\{N_1, N_2\}$. For $n > N$:

$$|a_nb_n - AB| < M\epsilon + |B|\epsilon = \epsilon(M + |B|)$$

By the $K - \epsilon$ Principle (where $K = M + |B|$ is a constant independent of n and ϵ), the sequence converges to AB. ■

Proof of the Quotient Law

To prove $a_n/b_n \to A/B$, it suffices to prove that the sequence of reciprocals $1/b_n \to 1/B$. Once established, we can view the quotient as a product: $a_n \cdot (1/b_n) \to A \cdot (1/B)$.

***Lemma* 4.1.1.** *Bounding the Denominator.* If $b_n \to B$ and $B \neq 0$, then there exists N such that for all $n > N$, $|b_n| > |B|/2$.

Proof. Let $\epsilon = |B|/2$. Since $|B| > 0$, $\epsilon > 0$. Because $b_n \to B$, there exists N such that for all $n > N$, $|b_n - B| < |B|/2$. By the Reverse Triangle Inequality, $|B| - |b_n| \leq |b_n - B| < |B|/2$. Rearranging yields $|b_n| > |B| - |B|/2 = |B|/2$. ■

This lemma ensures that the denominator is eventually bounded away from zero.

Proof of Reciprocal Convergence. We evaluate the difference $|\frac{1}{b_n} - \frac{1}{B}|$:

$$\left|\frac{1}{b_n} - \frac{1}{B}\right| = \left|\frac{B - b_n}{b_n B}\right| = \frac{|b_n - B|}{|b_n||B|}$$

Using the lemma, for sufficiently large n, $|b_n| > |B|/2$. Thus, $\frac{1}{|b_n|} < \frac{2}{|B|}$. Substituting this bound:

$$\left|\frac{1}{b_n} - \frac{1}{B}\right| < \frac{|b_n - B|}{\left(\frac{|B|}{2}\right)|B|} = \frac{2}{|B|^2}|b_n - B|$$

Since $b_n \to B$, the term $|b_n - B|$ can be made arbitrarily small. By the $K - \epsilon$ Principle (with $K = 2/|B|^2$), the sequence converges to $1/B$. ■

The Quotient Law follows immediately from the Product Law applied to (a_n) and $(1/b_n)$.

Limits and Order

The limit process preserves the non-strict order relations of the real numbers. If one sequence is consistently larger than another, their limits must respect this hierarchy.

***Theorem* 4.1.2. Order Preservation.** Let (a_n) and (b_n) be convergent sequences with limits A and B respectively. If $a_n \leq b_n$ for all $n \in \mathbb{N}$ (or for all sufficiently large n), then $A \leq B$.

Proof. We proceed by contradiction. Consider the sequence $c_n = b_n - a_n$. Since $a_n \leq b_n$, we have $c_n \geq 0$ for all n. By the Difference Law, $c_n \to B - A$. Let $L = B - A$. Assume for contradiction that $A > B$, which implies $L < 0$. Since $c_n \to L$, for $\epsilon = |L|/2$, there exists N such that for $n > N$:

$$|c_n - L| < \frac{|L|}{2}$$

This implies $L - (|L|/2) < c_n < L + (|L|/2)$. Since L is negative, $L = -|L|$. The rightmost term is $-|L| + |L|/2 = -|L|/2 < 0$. Thus, $c_n < 0$ for $n > N$. This contradicts the hypothesis that $c_n \geq 0$. Therefore, the assumption is false, and $A \leq B$. ■

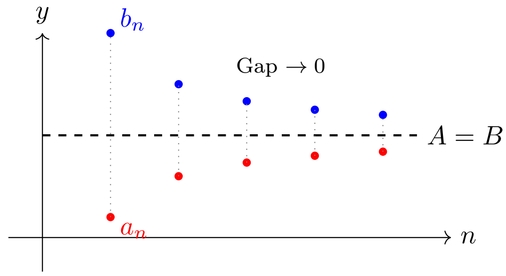


Figure 4.1: Although $a_n < b_n$ strictly for every n (shown by the gray gaps), the sequences converge to the same limit. The strict inequality is lost in the limit.

Note. Strict inequalities are not preserved by limits. For example, if $a_n = 0$ and $b_n = 1/n$, then $a_n < b_n$ for all n. However, $\lim a_n = 0$ and $\lim b_n = 0$, so the limits are equal ($A = B$), not strictly less.

***Theorem* 4.1.3. Sign Preservation Principle.** Let (a_n) be a convergent sequence with limit L.

1. If $L > 0$, then there exists $N \in \mathbb{N}$ such that $a_n > 0$ for all $n > N$.
2. More generally, if $\alpha < L < \beta$, then there exists $N \in \mathbb{N}$ such that $\alpha < a_n < \beta$ for all $n > N$.

Proof. For (1), let $\epsilon = L/2$. Since $L > 0$, $\epsilon > 0$. By the definition of convergence, there exists N such that for all $n > N$, $|a_n - L| < L/2$. This implies $L - L/2 < a_n < L + L/2$, so $a_n > L/2 > 0$. Statement (2) follows similarly by choosing $\epsilon = \min(L - \alpha, \beta - L)$. ■

The Squeeze Theorem

Frequently, we encounter sequences that cannot be directly simplified using algebraic limit laws (e.g., involving oscillating terms like $\sin n$ or factorials). The Squeeze Theorem (also known as the Sandwich Theorem or Pinching Theorem) allows us to determine the limit of such a sequence by trapping it between two simpler sequences that converge to the same value.

***Theorem* 4.1.4. The Squeeze Theorem.** Let (a_n), (b_n), and (c_n) be sequences such that for all sufficiently large n:

$$a_n \leq c_n \leq b_n$$

If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = L$, then (c_n) must also converge, and $\lim_{n\to\infty} c_n = L$.

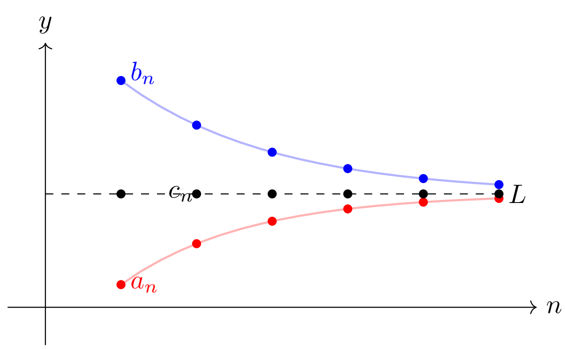


Figure 4.2: The Squeeze Theorem. Since a_n and b_n both converge to L, the sequence c_n, trapped between them, is forced to converge to L as well.

Proof. Let $\epsilon > 0$. We need to show that $|c_n - L| < \epsilon$ eventually. Subtracting L from the inequality $a_n \leq c_n \leq b_n$ gives:

$$a_n - L \leq c_n - L \leq b_n - L$$

From the convergence of the outer sequences:

1. Since $a_n \to L$, there exists N_1 such that for $n > N_1$, $|a_n - L| < \epsilon$, which implies $-\epsilon < a_n - L < \epsilon$. Specifically, we need $a_n - L > -\epsilon$.
2. Since $b_n \to L$, there exists N_2 such that for $n > N_2$, $|b_n - L| < \epsilon$, which implies $-\epsilon < b_n - L < \epsilon$. Specifically, we need $b_n - L < \epsilon$.

Let $N = \max\{N_1, N_2\}$. For $n > N$:

$$-\epsilon < a_n - L \leq c_n - L \leq b_n - L < \epsilon$$

Thus, $-\epsilon < c_n - L < \epsilon$, or $|c_n - L| < \epsilon$. ■

***Corollary* 4.1.1.** *Null Sequence Squeeze.* Suppose that $a \in \mathbb{R}$ and $(a_n), (x_n)$ are sequences such that:

$$|a_n - a| \leq x_n \quad \text{for all } n, \quad \text{and} \quad \lim_{n\to\infty} x_n = 0.$$

Then $\lim_{n\to\infty} a_n = a$.

Proof. We have squeezed the sequence $|a_n - a|$ between the constant sequence 0 and the sequence (x_n), both of which converge to 0. Thus $|a_n - a| \to 0$, which implies $a_n \to a$. ■

Example 4.1.1. Squeeze Theorem Application. Evaluate $\lim_{n\to\infty} (\sin n)/n$. The sine function oscillates, so we cannot use simple algebraic laws. However, we know that $-1 \leq \sin n \leq 1$ for all n. Dividing by n (where $n > 0$):

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$$

We identify the bounding sequences:

$$a_n = -\frac{1}{n} \quad \text{and} \quad b_n = \frac{1}{n}$$

We know that $\lim_{n\to\infty} (-1/n) = 0$ and $\lim_{n\to\infty} (1/n) = 0$. Since the limits are equal ($L = 0$), by the Squeeze Theorem:

$$\lim_{n\to\infty} \frac{\sin n}{n} = 0$$

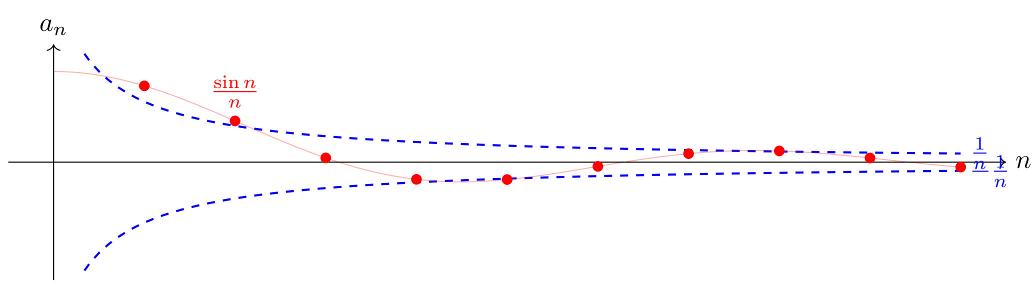


Figure 4.3: The sequence $\frac{\sin n}{n}$ oscillates boundedly but is trapped between the envelopes $\pm\frac{1}{n}$, forcing convergence to 0.

4.1.1 Null Sequences

In the analysis of convergence, it is often advantageous to centre our discussion around zero. A sequence (z_n) that converges to 0 is called a null sequence (or an infinitesimal sequence). The study of general convergence can be reduced to the study of null sequences by a simple translation:

$$\lim_{n\to\infty} x_n = L \iff \lim_{n\to\infty} (x_n - L) = 0$$

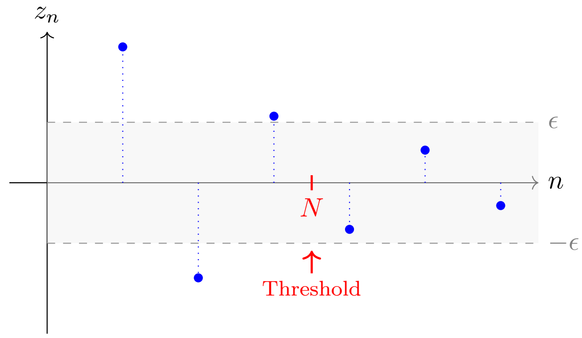


Figure 4.4: A null sequence (z_n). Regardless of sign, the terms eventually enter and stay within the ϵ-strip around 0.

Properties of Null Sequences

Null sequences enjoy specific algebraic stability.

1. **Sum/Difference**: If (a_n) and (b_n) are null sequences, then $(a_n \pm b_n)$ is a null sequence.
2. **Absolute Value**: (a_n) is a null sequence if and only if $(|a_n|)$ is a null sequence.
3. **Powers**: If (a_n) is a null sequence and $k \in \mathbb{N}$, then (a_n^k) is a null sequence.

The Bounded-Null Principle

A standard pitfall in limit arithmetic is assuming that $\lim(a_n b_n)$ requires both a_n and b_n to converge. However, if one sequence effectively "destroys" the magnitude of the other, convergence is preserved.

***Theorem* 4.1.5. Product of Bounded and Null Sequences**. Let (z_n) be a null sequence and let (b_n) be a bounded sequence. Then the product sequence $(z_n b_n)$ is a null sequence.

$$\lim_{n\to\infty} z_n = 0 \quad \text{and} \quad |b_n| \le M \implies \lim_{n\to\infty}(z_n b_n) = 0$$

Proof. Since (b_n) is bounded, there exists $M > 0$ such that $|b_n| \le M$ for all n. Let $\epsilon > 0$. We seek to show $|z_n b_n| < \epsilon$ eventually. Since (z_n) is null, there exists N such that for all $n > N$, $|z_n| < \epsilon/M$. Consequently, for all $n > N$:

$$|z_n b_n| = |z_n||b_n| < \frac{\epsilon}{M} \cdot M = \epsilon$$

Thus, $(z_n b_n) \to 0$. ∎

This theorem is computationally more efficient than the Squeeze Theorem for damped oscillations.

Example 4.1.2. Damped Oscillation. Evaluate $\lim_{n\to\infty} \frac{\sin(n^2)+\cos(n)}{n^2}$. We rewrite the term as:

$$\frac{1}{n^2} \cdot (\sin(n^2) + \cos(n))$$

- The sequence $z_n = 1/n^2$ is a null sequence (by the Archimedean property).
- The sequence $b_n = \sin(n^2) + \cos(n)$ is divergent and oscillates chaotically. However, by the triangle inequality, $|b_n| \le |\sin(n^2)| + |\cos(n)| \le 1 + 1 = 2$. Thus, (b_n) is bounded.

By the Bounded-Null Principle, the product converges to 0.

Remark. Note that the standard Product Law ($\lim a_n b_n = \lim a_n \lim b_n$) is inapplicable here because $\lim b_n$ does not exist.

Arithmetic of Infinity

The Algebraic Limit Laws can be extended to cases where limits are infinite, provided we adopt specific conventions. If $\lim a_n = \infty$ and $\lim b_n = \infty$, the following operations are well-defined:

$$\begin{aligned} \infty + \infty &= \infty \\ \infty \cdot \infty &= \infty \\ C \cdot \infty &= \begin{cases} \infty & \text{if } C > 0 \\ -\infty & \text{if } C < 0 \end{cases} \quad (\text{for } C \neq 0) \\ \frac{C}{\infty} &= 0 \quad (\text{for any } C \in \mathbb{R}) \end{aligned}$$

Remark. Indeterminate Forms: Extreme care must be taken when operations oppose each other. The following expressions are undefined in the arithmetic of limits and require more detailed analysis to resolve:

$$\frac{\infty}{\infty}, \quad 0 \cdot \infty, \quad \infty - \infty, \quad \frac{0}{0}, \quad 1^{\infty}.$$

For example, if $a_n = n^2$ and $b_n = n$, then ∞/∞ results in ∞. But if $a_n = n$ and $b_n = n^2$, ∞/∞ results in 0. The symbol ∞/∞ itself gives no information about the limit.

4.2 Convergence of Subsequences

Recall that a subsequence is formed by composing the sequence with a strictly increasing index function (dfn 3.2.1). Formally, if $\sigma : \mathbb{N} \to \mathbb{N}$ is such a function (where $n_k = \sigma(k)$), the subsequence is denoted $(a_{\sigma(n)})$. Having established the algebraic limit laws, we now turn to the topological relationship between a sequence and its subsequences.

Inheritance of Convergence

The first result is intuitive: if the "parent" sequence approaches a limit, any "child" sequence (subsequence) constrained to the same path must approach the same limit.

***Theorem* 4.2.1. Subsequence Convergence.** If a sequence (a_n) converges to a limit L, then every subsequence of (a_n) also converges to L.

Proof. Let (a_{n_k}) be a subsequence of (a_n), where the index mapping $n_k = \sigma(k)$ is strictly increasing. A fundamental property of strictly increasing functions on natural numbers is that $n_k \geq k$ for all $k \in \mathbb{N}$.

Let $\epsilon > 0$. Since $a_n \to L$, there exists a threshold $N(\epsilon)$ such that for all $n > N(\epsilon)$, $|a_n - L| < \epsilon$. Consider the subsequence term a_{n_k}. If we choose the index $k > N(\epsilon)$, then by the property of indices, $n_k \geq k > N(\epsilon)$. Consequently, the term a_{n_k} satisfies the convergence condition:

$$|a_{n_k} - L| < \epsilon$$

Thus, the same threshold function $N(\epsilon)$ suffices to prove the convergence of the subsequence. ■

A Criterion for Divergence

While the direct theorem is useful, its logical contrapositive provides a powerful tool for establishing divergence.

***Corollary* 4.2.1.** *Divergence Criterion.* If a sequence (a_n) contains two subsequences converging to distinct limits, then (a_n) is divergent.

This formalises our earlier observation regarding the oscillating sequence $1, 0, 1, 0, \ldots$, which possesses subsequences converging to 1 and 0 respectively. By the Uniqueness of Limits, the parent sequence cannot converge.

Example 4.2.1. Divergence of Trigonometric Sequences. Prove that the sequence $x_n = \sin n$ diverges.

Proof. Assume for contradiction that $\lim_{n\to\infty} \sin n = L$. We utilise the addition formula: $\sin(n+1)-\sin(n-1) = 2\sin(1)\cos(n)$. Taking the limit as $n \to \infty$:

$$L - L = 2\sin(1) \lim_{n\to\infty} \cos n \implies 0 = 2\sin(1) \lim_{n\to\infty} \cos n$$

Since $\sin(1) \neq 0$ (as 1 radian is not a multiple of π), it implies $\lim_{n\to\infty} \cos n = 0$. However, we also have the identity $\cos(2n) = 1 - 2\sin^2(n)$ or $\cos^2 n + \sin^2 n = 1$. Taking limits of $\cos^2 n + \sin^2 n = 1$:

$$(\lim \cos n)^2 + (\lim \sin n)^2 = 1 \implies 0^2 + L^2 = 1 \implies L^2 = 1$$

Taking limits of $\sin(2n) = 2\sin n \cos n$:If $\sin n \to L$ and $\cos n \to 0$, then the subsequence $\sin(2n)$ must converge to L (by uniqueness) but also to $2(L)(0) = 0$.Thus $L = 0$.We have reached a contradiction: $L^2 = 1$ and $L = 0$. Therefore, the sequence diverges. ■

***Proposition* 4.2.1.** *Odd-Even Decomposition.* A sequence (x_n) converges to L if and only if the subsequences of odd terms (x_{2n-1}) and even terms (x_{2n}) both converge to L.

Proof.

($\Rightarrow$) This follows immediately from the Subsequence Convergence Theorem.

($\Leftarrow$) Let $\epsilon > 0$. Since $x_{2n-1} \to L$, there exists N_{odd} such that $|x_{2n-1} - L| < \epsilon$ for all $2n-1 > N_{odd}$. Since $x_{2n} \to L$, there exists N_{even} such that $|x_{2n} - L| < \epsilon$ for all $2n > N_{even}$. Let $N = \max(N_{odd}, N_{even})$. For any $k > N$:
- If k is odd, $k > N_{odd}$, so $|x_k - L| < \epsilon$.
- If k is even, $k > N_{even}$, so $|x_k - L| < \epsilon$.

Thus, $|x_k - L| < \epsilon$ for all $k > N$, so the sequence converges. ■

The Subsequence Cover Property

The converse of the inheritance theorem ("if every subsequence converges to L, then the sequence converges to L"), is trivially true because the sequence is a subsequence of itself. However, a more subtle and powerful version exists. It states that we do not need *every* subsequence to converge; we only need to know that every subsequence *contains* a convergent sub-subsequence.

***Theorem* 4.2.2. The Subsubsequence Criterion**. Let (a_n) be a sequence. If every subsequence of (a_n) has a further subsequence that converges to L, then the sequence (a_n) converges to L.

Proof. We proceed by contradiction. Assume that every subsequence of (a_n) has a sub-subsequence converging to L, but the sequence (a_n) does not converge to L.

Step 1: Negating Convergence. The statement $a_n \not\to L$ implies that there exists a specific $\epsilon_0 > 0$ such that for any threshold $N \in \mathbb{N}$, there is at least one index $n > N$ where the term a_n lies outside the neighbourhood $(L - \epsilon_0, L + \epsilon_0)$.

$$\exists \epsilon_0 > 0, \forall N \in \mathbb{N}, \exists n > N \text{ such that } |a_n - L| \geq \epsilon_0$$

Step 2: Constructing a "Bad" Subsequence. We claim that the set of indices $S = \{n \in \mathbb{N} : |a_n - L| \geq \epsilon_0\}$ is infinite. If S were finite, we could choose $N > \max(S)$, and for all subsequent terms the inequality

$|a_n - L| < \epsilon_0$ would hold, implying convergence. Since S is infinite, we can construct a subsequence (a_{n_k}) consisting entirely of terms from S. We define the indices inductively:

$$\begin{aligned} n_1 &= \inf S \\ n_2 &= \inf(S \setminus \{n_1\}) \\ &\vdots \\ n_k &= \inf(S \setminus \{n_1, \ldots, n_{k-1}\}) \end{aligned}$$

This yields a strictly increasing sequence of indices n_k. By construction, for all k, $|a_{n_k} - L| \geq \epsilon_0$.

Step 3: The Contradiction. Consider this specific subsequence (a_{n_k}). By the hypothesis of the theorem, this subsequence must contain a further subsequence $(a_{n_{k_j}})$ that converges to L. However, every term in our constructed subsequence satisfies $|a_{n_k} - L| \geq \epsilon_0$. This property is inherited by any sub-subsequence. Thus, for all j:

$$|a_{n_{k_j}} - L| \geq \epsilon_0$$

It is impossible for a sequence to converge to L if all its terms remain at least a distance ϵ_0 away from L. This contradicts the hypothesis that a convergent sub-subsequence exists.

Therefore, the initial assumption must be false, and (a_n) converges to L. ■

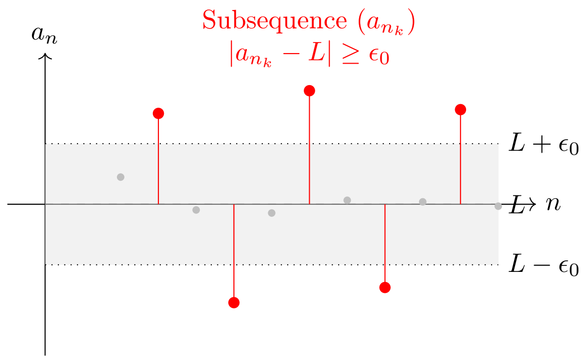


Figure 4.5: Visualisation of the contradiction proof. If $a_n \not\to L$, we can extract a subsequence (red points) strictly outside the ϵ_0-band. This red subsequence cannot have any sub-subsequence converging to L, contradicting the theorem's hypothesis.

4.3 Special Limits II

Having established the algebraic limit laws and the Squeeze Theorem, and having examined fundamental power functions in the section 3.4, we focus on some more specific sequences that appear frequently in analysis.

Geometric Series

In section 3.4, we proved that for a geometric sequence with ratio $|b| < 1$, the terms decay to zero ($\lim\limits_{n\to\infty} b^n = 0$). Conversely, if $b > 1$, we can write $b = 1 + r$ with $r > 0$. By the Binomial expansion, $(1 + r)^n \geq 1 + nr$, which grows without bound. Thus, $b^n \to \infty$.

This behaviour allows us to sum the Geometric Series. Let S_n be the sequence of partial sums:

$$S_n = 1 + b + b^2 + \cdots + b^n$$

We utilize the algebraic identity $(1-b)(1+b+\cdots+b^n) = 1-b^{n+1}$. Assuming $|b|<1$, we can solve for S_n:

$$S_n = \frac{1-b^{n+1}}{1-b}$$

Since $|b|<1$, we have $b^{n+1} \to 0$. Thus:

$$\lim_{n\to\infty} \sum_{k=0}^{n} b^k = \frac{1}{1-b}$$

Parametrized Limits

Occasionally, a limit depends on an external parameter x. This requires careful case analysis. Consider the sequence $a_n(x) = \frac{1}{1+n^2x}$ for a fixed $x \in \mathbb{R}$.

- If $x = 0$, then $a_n(0) = 1/1 = 1$ for all n. Thus, $\lim_{n\to\infty} a_n(0) = 1$.
- If $x > 0$, then $n^2x \to \infty$, so the denominator diverges to infinity and $a_n(x) \to 0$.
- If $x < 0$, the sequence is undefined when $1 + n^2x = 0$. Assuming x is such that the denominator is non-zero for large n, $|n^2x| \to \infty$, so the limit is 0.

The limit function is 1 at $x = 0$ and 0 elsewhere. This discontinuity presages the distinction between pointwise and uniform convergence.

Roots of Constants

We previously proved that $\lim_{n\to\infty} \sqrt[n]{n} = 1$ using the AM-GM inequality. We can apply similar logic (or Bernoulli's inequality) to constants.

***Theorem* 4.3.1. Limit of n-th Roots of Constants**. For any $P > 0$, $\lim_{n\to\infty} \sqrt[n]{P} = 1$.

Proof. If $P = 1$, the result is trivial.

Case 1: $P > 1$. Let $x_n = \sqrt[n]{P} - 1$. Since $P > 1$, we have $x_n > 0$. Raising both sides to the power n and applying Bernoulli's Inequality (or the first term of the Binomial expansion):

$$P = (1 + x_n)^n \geq 1 + nx_n$$

Rearranging yields $0 < x_n \leq \frac{P-1}{n}$. By the Squeeze Theorem, $x_n \to 0$, implying $\sqrt[n]{P} \to 1$.

Case 2: $0 < P < 1$. Let $Q = 1/P > 1$. Then $\sqrt[n]{P} = 1/\sqrt[n]{Q}$. Since we just proved $\sqrt[n]{Q} \to 1$, the reciprocal converges to $1/1 = 1$.

∎

The Hierarchy of Growth

We conclude with a rigorous comparison of polynomial, exponential, and factorial growth. The colloquial wisdom is that "exponentials beat polynomials" and "factorials beat exponentials."

***Theorem* 4.3.2. Polynomial vs. Exponential**. Let $a > 1$ and $k \in \mathbb{N}$. Then:

$$\lim_{n\to\infty} \frac{n^k}{a^n} = 0 \quad \left(\text{equivalently } \frac{a^n}{n^k} \to \infty\right)$$

Proof. Let $a = 1 + b$ with $b > 0$. We expand $(1 + b)^n$ using the Binomial Theorem. For $n > k + 1$, we look specifically at the term with index $k + 1$:

$$(1+b)^n = \sum_{j=0}^{n} \binom{n}{j} b^j > \binom{n}{k+1} b^{k+1} = \frac{n(n-1)\cdots(n-k)}{(k+1)!} b^{k+1}$$

The expression on the right behaves like a polynomial in n of degree $k + 1$. Specifically, for large n, the product $n(n - 1)\cdots(n - k)$ grows as n^{k+1}. Thus, there exists a constant C such that $(1 + b)^n > Cn^{k+1}$. Dividing by n^k:

$$\frac{a^n}{n^k} > C\frac{n^{k+1}}{n^k} = Cn$$

Since $Cn \to \infty$, the reciprocal n^k/a^n converges to 0. ■

***Theorem* 4.3.3. Exponential vs. Factorial**. For any $r \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{r^n}{n!} = 0$$

Proof. Assume $r \neq 0$. Fix an integer $N_0 > 2|r|$. For any $n > N_0$, we split the product into a constant part and a "tail":

$$\left|\frac{r^n}{n!}\right| = \underbrace{\frac{|r|^{N_0}}{N_0!}}_{\text{Constant } C} \cdot \underbrace{\frac{|r|}{N_0+1} \cdot \frac{|r|}{N_0+2} \cdots \frac{|r|}{n}}_{n-N_0 \text{ terms}}$$

In the second part, every denominator is greater than $N_0 > 2|r|$, so each factor is strictly less than $\frac{|r|}{2|r|} = \frac{1}{2}$.

$$0 < \left|\frac{r^n}{n!}\right| < C \cdot \left(\frac{1}{2}\right)^{n-N_0} = (C2^{N_0})\frac{1}{2^n}$$

Since we proved earlier that $q^n \to 0$ for $|q| < 1$, $(1/2)^n \to 0$. By the Squeeze Theorem, the limit is 0. ■

This establishes the standard hierarchy of dominance as $n \to \infty$:

$$\log n \ll n^k \ll a^n \ll n! \ll n^n$$

where $A \ll B$ implies $\lim_{n\to\infty}(A/B) = 0$.

4.4 Monotone Convergence and Bolzano-Weierstrass

We have established the definition of convergence and the algebraic laws that govern limits. However, using the definition directly requires us to know the limit *a priori*. In this section, we develop powerful tools to establish the convergence of a sequence based solely on its internal properties (specifically, its boundedness and monotonicity), without needing to guess the limit value beforehand.

The Monotone Convergence Theorem

Intuitively, if a sequence moves in only one direction (monotonicity) and is prevented from escaping to infinity by a barrier (boundedness), it must eventually bunch up against that barrier. This geometric intuition is formalised by the Completeness Axiom.

***Theorem* 4.4.1. Monotone Convergence Theorem (MCT)**. A monotone sequence of real numbers is convergent if and only if it is bounded. Specifically:

1. If (a_n) is increasing and bounded above, it converges to $\sup\{a_n\}$.
2. If (a_n) is decreasing and bounded below, it converges to $\inf\{a_n\}$.

Proof.

($\Rightarrow$) If a sequence converges, we have already shown it must be bounded (see Properties of Convergent Sequences).

($\Leftarrow$) We prove the case where (a_n) is increasing and bounded above. The decreasing case is analogous.Since the set of terms $A = \{a_n \mid n \in \mathbb{N}\}$ is non-empty and bounded above, the The Completeness Axiom ensures the existence of a supremum $L = \sup A$. We claim $\lim_{n\to\infty} a_n = L$. Let $\epsilon > 0$. By the characterisation of the supremum, L is the *least* upper bound, so $L - \epsilon$ is not an upper bound. Therefore, there exists some index N such that $a_N > L - \epsilon$. Since the sequence is increasing, for all $n \geq N$, we have $a_n \geq a_N$. Combining this with the fact that L is an upper bound ($a_n \leq L$):

$$L - \epsilon < a_N \leq a_n \leq L < L + \epsilon$$

Thus, $|a_n - L| < \epsilon$ for all $n \geq N$. ■

Example 4.4.1. Recursive Sequences. Consider the sequence defined by $x_1 = 1$ and $x_{n+1} = \sqrt{2 + x_n}$.

Boundedness: We claim $x_n < 2$. For $n = 1$, $1 < 2$. If $x_k < 2$, then $x_{k+1} = \sqrt{2 + x_k} < \sqrt{2 + 2} = 2$. By induction, the sequence is bounded above.

Monotonicity: We claim $x_{n+1} > x_n$. This is equivalent to $\sqrt{2 + x_n} > x_n$, or $2 + x_n > x_n^2$, or $x_n^2 - x_n - 2 < 0$. Since the roots of $t^2 - t - 2$ are 2 and -1, and $0 < x_n < 2$, the inequality holds.

By the MCT, the sequence converges to a limit L. Passing the limit through the recurrence

$$L = \sqrt{2 + L} \implies L^2 - L - 2 = 0 \implies (L - 2)(L + 1) = 0$$

Since $x_n > 0$, the limit must be $L = 2$.

4.4.1 The Bolzano-Weierstrass Theorem

While the MCT is powerful, not all bounded sequences are monotone. For instance, the sequence $((-1)^n)$ is bounded but oscillates. However, it contains a convergent subsequence (e.g., the constant sequence of 1s). The Bolzano-Weierstrass Theorem generalises this observation, asserting that *every* bounded sequence contains a convergent subsequence. This is a cornerstone result in analysis, linking the algebraic property of boundedness to the topological property of compactness.We present two proofs: one relying on the construction of a monotone subsequence (Newman's approach), and one utilising the Nested Interval Property (the Bisection method).

Proof I: The Peak Point Lemma

This approach relies on extracting order from chaos. We show that every sequence, no matter how erratic, contains a monotone subsequence.

Definition 4.4.1. ***Peak Point***. Let (a_n) be a sequence. A term a_n is called a *peak point* (or simply a peak) if it is at least as large as every term that follows it $a_n \geq a_m$for all $m > n$.

Intuitively, if we view the sequence as a landscape, a peak point is a location from which "looking forward" (to higher indices) reveals no higher ground.

***Lemma* 4.4.1.** *Monotone Subsequence Lemma.* Every sequence of real numbers possesses a monotone subsequence.

Proof. We consider two cases regarding the number of peak points in the sequence (a_n).

Case 1: Infinite Peak Points. Suppose there are infinitely many peak points. We list their indices in increasing order: $n_1 < n_2 < n_3 < \dots$. Since a_{n_1} is a peak point and $n_2 > n_1$, we have $a_{n_1} \geq a_{n_2}$. Similarly, since a_{n_2} is a peak point and $n_3 > n_2$, we have $a_{n_2} \geq a_{n_3}$. In general, $a_{n_k} \geq a_{n_{k+1}}$. Thus, (a_{n_k}) is a decreasing subsequence.

Case 2: Finite Peak Points. Suppose there are only finitely many peak points (possibly none). Let N be the largest index among the peak points (set $N = 0$ if there are none). For any $n > N$, the term a_n is *not* a peak point. We construct an increasing subsequence as follows:

- Choose any index $n_1 > N$. Since a_{n_1} is not a peak point, there must exist some $n_2 > n_1$ such that $a_{n_2} > a_{n_1}$.
- Since $n_2 > N$, a_{n_2} is not a peak point. Thus, there exists $n_3 > n_2$ such that $a_{n_3} > a_{n_2}$.
- Proceeding inductively, we find $n_{k+1} > n_k$ such that $a_{n_{k+1}} > a_{n_k}$.

This yields a strictly increasing subsequence (a_{n_k}). ■

***Theorem* 4.4.2. Bolzano-Weierstrass**. Every bounded sequence of real numbers has a convergent subsequence.

Proof via Monotone Subsequences. Let (a_n) be a bounded sequence. By the Monotone Subsequence Lemma, (a_n) contains a monotone subsequence (a_{n_k}). Since the original sequence is bounded, the subsequence (a_{n_k}) is also bounded. By the Monotone Convergence Theorem, a bounded monotone sequence must converge. ■

Proof II: The Method of Bisection

The second proof is constructive and geometric. It traps the terms of the subsequence within successively smaller intervals, invoking the Nested Interval Property.

Proof via Bisection. Let (a_n) be a bounded sequence. Thus, there exists $M > 0$ such that $a_n \in [-M, M]$ for all n. Let $I_1 = [-M, M]$.

Step 1: Constructing Nested Intervals. We divide I_1 into two closed sub-intervals of equal length, $J_1 = [-M, 0]$ and $J_2 = [0, M]$. The set of indices $\mathbb{N}$ is infinite. By the Pigeonhole Principle (extended to infinite sets), at least one of these sub-intervals must contain terms a_n for infinitely many indices n.Let I_2 be the half containing infinitely many terms. We repeat this process. Given an interval I_k containing infinitely many terms of the sequence, we split it into two equal closed halves. We select one half, I_{k+1}, that contains infinitely many terms of the sequence.This generates a sequence of closed intervals $I_1 \supseteq I_2 \supseteq I_3 \ldots$ with the following properties:

1. The length of I_k is $2M/2^{k-1}$.
2. Each I_k contains a_n for infinitely many values of n.

Step 2: Finding the Limit. By the Nested Interval Property (see Number System notes), the intersection $\bigcap_{k=1}^{\infty} I_k$ is non-empty. Since the lengths converge to 0 (Archimedean Property), the intersection contains a unique point L.

Step 3: Constructing the Subsequence. We construct the subsequence (a_{n_k}) recursively:

- Choose n_1 such that $a_{n_1} \in I_1$. (Always possible).
- Choose n_2 such that $n_2 > n_1$ and $a_{n_2} \in I_2$. (Possible because I_2 contains infinitely many indices).
- Generally, having chosen n_k, choose $n_{k+1} > n_k$ such that $a_{n_{k+1}} \in I_{k+1}$.

For any k, both a_{n_k} and L lie within the interval I_k. Therefore:

$$|a_{n_k} - L| \leq \text{length}(I_k) = \frac{2M}{2^{k-1}}$$

As $k \to \infty$, the length approaches 0. By the Squeeze Theorem, $\lim_{k\to\infty} a_{n_k} = L$.

■

4.4.2 Applications of Monotone Convergence

The Monotone Convergence Theorem allows us to define constants and algorithms rigorously that would otherwise rely on intuition. We examine two classical applications: the definition of Euler's number and the algorithmic computation of square roots.

The Euler Number

Consider the sequence $e_n = (1 + \frac{1}{n})^n$. While it is well-known that this converges to $e \approx 2.718$, proving existence requires the MCT. To do so, we introduce an auxiliary sequence $y_n = (1 + \frac{1}{n})^{n+1}$.

***Theorem* 4.4.3. Existence of e.** The sequence $e_n = (1 + \frac{1}{n})^n$ is strictly increasing and bounded above. Consequently, it converges to a limit, denoted by e.

Proof. We first analyse the auxiliary sequence $y_n = (1+\frac{1}{n})^{n+1} = (\frac{n+1}{n})^{n+1}$. Consider the ratio of consecutive terms for $n \geq 1$:

$$\begin{aligned}\frac{y_{n-1}}{y_n} &= \frac{(\frac{n}{n-1})^n}{(\frac{n+1}{n})^{n+1}} = \left(\frac{n}{n-1}\right)^n \left(\frac{n}{n+1}\right)^{n+1} \\ &= \frac{n}{n+1}\left(\frac{n^2}{n^2-1}\right)^n = \frac{n}{n+1}\left(1 + \frac{1}{n^2-1}\right)^n\end{aligned}$$

By Bernoulli's Inequality, $(1 + x)^n \geq 1 + nx$. With $x = \frac{1}{n^2-1}$:

$$\left(1 + \frac{1}{n^2-1}\right)^n \geq 1 + \frac{n}{n^2-1} > 1 + \frac{n}{n^2} = \frac{n+1}{n}$$

Substituting this back:

$$\frac{y_{n-1}}{y_n} > \frac{n}{n+1} \cdot \frac{n+1}{n} = 1 \implies y_{n-1} > y_n$$

Thus, (y_n) is strictly decreasing. Since $y_n > 1$, it is bounded below, so $\lim y_n$ exists. Now consider e_n. We observe that:

$$e_n = y_n \cdot \frac{1}{1 + 1/n} = y_n \cdot \frac{n}{n+1}$$

Since $\lim_{n\to\infty} \frac{n}{n+1} = 1$, the limits of (e_n) and (y_n) are identical. Furthermore, since y_n decreases to the limit from above, and $e_n < y_n$, e_n is bounded above (specifically by $y_1 = (1 + 1)^2 = 4$). It can also be shown via the AM-GM inequality that e_n is strictly increasing. Thus, the limit exists. ■

The Babylonian Method

We apply the MCT to a sequence defined recursively by the Newton-Raphson method (known historically as the Babylonian method) to compute square roots. To calculate $\sqrt{\alpha}$ for $\alpha > 0$, we define:

$$x_1 > \sqrt{\alpha}, \quad x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) \tag{4.1}$$

***Theorem* 4.4.4. Convergence to $\sqrt{\alpha}$.** The sequence defined above converges to $\sqrt{\alpha}$.

Proof. **Boundedness:** Using the AM-GM inequality on the terms x_n and α/x_n:

$$x_{n+1} = \frac{x_n + \alpha/x_n}{2} \geq \sqrt{x_n \cdot \frac{\alpha}{x_n}} = \sqrt{\alpha}$$

Thus, $x_n \geq \sqrt{\alpha}$ for all $n \geq 2$.

Monotonicity: Since $x_n \geq \sqrt{\alpha}$, we have $x_n^2 \geq \alpha$, which implies $\alpha/x_n \leq x_n$. Therefore:

$$x_{n+1} = \frac{1}{2}(x_n + \frac{\alpha}{x_n}) \leq \frac{1}{2}(x_n + x_n) = x_n$$

The sequence is decreasing and bounded below by $\sqrt{\alpha}$. By the MCT, it converges to a limit $L \geq \sqrt{\alpha}$. Passing the limit through the recurrence:

$$L = \frac{1}{2}\left(L + \frac{\alpha}{L}\right) \implies 2L = L + \frac{\alpha}{L} \implies L^2 = \alpha \implies L = \sqrt{\alpha}$$

■

4.5 Cauchy Sequences

In our study of convergence thus far, the definition of a limit, $\lim_{n\to\infty} a_n = L$, requires *a priori* knowledge of the value L. We verify convergence by testing whether the terms of the sequence eventually reside within an arbitrary ϵ-neighbourhood of this candidate limit. However, in many practical and theoretical contexts, the limit is unknown or difficult to compute explicitly. We therefore require an intrinsic criterion for convergence — one that depends solely on the internal behaviour of the sequence's terms relative to one another, rather than their proximity to an external point.

Definition and Intuition

Intuitively, if a sequence converges to a specific point L, the terms must eventually cluster around L. Consequently, as the terms crowd closer to L, they must essentially crowd closer to *each other*. If we consider the tail of the sequence, all terms are confined within a small interval; thus, the distance between any two terms in that tail must be small.

Definition 4.5.1. ***Cauchy Sequence***. A sequence (a_n) is called a Cauchy sequence if for every $\epsilon > 0$, there exists a natural number $N \in \mathbb{N}$ such that for all $n, m > N$:

$$|a_n - a_m| < \epsilon$$

This definition formalises the notion of mutual proximity. Unlike the definition of convergence, which anchors the sequence to a static limit L, the Cauchy condition requires that the terms eventually become indistinguishable from one another at any given scale of precision.

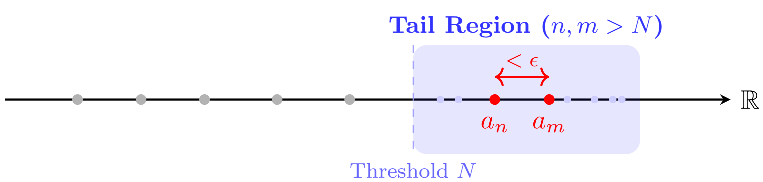


Figure 4.6: The Cauchy Condition. Beyond the index threshold N, the terms of the sequence are trapped in a "tail" where they bunch together.

Examples of Cauchy Analysis

To verify the Cauchy criterion, we typically assume without loss of generality that $m > n$ and write $m = n+p$ for some $p \geq 1$. We then attempt to bound the difference $|a_{n+p} - a_n|$ independently of p.

Example 4.5.1. Geometric Sequences. Let $|q| < 1$. We show that the sequence $a_n = q^n$ is Cauchy. Consider the difference for $m > n$:

$$|a_m - a_n| = |q^{n+p} - q^n| = |q^n(q^p - 1)| = |q|^n|1 - q^p|$$

Using the triangle inequality and the fact that $|q| < 1$ (implies $|q|^p < 1$):

$$|1 - q^p| \leq 1 + |q|^p < 2$$

Thus:

$$|a_m - a_n| < 2|q|^n$$

Since $|q| < 1$, the sequence $(2|q|^n)$ is a null sequence. For any $\epsilon > 0$, there exists N such that $2|q|^n < \epsilon$ for all $n > N$. Consequently, (a_n) is Cauchy.

Example 4.5.2. The Alternating Sequence. Let $a_n = (-1)^n$. Consider the distance between consecutive terms ($p = 1$):

$$|a_{n+1} - a_n| = |(-1)^{n+1} - (-1)^n| = |(-1)^n(-1 - 1)| = |-1 - 1| = 2$$

If we choose $\epsilon = 1$, there is no N such that $|a_m - a_n| < 1$ for all $m, n > N$, since consecutive terms always differ by 2. Thus, the sequence is not Cauchy (and therefore divergent).

Example 4.5.3. Convergence of a Finite Series. Consider the sequence of partial sums defined by $a_n = \sum_{k=1}^{n} \frac{1}{k^2}$. For $m > n$:

$$|a_m - a_n| = \sum_{k=1}^{m} \frac{1}{k^2} - \sum_{k=1}^{n} \frac{1}{k^2} = \sum_{k=n+1}^{m} \frac{1}{k^2}$$

We utilise the inequality $\frac{1}{k^2} < \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$ for $k \geq 2$. Applying this to the sum (telescoping sum):

$$|a_m - a_n| < \sum_{k=n+1}^{m} \left(\frac{1}{k-1} - \frac{1}{k}\right) = \left(\frac{1}{n} - \frac{1}{n+1}\right) + \cdots + \left(\frac{1}{m-1} - \frac{1}{m}\right)$$

Most terms cancel, leaving:

$$|a_m - a_n| < \frac{1}{n} - \frac{1}{m} < \frac{1}{n}$$

For any $\epsilon > 0$, choosing $N > 1/\epsilon$ ensures that $|a_m - a_n| < 1/N < \epsilon$. Thus, the sequence converges.

Example 4.5.4. Divergence of the Harmonic Sums. Consider $a_n = \sum_{k=1}^{n} \frac{1}{k}$. We test the Cauchy condition with $m = 2n$.

$$|a_{2n} - a_n| = \sum_{k=n+1}^{2n} \frac{1}{k} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$

This sum contains n terms. The smallest term is the last one, $1/2n$.

$$|a_{2n} - a_n| \geq \sum_{k=n+1}^{2n} \frac{1}{2n} = n \cdot \frac{1}{2n} = \frac{1}{2}$$

For $\epsilon = 1/2$, the condition fails. No matter how large N is, we can always find $n > N$ and $m = 2n$ such that the distance is at least $1/2$. Therefore, the sequence is not Cauchy and diverges to infinity.

Boundedness of Cauchy Sequences

Before establishing the relationship between Cauchy sequences and convergent sequences, we prove a fundamental property: Cauchy sequences cannot oscillate wildly or diverge to infinity.

***Proposition* 4.5.1.** *Boundedness of Cauchy Sequences.* Every Cauchy sequence is bounded.

Proof. Let (a_n) be a Cauchy sequence. We apply the definition with a fixed tolerance, say $\epsilon = 1$. There exists $N \in \mathbb{N}$ such that for all $n, m > N$, $|a_n - a_m| < 1$. Fix $m = N + 1$. Then for all $n > N$:

$$|a_n - a_{N+1}| < 1$$

By the Reverse Triangle Inequality, $|a_n| - |a_{N+1}| \leq |a_n - a_{N+1}| < 1$, which implies:

$$|a_n| < 1 + |a_{N+1}| \quad \text{for all } n > N.$$

The sequence consists of the tail (bounded by $1 + |a_{N+1}|$) and the finite set of initial terms $\{a_1, \ldots, a_N\}$. Let M be the maximum of these magnitudes:

$$M = \max\{|a_1|, |a_2|, \ldots, |a_N|, 1 + |a_{N+1}|\}$$

Then $|a_n| \leq M$ for all $n \in \mathbb{N}$. Thus, the sequence is bounded. ∎

The Cauchy Convergence Criterion

We now state and prove the central theorem regarding Cauchy sequences. In the real number system, the property of being Cauchy is equivalent to the property of being convergent.

***Theorem* 4.5.1. Cauchy Convergence Criterion**. A sequence of real numbers is convergent if and only if it is a Cauchy sequence.

Proof. The proof consists of two directions.

Convergent $\implies$ Cauchy. Suppose $\lim_{n\to\infty} a_n = L$. Let $\epsilon > 0$ be given. We must show that mutual distances $|a_n - a_m|$ can be made less than ϵ. Since $a_n \to L$, there exists $N \in \mathbb{N}$ such that for all $k > N$, $|a_k - L| < \epsilon/2$. Let $n, m > N$. We introduce the limit L into the expression via the Triangle Inequality:

$$\begin{aligned} |a_n - a_m| &= |(a_n - L) + (L - a_m)| \\ &\leq |a_n - L| + |L - a_m| \\ &= |a_n - L| + |a_m - L| \end{aligned}$$

Since both $n, m > N$, both terms are bounded by $\epsilon/2$.

$$|a_n - a_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus, (a_n) is Cauchy.

Cauchy $\implies$ Convergent. This direction relies on the completeness of $\mathbb{R}$. Suppose (a_n) is a Cauchy sequence.

i **Existence of a Candidate Limit.** By the previous proposition, (a_n) is bounded. By Bolzano-Weierstrass, every bounded sequence contains a convergent subsequence. Let (a_{n_k}) be a subsequence converging to a limit L. We claim that the entire sequence (a_n) converges to L.

ii **Convergence of the Whole Sequence.** Let $\epsilon > 0$. We use the convergent subsequence as an "anchor".

- Since (a_n) is Cauchy, there exists N_1 such that for all $n, m > N_1$, $|a_n - a_m| < \epsilon/2$.
- Since $a_{n_k} \to L$, there exists K such that for all $k > K$, $|a_{n_k} - L| < \epsilon/2$.

We choose an index from the subsequence, n_k, that is sufficiently large. Specifically, we choose k such that $k > K$ and $n_k > N_1$. (This is possible because indices of a subsequence tend to infinity). Now, let $n > N_1$. We estimate the distance from a_n to L:

$$|a_n - L| \leq |a_n - a_{n_k}| + |a_{n_k} - L|$$

The first term $|a_n - a_{n_k}|$ is less than $\epsilon/2$ because both indices n and n_k are greater than N_1 (using the Cauchy property). The second term $|a_{n_k} - L|$ is less than $\epsilon/2$ because of the convergence of the subsequence.

$$|a_n - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus, $\lim_{n\to\infty} a_n = L$.

■

This theorem is profound because it allows us to prove convergence without knowing the limit. It is the defining characteristic of a *complete* metric space.

Completeness via Monotone Convergence

In our construction of the real numbers, we posited the Completeness Axiom (Least Upper Bound Property) as a foundational axiom. We then derived the Monotone Convergence Theorem (MCT), the Archimedean Property, and the Nested Interval Property (NIP). It is instructive to note that these properties are deeply interlinked. In fact, if we assume the MCT as an axiom, we can derive the other forms of completeness.

***Theorem* 4.5.2. MCT implies the Archimedean Property**. Assume the Monotone Convergence Theorem (MCT) holds. Then for any $x, y > 0$, there exists $n \in \mathbb{N}$ such that $nx > y$.

Proof. Assume, for contradiction, that the Archimedean Property fails. Then there exist $x, y > 0$ such that $nx \le y$ for all $n \in \mathbb{N}$. Consider the sequence $a_n = nx$. Since $x > 0$, the sequence is strictly increasing $(a_{n+1} - a_n = x > 0)$. By our assumption, $a_n \le y$ for all n, so the sequence is bounded above. By the MCT, the sequence must converge to a limit L. However, if a sequence converges, it must be Cauchy. Let us test the difference between consecutive terms:

$$|a_{n+1} - a_n| = |(n+1)x - nx| = x$$

For the sequence to be Cauchy, this difference must eventually be less than any ϵ. If we choose $\epsilon = x/2$, we see that $|a_{n+1} - a_n| = x > x/2$ for all n. Thus, the sequence is not Cauchy and therefore not convergent. This contradicts the conclusion of the MCT. Thus, the sequence cannot be bounded, and the Archimedean Property must hold. ■

***Theorem* 4.5.3. MCT implies the Nested Interval Property**. Assume the Monotone Convergence Theorem holds. Let $[a_n, b_n]$ be a sequence of nested closed intervals ($[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$). Then $\bigcap_{n=1}^{\infty}[a_n, b_n] \neq \emptyset$.

Proof. Consider the sequence of left endpoints (a_n). Since the intervals are nested, $a_n \le a_{n+1}$ for all n, so (a_n) is increasing. Furthermore, for any n, $a_n \le b_n \le b_1$. Thus, (a_n) is bounded above by b_1. By the MCT, the sequence (a_n) converges to a limit $A = \sup\{a_n\}$. We claim $A \in \bigcap[a_n, b_n]$. Since A is the supremum of the left endpoints, $a_n \le A$ for all n. Also, since every b_k is an upper bound for the set of all a_n (due to the nested property, any left endpoint is less than any right endpoint), we must have $A \le b_k$ for all k. Thus, $a_k \le A \le b_k$ for all k, which implies $A \in [a_k, b_k]$ for all k. The intersection is non-empty. ■

4.6 Exercises

Part I: Computational Limits and Basic Laws

1. Use the Limit Laws and the Squeeze Theorem to determine the limits of the following sequences, or prove that they diverge.

(a) $a_n = \frac{3^n+(-2)^n}{3^{n+1}+(-2)^{n+1}}$
(b) $b_n = \sqrt{n}(\sqrt{n+1} - \sqrt{n})$
(c) $c_n = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\cdots\left(1 - \frac{1}{n^2}\right)$
(d) $d_n = \frac{1}{n^2}(1 + 2 + \cdots + n)$

2. **Limits of n-th roots.** Using the fact that $\lim_{n\to\infty} n^{1/n} = 1$ and the Squeeze Theorem, evaluate:

   (a) $\lim_{n\to\infty} (2^n + 3^n)^{1/n}$
   (b) $\lim_{n\to\infty} (1^n + 2^n + \cdots + 100^n)^{1/n}$
   (c) $\lim_{n\to\infty} (n^2 - n + 2)^{1/n}$

3. **Rationalisation.** Evaluate the limit of the sequence given by:

$$x_n = n\left(\sqrt{1 + \frac{1}{n}} - 1\right).$$

   Remark. Multiply by the conjugate expression $\frac{\sqrt{1+1/n}+1}{\sqrt{1+1/n}+1}$. This linearisation technique is the precursor to the derivative.

4. **Infinite Products.** Consider the sequence $P_n = \prod_{k=2}^{n}\left(1 - \frac{1}{k}\right)$.

   (a) Write out the first few terms of the product and simplify the expression for P_n algebraically.
   (b) Prove that $\lim_{n\to\infty} P_n = 0$.
   (c) Now consider $Q_n = \prod_{k=2}^{n}\left(1 - \frac{1}{k^2}\right)$. Prove that $\lim_{n\to\infty} Q_n = \frac{1}{2}$.

5. **Divergence by Summation.** Prove that the sequence $S_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$ diverges to $+\infty$.

   Remark. Bound the terms $\frac{1}{\sqrt{k}}$ from below by $\frac{1}{\sqrt{n}}$.

Part II: Theoretical Properties

6. **Monotone Convergence in Action.** Let $x_1 = \sqrt{2}$ and define the sequence recursively by $x_{n+1} = \sqrt{2 + x_n}$.

   (a) Prove by induction that $x_n < 2$ for all n.
   (b) Prove by induction that $x_{n+1} > x_n$ for all n.
   (c) **Monotone Convergence Theorem:** This theorem states that any bounded, monotone sequence converges. Accepting this result (without proof) implies that the limit $L = \lim_{n\to\infty} x_n$ exists. Determine the exact value of L by taking the limit as $n \to \infty$ on both sides of the defining relation $x_{n+1} = \sqrt{2 + x_n}$.

7. **Ratio Tests for Sequences.** Let (a_n) be a sequence of positive terms.

   (a) Prove that if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1$, then $\lim_{n\to\infty} a_n = 0$.
   (b) Show by counter-example that if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, the sequence may converge or diverge. *Consider $a_n = n$ and $a_n = 1/n$.*

8. **Cesàro Means (Arithmetic Mean Limit).** Let (a_n) be a sequence that converges to a. Define a new sequence of averages $\sigma_n = \frac{a_1+a_2+\cdots+a_n}{n}$.

   (a) Prove that $\lim_{n\to\infty} \sigma_n = a$.

Remark. Split the sum into two parts. Since $a_n \to a$, for any ϵ, there is an N such that $a_k \approx a$ for $k > N$. The terms before N become negligible when divided by a large n.

(b) Give an example of a divergent sequence (a_n) for which the sequence of averages (σ_n) converges.

9. Geometric Mean Limit. Using the result from the previous exercise and the identity $\ln(a_1 \dots a_n) = \sum \ln(a_i)$, prove that if (a_n) is a sequence of positive numbers converging to $a > 0$, then:

$$\lim_{n\to\infty} \sqrt[n]{a_1 a_2 \cdots a_n} = a.$$

10. Cauchy's Root-Ratio Theorem. Using the previous exercise, prove that for a sequence of positive numbers (a_n):

$$\text{If } \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L, \quad \text{then } \lim_{n\to\infty} \sqrt[n]{a_n} = L.$$

Use this to verify that $\lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = e$.

Remark. Let $x_n = \frac{n^n}{n!}$.

Part III: Advanced Challenges

11. The Dominant Term (Max-Norm). Let $a_1, a_2, \dots, a_m$ be non-negative real numbers. Let $M = \max\{a_1, \dots, a_m\}$. Prove that:

$$\lim_{n\to\infty} (a_1^n + a_2^n + \cdots + a_m^n)^{1/n} = M.$$

Remark. Factor out M^n from the summation and use the Squeeze Theorem. Note that $1 \le \sum (a_k/M)^n \le m$.

12. Contractive Sequences. A sequence (x_n) is called *contractive* if there exists a constant $k \in (0, 1)$ such that $|x_{n+2} - x_{n+1}| \le k|x_{n+1} - x_n|$ for all n.

(a) Show that $|x_{n+1} - x_n| \le k^{n-1}|x_2 - x_1|$.

(b) Use the Triangle Inequality to show that for $m > n$:

$$|x_m - x_n| \le |x_2 - x_1| \frac{k^{n-1}}{1-k}.$$

(c) Conclude that (x_n) is a Cauchy sequence (and therefore convergent).

13. ⋆ Stability Analysis of Non-Linear Recurrence. Let c be a real number. Consider the recursive sequence defined by $a_1 = c/2$ and

$$a_{n+1} = \frac{c}{2} + \frac{a_n^2}{2}.$$

(a) Suppose the sequence converges to a limit L. Show that L must satisfy the quadratic equation $L^2 - 2L + c = 0$.

(b) For which values of c does this equation have real solutions? What are the possible values for L?

(c) **Case $0 < c \le 1$:** Prove by induction that the sequence is increasing and bounded above by $1 - \sqrt{1-c}$. Conclude that the limit is indeed $1 - \sqrt{1-c}$.

(d) **Case $c > 1$:** Prove that $a_n \to +\infty$.

Remark. Show that if the sequence were bounded, it would converge to a root of the quadratic, which is impossible for $c > 1$.

14. ⋆ The Toeplitz Theorem (Generalised Averaging). This is a generalisation of the Arithmetic Mean exercise. Let $t_{n,k}$ be a grid of non-negative weights ($1 \le k \le n$) such that for every fixed k, $\lim_{n\to\infty} t_{n,k} = 0$, and for every n, $\sum_{k=1}^{n} t_{n,k} = 1$. Let (a_n) converge to L. Define the transformed sequence $x_n = \sum_{k=1}^{n} t_{n,k} a_k$. Prove that $\lim_{n\to\infty} x_n = L$.

Remark. Write $x_n - L = \sum_{k=1}^{n} t_{n,k}(a_k - L)$. Split the sum into a "head" (small indices k) where $t_{n,k}$ vanishes as $n \to \infty$, and a "tail" (large indices k) where $|a_k - L|$ is small.

15. $\star$ **A Pseudo-Cauchy Condition.** *A fundamental property of real numbers is that if terms get close to each other, the sequence converges (The Cauchy Criterion).* However, "close" must be defined carefully.

(a) Give an example of a divergent sequence (a_n) such that $\lim_{n\to\infty} |a_{n+1} - a_n| = 0$. *This shows that distance between consecutive terms going to zero is insufficient.*

(b) Now, suppose a sequence satisfies the stronger condition:

$$|a_{n+p} - a_n| \leq \frac{p}{n^2}$$

for all positive integers p. Prove that this sequence converges.

Remark. Fix n and let p vary to bound the tail of the sequence. Recall that $\sum 1/n^2$ converges, which suggests the "total distance" remaining is finite.

16. $\star$ **Quasi-Monotone Convergence.** Let (x_n) be a non-negative sequence satisfying the inequality:

$$x_{n+1} \leq x_n + \frac{1}{n^2} \quad \text{for all } n \in \mathbb{N}.$$

Prove that (x_n) converges.

Remark. Consider the auxiliary sequence $y_n = x_n - \sum_{k=1}^{n-1} \frac{1}{k^2}$. Show that (y_n) is monotone decreasing and bounded below. You may assume that $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges to a finite limit.

17. $\star\star$ **Binomial Averaging.** Let (a_n) be a sequence converging to L. Prove that:

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} a_k = L.$$

Remark. This is a specific application of the Toeplitz Theorem. Identify the weights $t_{n,k} = \binom{n}{k}2^{-n}$. You must justify why for fixed k, $\binom{n}{k}2^{-n} \to 0$ as $n \to \infty$. Stirling's approximation or simple factorial bounds may help.

18. $\star\star$ **The Trapped Recurrence.** Let (a_n) be a sequence such that $0 < a_n < 1$ for all n, satisfying the inequality:

$$(1 - a_n)a_{n+1} > \frac{1}{4}.$$

Prove that $\lim_{n\to\infty} a_n = \frac{1}{2}$.

Remark. First, recall the algebraic inequality $x(1-x) \leq \frac{1}{4}$ for all real x. Use the given inequality to show that $a_{n+1} > a_n$, making the sequence strictly increasing. Then apply the Monotone Convergence Theorem and solve for the limit.

19. $\star\star$ **Parameter Dependent Convergence.** Let $u_1 = b$. Define a sequence recursively by:

$$u_{n+1} = u_n^2 + (1 - 2a)u_n + a^2 \quad (n \geq 1).$$

Discuss the convergence of (u_n) for different values of the parameters a and b.

Remark. Complete the square to rewrite the recurrence in the form $u_{n+1} - a = (u_n - a)^2 + (u_n - a)$. Let $v_n = u_n - a$ to simplify the system to $v_{n+1} = v_n(1 + v_n)$. Under what conditions does $v_n \to 0$?

20. $\star\star\star$ **Characterisation of Divergence.** Let (a_n) be a bounded sequence. Prove that (a_n) diverges if and only if it contains two subsequences converging to distinct limits.

Remark. The "if" direction is the Divergence Criterion. For the "only if" direction: if (a_n) diverges, it has no single limit. By Bolzano-Weierstrass, it has *at least one* convergent subsequence with limit L. Since the whole sequence does not converge to L, construct a subsequence entirely outside a neighbourhood of L. Apply Bolzano-Weierstrass to this "outer" subsequence to find a second limit $L' \neq L$.

Chapter 5

Infinite Sums

Having established a rigorous framework for sequences and their limits, we now turn our attention to the summation of infinitely many terms. While the concept of adding an infinite list of numbers appears intuitive, it requires a precise definition based on the theory of sequences to avoid the paradoxes often associated with infinity.

5.1 Infinite Series

An infinite series is essentially a sequence of sums. Given a sequence of real numbers (b_n), we do not simply "add them all up" in one step; rather, we define the sum as the limit of partial additions.

Definition 5.1.1. ***Infinite Series***. Let $(b_n)_{n=1}^{\infty}$ be a sequence of real numbers. The formal expression $\sum_{n=1}^{\infty} b_n$ is called an infinite series. We define the m-th *partial sum* S_m as:

$$S_m := \sum_{n=1}^{m} b_n = b_1 + b_2 + \cdots + b_m$$

The series is said to converge to a limit L if the sequence of partial sums (S_m) converges to L. In this case, we write:

$$\sum_{n=1}^{\infty} b_n = L$$

If the sequence (S_m) diverges, the series is said to *diverge*.

It is crucial to distinguish between the sequence of terms (b_n) and the sequence of partial sums (S_m). The behaviour of the series is determined strictly by (S_m).

***Proposition* 5.1.1.** *The Divergence Test.* If the series $\sum_{n=1}^{\infty} b_n$ converges, then $\lim_{n\to\infty} b_n = 0$.

Proof. Let S_m converge to L. We observe that $b_m = S_m - S_{m-1}$. By the Algebraic Limit Laws:

$$\lim_{m\to\infty} b_m = \lim_{m\to\infty} S_m - \lim_{m\to\infty} S_{m-1} = L - L = 0$$

∎

Remark. The condition $b_n \to 0$ is necessary but not sufficient for convergence. A series may have terms tending to zero yet still diverge, as we shall demonstrate with the harmonic series.

Series with Non-Negative Terms

The analysis of infinite series is significantly simplified when the terms b_n are non-negative ($b_n \geq 0$). In this scenario, the sequence of partial sums is monotonic.

$$S_{m+1} = S_m + b_{m+1} \geq S_m$$

Since (S_m) is increasing, we may apply the Monotone Convergence Theorem (MCT).

***Theorem* 5.1.1. Convergence of Non-Negative Series.** A series $\sum_{n=1}^{\infty} b_n$ with $b_n \geq 0$ converges if and only if the sequence of partial sums (S_m) is bounded above.

Proof. Let S_m be the partial sums. Since $b_n \geq 0$, we have $S_{m+1} - S_m = b_{m+1} \geq 0$. Thus, (S_m) is a monotonically increasing sequence. The result follows immediately from the Monotone Convergence Theorem (MCT): an increasing sequence converges if and only if it is bounded above. ■

This theorem allows us to determine convergence without explicit knowledge of the limit value L. We usually establish boundedness by comparing the series to a known quantity.

Example: The Inverse Squares

Consider the series of reciprocal squares:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$$

Since the terms are positive, the partial sums S_m are increasing. To prove convergence, we must demonstrate that S_m is bounded above. We employ a technique of estimation by finding a larger series that is easier to sum (a telescoping series).

Proof. We seek an upper bound for $S_m = \sum_{n=1}^{m} \frac{1}{n^2}$. For $n \geq 2$, we have the inequality $n^2 > n(n-1)$. Consequently:

$$\frac{1}{n^2} < \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$$

We can now estimate the partial sum:

$$\begin{aligned} S_m &= 1 + \sum_{n=2}^{m} \frac{1}{n^2} \\ &< 1 + \sum_{n=2}^{m} \left(\frac{1}{n-1} - \frac{1}{n} \right) \end{aligned}$$

The summation on the right is a telescoping sum. Expanding the terms reveals the cancellation:

$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{m-1} - \frac{1}{m}\right) = 1 - \frac{1}{m}$$

Thus, we have the bound:

$$S_m < 1 + \left(1 - \frac{1}{m}\right) = 2 - \frac{1}{m} < 2$$

Since S_m is increasing and bounded above by 2, the series converges by the MCT. ■

Note. The heuristic used here involves increasing individual terms to form a simpler, summable expression. While this proof confirms convergence, it does not reveal the exact sum, which Euler famously proved to be $\pi^2/6$.

Example: The Harmonic Series

We now consider the harmonic series:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

Although the terms approach zero ($\lim 1/n = 0$), the series diverges. We prove this by showing that the partial sums are unbounded. The proof utilises a technique of grouping terms, often attributed to Nicole Oresme (c. 1350).

Proof. Let us examine specific partial sums at indices that are powers of 2. Let $m = 2^k$.

$$S_{2^k} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \dots + \frac{1}{8}\right) + \dots + \left(\frac{1}{2^{k-1}+1} + \dots + \frac{1}{2^k}\right)$$

We define groups of terms ending at powers of 2. The j-th group (where j runs from 1 to k) sums terms from $n = 2^{j-1}+1$ to $n = 2^j$. The number of terms in the j-th group is $2^j - 2^{j-1} = 2^{j-1}$. In each group, the smallest term is the last one, $1/2^j$. Therefore, we can bound the sum of the group from below:

$$\sum_{n=2^{j-1}+1}^{2^j} \frac{1}{n} \geq 2^{j-1} \cdot \frac{1}{2^j} = \frac{1}{2}$$

Applying this lower bound to the expression for S_{2^k}:

$$S_{2^k} \geq 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{2}\right)}_{\text{Group 2}} + \underbrace{\left(\frac{1}{2}\right)}_{\text{Group 3}} + \dots + \underbrace{\left(\frac{1}{2}\right)}_{\text{Group } k}$$

There are $k-1$ groups following the first $1/2$. Thus:

$$S_{2^k} \geq 1 + \frac{1}{2} + (k-1)\frac{1}{2} = 1 + \frac{k}{2}$$

As $k \to \infty$, $1 + k/2 \to \infty$. Consequently, the sequence of partial sums (S_m) is unbounded. By the MCT (contrapositive), the series diverges. ∎

Remark. The divergence of the harmonic series is extremely slow. For the sum to exceed 100, one would require approximately 1.5×10^{43} terms. Nonetheless, it diverges to infinity.

This grouping technique (estimating sums over blocks of size 2^k), is generalised by the Cauchy Condensation Test, which provides a powerful criterion for the convergence of monotone decreasing series.

5.2 The Cauchy Condensation Test

The technique employed to establish the divergence of the harmonic series—grouping terms into blocks of powers of two—admits a powerful generalisation known as the Cauchy Condensation Test. This criterion allows us to test the convergence of a series with monotonically decreasing terms by replacing it with a "condensed" series that is often far easier to analyse (typically a geometric series).

***Theorem* 5.2.1. Cauchy Condensation Test**. Let (b_n) be a non-increasing sequence of non-negative real numbers ($b_1 \geq b_2 \geq \dots \geq 0$). Then the series $\sum_{n=1}^{\infty} b_n$ converges if and only if the condensed series

$$\sum_{k=0}^{\infty} 2^k b_{2^k} = b_1 + 2b_2 + 4b_4 + 8b_8 + \dots$$

converges.

Remark. The intuition here is sparsity versus magnitude. The terms b_{2^k} are selected exponentially far apart, but we weight them by 2^k to compensate for the terms we skipped. The test asserts that these two effects balance perfectly for monotone sequences.

Proof. We examine the partial sums of the original series, $S_m = \sum_{n=1}^{m} b_n$, and the partial sums of the condensed series, $T_k = \sum_{j=0}^{k} 2^j b_{2^j}$. Since $b_n \geq 0$, both sequences of partial sums are non-decreasing. By the MCT, convergence is equivalent to boundedness.

($\Leftarrow$) **Convergence Implication.** Suppose the condensed series converges to a limit T. We wish to show that (S_m) is bounded. Fix $m \in \mathbb{N}$. Choose k sufficiently large such that $m \leq 2^{k+1} - 1$. We group the terms of S_m as follows:

$$\begin{aligned} S_m &\leq S_{2^{k+1}-1} \\ &= b_1 + (b_2 + b_3) + (b_4 + b_5 + b_6 + b_7) + \cdots + (b_{2^k} + \cdots + b_{2^{k+1}-1}) \end{aligned}$$

In the group $(b_{2^j} + \cdots + b_{2^{j+1}-1})$, there are 2^j terms. Since (b_n) is non-increasing, the largest term in this group is the first one, b_{2^j}. However, to obtain an upper bound involving the condensed series, we observe that for the convergence direction, we essentially bounded $b_2 + b_3 \leq 2b_2$, $b_4 + \cdots + b_7 \leq 4b_4$, and so on. Thus:

$$S_m \leq b_1 + 2b_2 + 4b_4 + \cdots + 2^k b_{2^k} = T_k \leq T$$

Since S_m is bounded by T for all m, the original series converges.

($\Rightarrow$) **Divergence Implication.** Suppose the condensed series diverges ($T_k \to \infty$). We show that S_m must also be unbounded. We group terms to find a lower bound. Consider the partial sum up to 2^k:

$$\begin{aligned} S_{2^k} &= b_1 + b_2 + (b_3 + b_4) + (b_5 + \cdots + b_8) + \cdots + (b_{2^{k-1}+1} + \cdots + b_{2^k}) \\ &\geq b_1 + b_2 + 2b_4 + 4b_8 + \cdots + 2^{k-1} b_{2^k} \\ &= b_1 + \frac{1}{2}\left(2b_2 + 4b_4 + 8b_8 + \cdots + 2^k b_{2^k}\right) \\ &= b_1 + \frac{1}{2}(T_k - b_1) \end{aligned}$$

If $T_k \to \infty$, then $S_{2^k} \to \infty$. Thus the original series diverges.

∎

5.2.1 The p-series Test

The primary application of the Condensation Test is to determine the convergence of the generalised harmonic series, commonly known as the p-series.

***Theorem* 5.2.2. Convergence of p-series**. The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Proof. Let $b_n = 1/n^p$. This sequence is non-negative and decreasing for all $p \in \mathbb{R}$. We apply the Cauchy Condensation Test. The condensed series is:

$$\sum_{k=0}^{\infty} 2^k b_{2^k} = \sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^p} = \sum_{k=0}^{\infty} \frac{2^k}{2^{kp}} = \sum_{k=0}^{\infty} (2^{1-p})^k$$

This is a geometric series $\sum r^k$ with ratio $r = 2^{1-p}$.

Case $p > 1$: Then $1 - p < 0$, so $r = 2^{1-p} < 1$. A geometric series with ratio strictly less than 1 converges. Thus, the original series converges.

Case $p \leq 1$: Then $1 - p \geq 0$, so $r = 2^{1-p} \geq 1$. A geometric series with ratio $r \geq 1$ diverges. Thus, the original series diverges.

∎

Remark. This result establishes the boundary between convergence and divergence for polynomial decay. The harmonic series ($p = 1$) is the threshold. For $p > 1$, the sum defines the **Riemann Zeta Function**, denoted $\zeta(p) = \sum_{n=1}^{\infty} n^{-p}$. While we have proven convergence, finding the exact sum is difficult. Euler famously proved $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$, though values for odd integers (like $\zeta(3)$) remain mysterious.

5.2.2 The Cauchy Criterion for Series

In the study of sequences, we established that a sequence converges if and only if it is a Cauchy sequence. As an infinite series is defined by its sequence of partial sums, we can translate the Cauchy Convergence Criterion directly to the context of series.

***Theorem* 5.2.3. Cauchy Criterion for Series**. The series $\sum_{n=1}^{\infty} b_n$ converges if and only if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > m > N$:

$$\left| \sum_{k=m+1}^{n} b_k \right| = |S_n - S_m| < \epsilon$$

Proof. Let $S_n = \sum_{k=1}^{n} b_k$ be the sequence of partial sums. By definition, the series converges if and only if (S_n) converges. By the Cauchy Convergence Criterion, the sequence (S_n) converges if and only if it is a Cauchy sequence. The condition for (S_n) to be Cauchy is exactly:

$$|S_n - S_m| < \epsilon \quad \text{for all } n, m > N$$

Substituting $S_n - S_m = \sum_{k=m+1}^{n} b_k$ (assuming $n > m$) yields the theorem statement immediately. ∎

This criterion allows us to prove the convergence of a series by showing that the "tail" of the sum becomes arbitrarily small, without needing to compute the limit or show monotonicity.

Example 5.2.1. Alternating Harmonic Series. Consider $\sum_{n=1}^{\infty} (-1)^{n+1}/n$. The partial sums are not monotone, so the previous tests do not apply directly. However, using the Cauchy Criterion, one can show that for large m, the alternating sum $\left| \frac{1}{m+1} - \frac{1}{m+2} + \cdots \pm \frac{1}{n} \right|$ is bounded by the first term $\frac{1}{m+1}$, which vanishes as $m \to \infty$. Thus, the series converges.

5.3 General Convergence Tests

Hitherto, our methods for determining the convergence of a series have relied on specific structural properties, such as monotonicity (Integral Test, Condensation Test) or the ability to compute partial sums explicitly (Telescoping Series). We now establish a broader class of tests based on the principle of *comparison*. The fundamental philosophy is simple: if a series is "smaller" than a convergent series, it converges; if it is "larger" than a divergent series, it diverges.

5.3.1 The Comparison Test

***Theorem* 5.3.1. Direct Comparison Test**. Let $\sum a_n$ and $\sum b_n$ be series with non-negative terms such that, for all $n \in \mathbb{N}$ (or for all n sufficiently large), $0 \leq a_n \leq b_n$.

(i) If $\sum b_n$ converges, then $\sum a_n$ converges.
(ii) If $\sum a_n$ diverges, then $\sum b_n$ diverges.

Proof. Part (ii) is the contrapositive of part (i). We prove (i) using the Cauchy Criterion for Series. Suppose $\sum b_n$ converges. Let $\epsilon > 0$. By the Cauchy Criterion for Series, there exists a threshold N such that for all $n > m > N$:

$$\sum_{k=m+1}^{n} b_k < \epsilon$$

(The absolute value is redundant as terms are non-negative). By the hypothesis $0 \leq a_k \leq b_k$, it follows immediately that:

$$\sum_{k=m+1}^{n} a_k \leq \sum_{k=m+1}^{n} b_k < \epsilon$$

Thus, the partial sums of $\sum a_n$ satisfy the Cauchy criterion and must converge. ■

Note. To apply the Direct Comparison Test effectively, one requires a "toolkit" of reference series. The most common references are the Geometric Series ($\sum r^n$) and the p-series ($\sum 1/n^p$), whose convergence properties we have fully characterised.

Remark. Asymptotic Dominance: The convergence of a series is determined solely by the behaviour of its "tail" (terms beyond some index N). Consequently, the condition $a_n \leq b_n$ need not hold for all n, but merely for all $n \geq N$ for some fixed N.

5.3.2 Absolute and Conditional Convergence

The Comparison Test requires non-negative terms. To handle series with mixed signs, we introduce the concept of absolute convergence.

Definition 5.3.1. ***Absolute Convergence***. A series $\sum a_n$ is said to be absolutely convergent if the series of absolute values $\sum |a_n|$ converges. If $\sum a_n$ converges but $\sum |a_n|$ diverges, the series is said to be *conditionally convergent.*

***Proposition* 5.3.1.** *Absolute Convergence Implies Convergence.* If a series $\sum a_n$ is absolutely convergent, then it is convergent.

Proof. This is a direct application of the Cauchy Criterion for Series. Since $\sum |a_n|$ converges, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > m > N$:

$$\sum_{k=m+1}^{n} |a_k| < \epsilon$$

By the Triangle Inequality for finite sums:

$$\left| \sum_{k=m+1}^{n} a_k \right| \leq \sum_{k=m+1}^{n} |a_k|$$

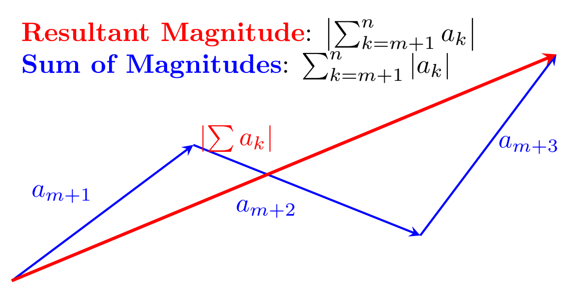


Figure 5.1: Visualisation of the Triangle Inequality. The length of the red vector (the modulus of the sum) is shorter than the total length of the blue path (the sum of the moduli).

Consequently, $\left|\sum_{k=m+1}^{n} a_k\right| < \epsilon$. Thus, $\sum a_n$ satisfies the Cauchy Criterion and is convergent. ■

5.3.3 The Ratio Test

Often, finding a direct comparison series is difficult. The Ratio Test allows us to compare a series to a geometric series by examining the rate at which its terms decay.

***Theorem* 5.3.2. d'Alembert's Ratio Test**. Let (a_n) be a sequence of non-zero terms. Suppose the limit

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

exists.

(i) If $L < 1$, the series $\sum a_n$ is absolutely convergent.
(ii) If $L > 1$, the series $\sum a_n$ is divergent.
(iii) If $L = 1$, the test is inconclusive.

Proof.

Case (i): $L < 1$. Since $L < 1$, we can choose a number r such that $L < r < 1$. By the definition of the limit, there exists $N \in \mathbb{N}$ such that for all $n \geq N$:

$$\left|\frac{a_{n+1}}{a_n}\right| < r \implies |a_{n+1}| < r|a_n|$$

By induction, $|a_{N+k}| < |a_N|r^k$. Thus, the tail of the series is dominated by a convergent geometric series $\sum |a_N|r^k$ (since $r < 1$). By the Direct Comparison Test, $\sum |a_n|$ converges.

Case (ii): $L > 1$. There exists N such that for all $n \geq N$, $|a_{n+1}|/|a_n| > 1$. This implies the terms $|a_n|$ are strictly increasing eventually, so $\lim a_n \neq 0$. By the The Divergence Test, the series diverges.

Case (iii): $L = 1$. We exhibit two examples where $L = 1$ but the behaviour differs:

- $\sum \frac{1}{n}$ diverges (Harmonic Series).
- $\sum \frac{1}{n^2}$ converges (p-series with $p = 2$).

In both cases, $\lim |\frac{a_{n+1}}{a_n}| = 1$. Thus, the test provides no information.

■

5.4 Applications of Convergence

We conclude our discussion on series by examining two celebrated applications: a number-theoretic proof regarding the distribution of prime numbers, and the mathematical resolution of a classical philosophical paradox.

The Divergence of Prime Reciprocals

Paul Erdős was renowned for solving a vast range of problems using elementary yet ingenious techniques. One of his most beautiful proofs concerns the harmonic series restricted to prime numbers. While the harmonic series $\sum 1/n$ diverges, and the series of squares $\sum 1/n^2$ converges, the distribution of primes is sufficiently dense that their sum also diverges.

***Theorem* 5.4.1. Erdős's Theorem on Primes**. The series of reciprocals of prime numbers diverges:

$$\sum_{p \text{ prime}} \frac{1}{p} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots = \infty$$

Proof. We proceed by contradiction. Assume the series converges. By the definition of convergence, the tail of the series must vanish. There exists an index k such that:

$$\sum_{j=k+1}^{\infty} \frac{1}{p_j} < \frac{1}{2}$$

We classify the prime numbers into two sets:

- **Small Primes:** $P_S = \{p_1, p_2, \dots, p_k\}$.
- **Big Primes:** $P_B = \{p_{k+1}, p_{k+2}, \dots\}$.

Let N be a large natural number (to be determined later). We partition the set of integers $\{1, 2, \dots, N\}$ into two disjoint sets based on their prime factorisation:

$$N_{big} = \{n \le N : n \text{ is divisible by at least one big prime}\}$$
$$N_{small} = \{n \le N : n \text{ is divisible } \textit{only} \text{ by small primes}\}$$

Note that $1 \in N_{small}$ (as the empty product of primes). Since these sets partition $\{1, \dots, N\}$, we must have $|N_{big}| + |N_{small}| = N$. We now estimate the size of these sets.

Estimating $|N_{big}|$: For any prime p, the number of multiples of p less than or equal to N is $\lfloor N/p \rfloor$. Thus:

$$|N_{big}| \le \sum_{j=k+1}^{\infty} \left\lfloor \frac{N}{p_j} \right\rfloor \le \sum_{j=k+1}^{\infty} \frac{N}{p_j} = N \sum_{j=k+1}^{\infty} \frac{1}{p_j}$$

Using our initial assumption that the tail sum is less than 1/2:

$$|N_{big}| < \frac{N}{2}$$

Estimating $|N_{small}|$: Let $n \in N_{small}$. By the Fundamental Theorem of Arithmetic, we can write n in the form $n = a_n^2 b_n$, where b_n is a square-free integer.

1. **The square-free part (b_n):** Since n has only small prime factors, b_n must be a product of distinct primes from $P_S = \{p_1, \dots, p_k\}$. There are 2^k possible subsets of P_S, so there are at most 2^k choices for b_n.
2. **The square part (a_n^2):** Since $a_n^2 \le n \le N$, we have $a_n \le \sqrt{N}$. There are at most $\sqrt{N}$ choices for a_n.

The total number of elements in N_{small} is bounded by the product of these choices:

$$|N_{small}| \le 2^k \sqrt{N}$$

The Contradiction: Combining the estimates:

$$N = |N_{big}| + |N_{small}| < \frac{N}{2} + 2^k \sqrt{N}$$

Rearranging yields $\frac{N}{2} < 2^k \sqrt{N}$, or $\sqrt{N} < 2^{k+1}$. However, k is fixed (determined by the convergence of the series), whereas N is arbitrary. If we choose $N = 2^{2k+2}$, we obtain:

$$2^{2k+2} < \frac{2^{2k+2}}{2} + 2^k \cdot 2^{k+1} = 2^{2k+1} + 2^{2k+1} = 2^{2k+2}$$

This implies $N < N$, a contradiction. Thus, the series diverges. ∎

***Corollary* 5.4.1.** *Infinitude of Primes.* The set of prime numbers is infinite.

Proof. If there were finitely many primes, the sum $\sum(1/p)$ would be a finite sum of rational numbers, which converges. Since the series diverges, the set of primes must be infinite. ∎

Resolution of Zeno's Paradox

The theory of infinite series provides a rigorous resolution to Zeno's Dichotomy Paradox. Zeno of Elea argued that motion is impossible because to travel a distance d, one must first travel $d/2$, then half the remaining distance $d/4$, then $d/8$, and so on. He claimed that because this process involves an infinite number of steps, it cannot be completed in finite time.

The Mathematical Resolution Zeno's fundamental error lies in the implicit assumption that the sum of infinitely many time intervals must be infinite. Let the total distance be d and the velocity be v. The time taken to cover the first half is $t_1 = \frac{d/2}{v}$. The time for the next quarter is $t_2 = \frac{d/4}{v}$, and generally $t_n = \frac{d}{2^n v}$. The total time T is the sum of these intervals:

$$T = \sum_{n=1}^{\infty} t_n = \frac{d}{v} \sum_{n=1}^{\infty} \frac{1}{2^n}$$

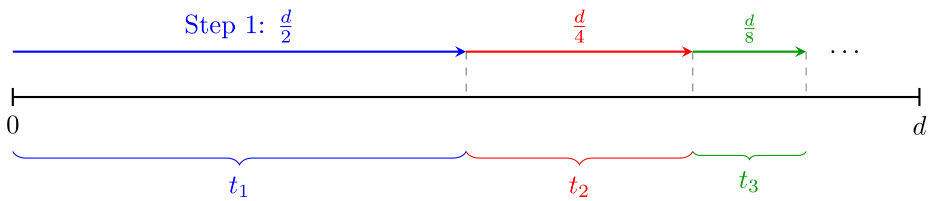


Figure 5.2: Visualisation of Zeno's Paradox.

This is a geometric series with $r = 1/2$.

$$T = \frac{d}{v} \left(\frac{1/2}{1 - 1/2} \right) = \frac{d}{v} \cdot 1 = \frac{d}{v}$$

The infinite series converges to a finite value. Thus, an infinite succession of events can indeed occur within a finite duration, resolving the paradox.

5.5 Conditional Convergence and Rearrangements

We have hitherto focused on series with non-negative terms. As defined in section 5.3, we distinguish between absolute convergence (where $\sum |a_n|$ converges) and conditional convergence (where $\sum a_n$ converges but $\sum |a_n|$ diverges). Recall that absolute convergence implies ordinary convergence. We now focus on the delicate nature of conditional convergence, where the sum exists solely due to the cancellation of signs.

5.5.1 The Alternating Series Test

The prototype of a conditionally convergent series is the alternating harmonic series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

While $\sum 1/n$ diverges, the alternating signs provide sufficient cancellation for the sum to exist. We formalise this observation in the Leibniz Criterion.

***Theorem* 5.5.1. Alternating Series Test (Leibniz).** Let (b_n) be a decreasing sequence of positive real numbers such that $\lim_{n\to\infty} b_n = 0$. Then the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ converges.

Proof. Let $S_n = \sum_{k=1}^{n} (-1)^{k+1} b_k$ denote the partial sums. We examine the behaviour of the even and odd partial sums separately. Consider the even partial sums S_{2n}:

$$S_{2n} = (b_1 - b_2) + (b_3 - b_4) + \dots + (b_{2n-1} - b_{2n})$$

Since (b_n) is decreasing, each parenthetical term is non-negative. Thus, (S_{2n}) is an increasing sequence. Furthermore, we can rewrite S_{2n} as:

$$S_{2n} = b_1 - (b_2 - b_3) - (b_4 - b_5) - \dots - b_{2n}$$

Since each term in parentheses is non-negative and $b_{2n} > 0$, we have $S_{2n} \leq b_1$. The sequence (S_{2n}) is increasing and bounded above, hence it converges to a limit L. Similarly, the odd partial sums can be written as $S_{2n+1} = S_{2n} + b_{2n+1}$. Since $S_{2n} \to L$ and $b_{2n+1} \to 0$ (by hypothesis), we have:

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} b_{2n+1} = L + 0 = L$$

Since both the even and odd subsequences of partial sums converge to the same limit L, the entire sequence of partial sums converges to L. ■

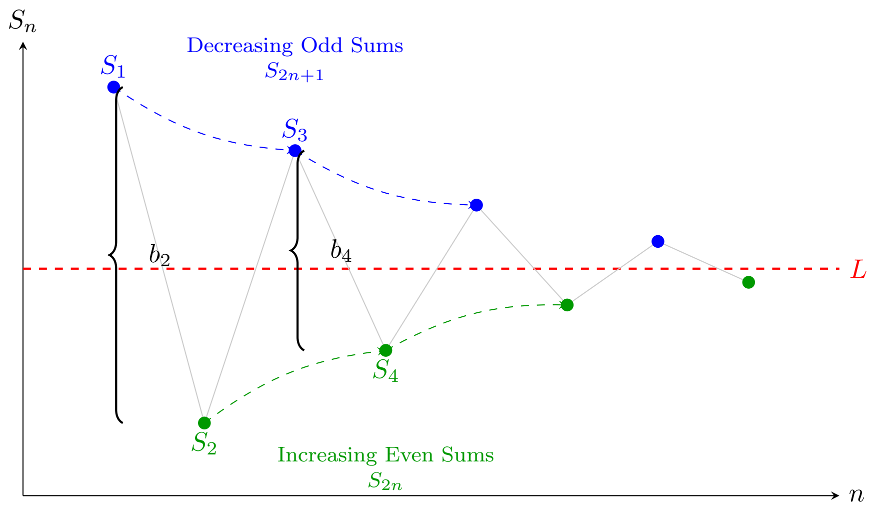


Figure 5.3: Visualisation of the Alternating Series Test. The partial sums oscillate around the limit L. The odd partial sums form a decreasing sequence bounded below by L, while the even partial sums form an increasing sequence bounded above by L.

5.5.2 Rearrangements of Series

A fundamental question in the theory of infinite series is whether the commutative law of addition holds. That is, does the order of summation affect the limit?

Definition 5.5.1. ***Rearrangement***. A series $\sum a'_n$ is a rearrangement of $\sum a_n$ if there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $a'_n = a_{\sigma(n)}$ for all n.

Stability of Absolute Convergence

For absolutely convergent series, the infinite sum behaves like a finite sum: the order of terms is irrelevant.

***Theorem* 5.5.2. Rearrangement of Absolutely Convergent Series.** If $\sum a_n$ converges absolutely to a limit L, then any rearrangement $\sum a_{\sigma(n)}$ also converges absolutely to L.

Proof. Let $\sum a_n$ be absolutely convergent. Let $\epsilon > 0$. Since $\sum |a_n|$ converges, there exists N such that $\sum_{k=N+1}^{\infty} |a_k| < \epsilon/2$. Consider the rearrangement $b_n = a_{\sigma(n)}$. We construct a threshold M for the new series. Choose M large enough such that the set of indices $\{1, 2, \ldots, N\}$ is contained within the set $\{\sigma(1), \sigma(2), \ldots, \sigma(M)\}$. This ensures that all the "critical" initial terms of the original series are included in the first M terms of the rearrangement. For any $m \geq M$, the difference between the partial sums is:

$$\left| \sum_{k=1}^{m} b_k - \sum_{k=1}^{\infty} a_k \right| = \left| \sum_{k=1}^{m} a_{\sigma(k)} - \sum_{k=1}^{\infty} a_k \right|$$

The terms $a_1, \ldots, a_N$ are present in both sums and cancel out. The remaining terms have indices greater than N in the original enumeration. Thus:

$$\left|\sum_{k=1}^{m} b_k - L\right| \leq \sum_{k=N+1}^{\infty} |a_k| < \frac{\epsilon}{2} < \epsilon$$

This proves convergence to L. The absolute convergence of the rearrangement follows similarly by bounding partial sums. ■

Riemann's Rearrangement Theorem

The behaviour of conditionally convergent series is starkly different. In the absence of absolute convergence, the value of the sum depends entirely on the order of cancellation. Consider the alternating harmonic series $\sum \frac{(-1)^{n+1}}{n} = L$. If we rearrange the terms to take one positive term followed by two negative terms:

$$\left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right) + \ldots$$

Simple arithmetic shows this sums to $L/2$. Riemann proved that this instability is total.

***Theorem* 5.5.3. Riemann's Rearrangement Theorem**. Let $\sum a_n$ be a conditionally convergent series. For any $L \in \mathbb{R} \cup \{-\infty, \infty\}$, there exists a rearrangement of $\sum a_n$ that converges to L.

Proof. **Step 1: Divergence of Positive and Negative Parts.** Let p_n be the sub-sequence of positive terms of a_n, and q_n be the sub-sequence of negative terms. If $\sum p_n$ were convergent, then since $\sum a_n$ converges, $\sum q_n = \sum a_n - \sum p_n$ would also converge. This would imply $\sum |a_n| = \sum p_n - \sum q_n$ converges, contradicting the assumption of conditional convergence. Thus, both $\sum p_n$ and $\sum q_n$ must diverge to ∞ and $-\infty$ respectively. Furthermore, since $\sum a_n$ converges, $a_n \to 0$, so both $p_n \to 0$ and $q_n \to 0$.

Step 2: The Algorithm. Let $L \in \mathbb{R}$. We construct the rearrangement inductively to oscillate around L.

1. Sum terms from (p_n) until the partial sum exceeds L. Let this require n_1 terms.

$$S_1 = \sum_{i=1}^{n_1} p_i > L$$

   This is possible because $\sum p_n \to \infty$.
2. Now add terms from (q_n) to S_1 until the sum drops below L. Let this require m_1 terms.

$$S_2 = S_1 + \sum_{j=1}^{m_1} q_j < L$$

   This is possible because $\sum q_n \to -\infty$.
3. Resume adding terms from (p_n) until the sum exceeds L again.

Step 3: Convergence. We repeat this process indefinitely. Let T_k denote the partial sums of this new rearrangement. The sequence T_k oscillates around L. The "error" $|T_k - L|$ is bounded by the magnitude of the last term added. Specifically, if we have just crossed L by adding a positive term p_{n_k}, then $|T_k - L| \leq p_{n_k}$. If we have just crossed L by adding a negative term q_{m_k}, then $|T_k - L| \leq |q_{m_k}|$. Since $a_n \to 0$, we have $p_n \to 0$ and $q_n \to 0$. Consequently, the oscillation amplitude shrinks to zero:

$$\lim_{k\to\infty} |T_k - L| = 0$$

Thus, the rearrangement converges to L. ■

5.6 Grouping and Products of Series

In our investigation of alternating series, we observed that conditionally convergent series are sensitive to rearrangement. However, grouping terms—inserting parentheses without altering the order—is a milder operation. While grouping can transform a divergent series into a convergent one (e.g., $(1-1)+(1-1)+\dots$ vs $1-1+1-1+\dots$), the converse is stable.

5.6.1 Grouping Terms

***Theorem* 5.6.1. Grouping Theorem**. If a series $\sum a_n$ converges to a sum A, then any series obtained by grouping terms of $\sum a_n$ (without changing order) also converges to A.

Proof. Grouping terms essentially corresponds to selecting a subsequence of the partial sums. Let $S_n = \sum_{k=1}^{n} a_k$ be the partial sums of the original series. Since the series converges to A, $S_n \to A$. A grouped series is defined by a strictly increasing sequence of indices $n_1 < n_2 < \dots$. The terms of the new series (b_k) are given by:

$$b_1 = a_1 + \dots + a_{n_1} = S_{n_1}$$

$$b_2 = a_{n_1+1} + \dots + a_{n_2} = S_{n_2} - S_{n_1}$$

The k-th partial sum of the grouped series, $T_k = \sum_{j=1}^{k} b_j$, is exactly S_{n_k}. Since (S_{n_k}) is a subsequence of the convergent sequence (S_n), it must converge to the same limit A. ■

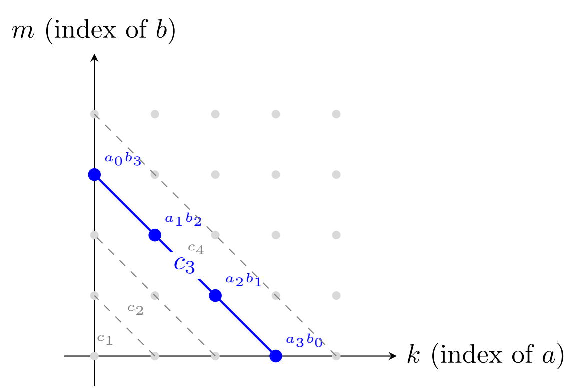
5.6.2 The Cauchy Product

Given two convergent series $\sum a_n = A$ and $\sum b_n = B$, it is natural to ask if there exists a "product series" that converges to AB. The naive term-wise product $\sum a_n b_n$ rarely works (e.g., if $a_n = b_n = 1/n$, the product is $\sum 1/n^2$, which converges, but A and B diverge).

Motivated by polynomial multiplication $(\sum a_i x^i)(\sum b_j x^j) = \sum c_n x^n$, we define the *convolution* of two series.

Definition 5.6.1. *Cauchy Product*. The **Cauchy product** of two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ is the series $\sum_{n=0}^{\infty} c_n$, where:

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0$$



Summing along diagonals

The convergence of the Cauchy product is subtle. Conditional convergence is insufficient to guarantee that $\sum c_n = AB$. However, absolute convergence provides the necessary stability.

***Theorem* 5.6.2. Mertens' Theorem (Simplified)**. If $\sum a_n$ and $\sum b_n$ converge absolutely to A and B respectively, then their Cauchy product $\sum c_n$ converges absolutely to AB.

Proof. Let A_n, B_n, C_n denote the partial sums of $\sum |a_k|, \sum |b_k|, \sum |c_k|$ respectively. Let A^*, B^* be the limits of the absolute series.

Part 1: Absolute Convergence. We bound the partial sum C_n:

$$\sum_{k=0}^{n} |c_k| = \sum_{k=0}^{n} \left| \sum_{j=0}^{k} a_j b_{k-j} \right| \leq \sum_{k=0}^{n} \sum_{j=0}^{k} |a_j||b_{k-j}|$$

The term on the right is a subset of the terms in the product of partial sums $A_n B_n$. Specifically, $\sum_{k=0}^{n} \sum_{j=0}^{k} |a_j||b_{k-j}|$ sums all terms $|a_i||b_j|$ where the indices sum to at most n (a triangle in the ij-plane). This is clearly bounded by the square product $A_n B_n$:

$$\sum_{k=0}^{n} |c_k| \leq \left(\sum_{i=0}^{n} |a_i| \right) \left(\sum_{j=0}^{n} |b_j| \right) = A_n B_n \leq A^* B^*$$

Since the partial sums of absolute values are bounded, $\sum c_n$ converges absolutely.

Part 2: Convergence to the Product. We wish to show $\left| \sum_{k=0}^{2n} c_k - AB \right| \to 0$. Observe that $\sum_{k=0}^{2n} c_k$ sums all products $a_i b_j$ where $i + j \leq 2n$. The product of partial sums $(\sum_{i=0}^{n} a_i)(\sum_{j=0}^{n} b_j)$ sums all $a_i b_j$ where $0 \leq i, j \leq n$. The difference includes terms where indices are "large". Consider the difference:

$$\Delta_n = \left| \sum_{k=0}^{2n} c_k - \left(\sum_{i=0}^{n} a_i \right) \left(\sum_{j=0}^{n} b_j \right) \right|$$

The terms in the first sum that are NOT in the second sum are those where $i + j \leq 2n$ but ($i > n$ or $j > n$). Let $\epsilon > 0$. Since the series converge absolutely, the "tails" are small. For large n, $\sum_{k=n+1}^{\infty} |a_k| < \epsilon$. The difference is bounded by sums of terms involving these tails:

$$\Delta_n \leq \left(\sum_{i=0}^{2n} |a_i| \right) \left(\sum_{j=n+1}^{2n} |b_j| \right) + \left(\sum_{j=0}^{2n} |b_j| \right) \left(\sum_{i=n+1}^{2n} |a_i| \right)$$

$$\Delta_n \leq A^*(\text{tail of } B) + B^*(\text{tail of } A)$$

As $n \to \infty$, the tails vanish, so $\Delta_n \to 0$. Since $(\sum_{i=0}^{n} a_i)(\sum_{j=0}^{n} b_j) \to AB$, the Cauchy product converges to AB. ■

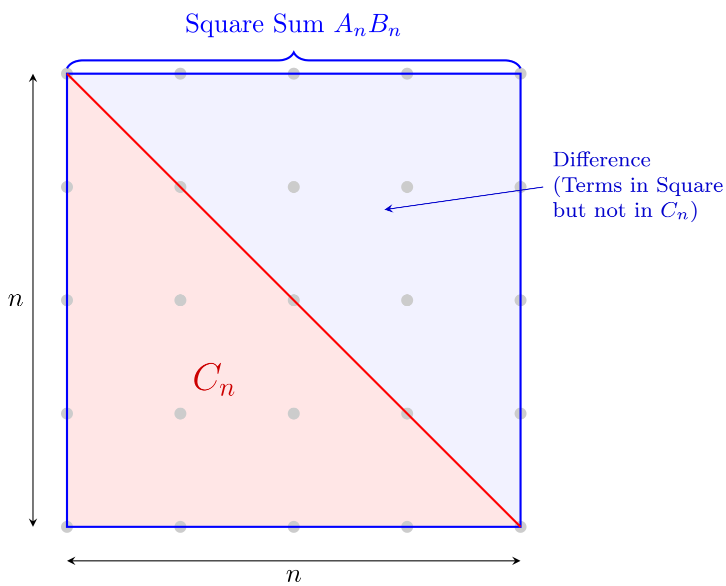


Figure 5.4: Visualisation of Mertens' Theorem. The bounds become clear by separating the partial sum C_n (red triangle) from the full product $A_n B_n$ (blue square).

Remark. It is sufficient for just *one* of the series to be absolutely convergent (and the other convergent) for the product to converge to AB. This stronger version is the full statement of Mertens' Theorem.

5.7 Power Series

Thus far, we have studied series of fixed numbers. A natural generalization is to replace the coefficients with functions of a variable x. The most fundamental class of such functional series is the Power Series, which can be viewed as "infinite polynomials."

Definition 5.7.1. *Power Series*. A power series centred at 0 is a series of the form:

$$S(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

where (a_n) is a sequence of real coefficients and $x \in \mathbb{R}$ is a variable. The *domain of convergence* is the set of all x for which the series converges.

Abel's Lemma

To determine the structure of the domain of convergence, we first establish a fundamental lemma: if a power series converges at a certain distance from the origin, it must converge at all points closer to the origin.

***Lemma* 5.7.1.** *Abel's Lemma.* If the power series $\sum a_n x^n$ converges at a point $x_0 \neq 0$, then it converges absolutely for all x satisfying $|x| < |x_0|$.

Proof. Since the series $\sum a_n x_0^n$ converges, the sequence of its terms must approach 0 (by the The Divergence Test). Consequently, the sequence $(a_n x_0^n)$ is bounded. There exists a constant $M > 0$ such that:

$$|a_n x_0^n| \leq M \quad \text{for all } n.$$

Now, consider any x such that $|x| < |x_0|$. We manipulate the term $|a_n x^n|$:

$$|a_n x^n| = |a_n x_0^n| \cdot \left|\frac{x}{x_0}\right|^n \leq M \cdot \left|\frac{x}{x_0}\right|^n$$

Let $\rho = |x/x_0|$. Since $|x| < |x_0|$, we have $0 \leq \rho < 1$. The series $\sum M\rho^n$ is a geometric series with ratio $\rho < 1$, which converges. By the Direct Comparison Test, the series $\sum |a_n x^n|$ converges. Thus, the power series converges absolutely at x. ∎

Radius of Convergence

Using Abel's Lemma, we can define the domain of convergence using a single parameter. The domain is not an arbitrary set, but a symmetric interval.

***Theorem* 5.7.1. Structure of the Domain.** For any power series $\sum a_n x^n$, there exists a unique $R \in [0, \infty]$ (called the Radius of Convergence) such that:

1. The series converges absolutely for all $|x| < R$.
2. The series diverges for all $|x| > R$.

At the boundaries $|x| = R$, the series may converge or diverge (requires specific testing).

Proof. Let D be the domain of convergence of the series. We consider the set of magnitudes for which the series converges:

$$S = \{|x| : x \in D\}$$

Note that $0 \in S$, so the set is non-empty. We examine two cases based on the The Completeness Axiom (Completeness Axiom).

Case 1: S is bounded above. By the Completeness Axiom, S has a supremum. Let $R = \sup S$.

- **Convergence for $|x| < R$:** Let x be such that $|x| < R$. By the definition of the supremum, there exists a real number $r \in S$ such that $|x| < r \leq R$. Since $r \in S$, there exists some $x_0 \in D$ with $|x_0| = r$ such that the series converges at x_0. By Abel's Lemma, since $|x| < |x_0|$, the series converges absolutely at x.
- **Divergence for $|x| > R$:** Suppose the series converges at some x with $|x| > R$. Then $|x| \in S$. This implies $|x| \leq \sup S = R$, which contradicts $|x| > R$. Thus, the series must diverge.

Case 2: S is unbounded. We define $R = \infty$. For any $x \in \mathbb{R}$, since S is unbounded, we can find $x_0 \in D$ such that $|x_0| > |x|$. By Abel's Lemma, the series converges absolutely at x. Thus, the series converges for all x.

■

Determining the Radius

We often use the Ratio Test to determine R. Applying the test to the terms $u_n = a_n x^n$:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = |x| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

For convergence, we require this limit to be strictly less than 1. Thus, if $L = \lim |a_{n+1}/a_n|$ exists, then $R = 1/L$.

Example 5.7.1. Examples of Radii.

- **Geometric Series:** $\sum x^n$. Here $a_n = 1$. The ratio is 1. Thus $R = 1$. It converges for $x \in (-1, 1)$.
- **The Exponential Series:** $\sum \frac{x^n}{n!}$. The ratio is $\frac{1/(n+1)!}{1/n!} = \frac{1}{n+1} \to 0$. Thus $|x| \cdot 0 < 1$ is true for all x. $R = \infty$.
- **Factorial Series:** $\sum n! x^n$. The ratio is $(n+1) \to \infty$. The series converges only at $x = 0$. $R = 0$.

5.8 Exercises

Part I: Convergence Calculations

1. **Basic Convergence Testing.** Determine whether the following series converge or diverge. If the series converges, specify if it is absolute or conditional.

   (a) $\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1}$

   (b) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

   (c) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

   (d) $\sum_{n=2}^{\infty} \frac{1}{n(\log_2 n)^2}$

   Remark. Use the Cauchy Condensation Test. Recall that $\log_2(2^k) = k$.

   (e) $\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n^2+1} - n\right)$

   Remark. Rationalise the expression first.

2. **Exact Summations.** Calculate the exact sum of the following series by identifying them as telescoping sums or geometric series.

   (a) $\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$

   (b) $\sum_{n=1}^{\infty} (\sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n})$

   (c) $\sum_{n=0}^{\infty} \frac{3^n+4^n}{12^n}$

3. **The Root Test.** While the Ratio Test is powerful, it is inconclusive when the limit is 1. A slightly stronger test is the Root Test. Let $\alpha = \limsup_{n\to\infty} |a_n|^{1/n}$. Prove that:

   (a) If $\alpha < 1$, the series $\sum a_n$ converges absolutely.

   (b) If $\alpha > 1$, the series $\sum a_n$ diverges.

   Remark. For (a), compare with a geometric series $\sum r^n$ where $\alpha < r < 1$. For (b), show that the terms do not tend to 0.

4. **Decimal Expansions.**

   (a) Express the repeating decimal $0.121212\ldots$ as an infinite geometric series and find its rational value p/q.

   (b) Prove formally that $0.999\cdots = 1$.

   Remark. Treat $0.999\ldots$ as $\sum_{n=1}^{\infty} 9 \cdot 10^{-n}$.

5. **Power Series Radii.** Find the radius of convergence R for the following power series using the Ratio or Root tests.

   (a) $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

   (b) $\sum_{n=1}^{\infty} \frac{n^3}{3^n} x^n$

   (c) $\sum_{n=1}^{\infty} \frac{x^{2n}}{n}$

   Remark. Let $y = x^2$ and find the radius for y first.

   (d) $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{n^2} x^n$

   Remark. Use the Root Test. Recall the definition of e.

Part II: Theoretical Extensions

6. **Abel's Summation Formula.** Just as we can rearrange finite sums, we can rearrange infinite series under specific conditions. Let (a_n) and (b_n) be sequences. Let $A_n = \sum_{k=1}^{n} a_k$ be the partial sums of a. Prove that for any $n \geq 1$:

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} - \sum_{k=1}^{n} A_k (b_{k+1} - b_k).$$

   Remark. Substitute $a_k = A_k - A_{k-1}$ (with $A_0 = 0$) into the sum and regroup the terms.

7. **Dirichlet's Test.** Use Abel's Summation Formula to prove the following powerful test for conditional convergence: If the partial sums of $\sum a_n$ are bounded (i.e., $|A_n| \leq M$), and (b_n) is a decreasing sequence converging to 0, then the series $\sum a_n b_n$ converges.

   (a) Prove the theorem.

   Remark. Apply Abel's formula. Show that the resulting series converges absolutely using the comparison test and the telescoping nature of $\sum(b_k - b_{k+1})$.

   (b) Apply this to show that $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$ converges for any $x \in \mathbb{R}$ where x is not a multiple of 2π.

   Remark. You may assume the trigonometric identity $|\sum_{k=1}^{n} \sin(kx)| \leq \frac{1}{|\sin(x/2)|}$.

8. **The Cauchy-Schwarz Inequality for Series.** Let $\sum a_n^2$ and $\sum b_n^2$ be convergent series of real numbers. Prove that the series $\sum a_n b_n$ converges absolutely, and:

$$\left(\sum_{n=1}^{\infty} a_n b_n\right)^2 \leq \left(\sum_{n=1}^{\infty} a_n^2\right)\left(\sum_{n=1}^{\infty} b_n^2\right).$$

Remark. Apply the finite Cauchy-Schwarz inequality to the partial sums and take limits.

9. **Raabe's Test.** The Ratio Test fails when $|\frac{a_{n+1}}{a_n}| \to 1$. Raabe's Test provides a finer instrument for these cases. Let (a_n) be a sequence of positive terms. Suppose that:

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = p.$$

Prove that if $p > 1$, the series converges.

Remark. Let $1 < s < p$. Compare the terms a_n with the sequence $b_n = \frac{1}{(n-1)^s}$. Use the Bernoulli/Binomial approximation $(1 - 1/n)^s \approx 1 - s/n$ to show that $\frac{a_{n+1}}{a_n} < \frac{b_{n+1}}{b_n}$ for large n.

10. $\star$ **Failure of the Cauchy Product.** We proved Mertens' Theorem for absolutely convergent series. Show that if we drop the requirement of absolute convergence, the Cauchy product may diverge. Consider the series $a_n = b_n = \frac{(-1)^n}{\sqrt{n+1}}$.

    (a) Verify that $\sum a_n$ converges using the Alternating Series Test.

    (b) Form the Cauchy product $c_n = \sum_{k=0}^n a_k b_{n-k}$. Show that $|c_n| \geq 1$ for all n.

    Remark. Observe that each term in the sum for c_n has the same sign. Use the minimum value of the denominator.

    (c) Conclude that $\sum c_n$ diverges.

11. $\star$ **Irrationality of e.** We have defined $e = \sum_{n=0}^\infty \frac{1}{n!}$.

    (a) Let s_n be the n-th partial sum. Prove that $0 < e - s_n < \frac{1}{n!n}$.

    Remark. Bound the tail $\sum_{k=n+1}^\infty \frac{1}{k!}$ by a geometric series with ratio $1/(n+1)$.

    (b) Assume $e = p/q$ for integers $p, q > 0$. Show that $q!e$ must be an integer.

    (c) Use the inequality from part (a) to show that $0 < q!e - q!s_q < 1$.

    (d) Conclude that e is irrational (as there are no integers strictly between 0 and 1).

12. $\star$ **Comparison of Divergence.** Suppose (a_n) is a decreasing sequence of positive real numbers converging to 0. Prove that if $\sum a_n$ diverges, then the series $\sum \min(a_n, \frac{1}{n})$ also diverges.

Remark. This shows that the harmonic series is, in a sense, the "boundary" of divergence. If $\sum a_n$ diverges, it cannot vanish significantly faster than $1/n$.

13. $\star$ **Infinite Products.** An infinite product $\prod_{n=1}^\infty (1 + a_n)$ is said to converge if the sequence of partial products $P_N = \prod_{n=1}^N (1 + a_n)$ converges to a non-zero limit. Prove that if $a_n \geq 0$, then $\prod_{n=1}^\infty (1 + a_n)$ converges if and only if the series $\sum_{n=1}^\infty a_n$ converges.

Remark. Use the inequality $1 + x \leq e^x$ to bound the partial products from above by the exponentials of the partial sums. For the reverse direction, note that $P_N \geq 1 + \sum_{n=1}^N a_n$.

14. $\star\star$ **The Square of a Series.**

    (a) Prove that if $\sum a_n$ is a convergent series of positive terms, then $\sum a_n^2$ also converges.

    (b) Give a counter-example to show that this is not true if the terms are not necessarily positive (i.e., find a conditionally convergent series $\sum a_n$ such that $\sum a_n^2$ diverges).

15. $\star\star\star$ **Square Roots of Divergence.** This is a problem posed by Abel. Let $\sum a_n$ be a divergent series of positive terms. Let S_n be the partial sums.

    (a) Prove that the series $\sum \frac{a_n}{S_n^2}$ converges.

    Remark. Observe that $\frac{a_n}{S_n^2} \leq \frac{S_n - S_{n-1}}{S_n S_{n-1}} = \frac{1}{S_{n-1}} - \frac{1}{S_n}$. This is a telescoping sum.

    (b) Prove that the series $\sum \frac{a_n}{S_n}$ diverges.

    Remark. Use the Cauchy Criterion. For large m, n, approximate the sum by noticing that for terms close together, $S_k \approx S_m$. The sum behaves like $\frac{S_n - S_m}{S_n} = 1 - \frac{S_m}{S_n}$. Since $S_n \to \infty$, this does not vanish.

Appendix A

Advanced Sequence Analysis

While the standard definition of convergence covers most introductory cases, analysis often requires tools to handle sequences that oscillate or behave erratically. In this chapter, we formalize the behavior of bounded but divergent sequences and introduce powerful computational tools for ratio limits.

A.1 Limit Superior and Limit Inferior

We have established that while every convergent sequence is bounded, not every bounded sequence converges. For example, $a_n = (-1)^n$ oscillates between -1 and 1. Although it has no single limit, it does have "bounds" that it eventually respects. To formalize this, we introduce the concepts of Limit Superior ($\limsup$) and Limit Inferior ($\liminf$).

Definition via Eventual Bounds

For a bounded sequence (a_n), consider the tail of the sequence starting from index n: $\{a_n, a_{n+1}, a_{n+2}, \dots\}$. We define two auxiliary sequences based on the supremum and infimum of these tails.

Definition A.1.1. *Limit Superior and Inferior*. Let (a_n) be a bounded sequence. We define the sequence of "eventual supremums" as $\beta_n = \sup\{a_k : k \geq n\}$. We define the sequence of "eventual infimums" as $\alpha_n = \inf\{a_k : k \geq n\}$.

- Since the tail shrinks as n increases ($\{a_{n+1}, \dots\} \subset \{a_n, \dots\}$), (β_n) is decreasing and (α_n) is increasing.
- Since (a_n) is bounded, both (β_n) and (α_n) are bounded and therefore convergent.

We define the *Limit Superior* and *Limit Inferior* as:

$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \left(\sup_{k\geq n} a_k \right)$$

$$\liminf_{n\to\infty} a_n := \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \left(\inf_{k\geq n} a_k \right)$$

Characterization via Subsequences

While the definition via eventual bounds is computationally useful, the limit superior has a profound topological meaning. Let E be the set of all subsequential limits (also called cluster points) of (a_n).

- The set E is non-empty (by Bolzano-Weierstrass) and closed.

- The limit superior is the maximum of this set: $\limsup_{n\to\infty} a_n = \sup E$.
- The limit inferior is the minimum of this set: $\liminf_{n\to\infty} a_n = \inf E$.

This characterization explains why lim sup is often called the "maximal limit."

Properties

The limit superior represents the "highest" value that the sequence visits infinitely often, while the limit inferior is the "lowest".

***Theorem* A.1.1. Consistency of Limits**. Let (a_n) be a bounded sequence.

1. The sequence converges to L if and only if the limit superior and limit inferior coincide:

$$\lim_{n\to\infty} a_n = L \iff \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = L$$

2. **Super-additivity and Sub-additivity:** For any two bounded sequences (a_n) and (b_n):

$$\liminf a_n + \liminf b_n \le \liminf(a_n + b_n) \le \limsup(a_n + b_n) \le \limsup a_n + \limsup b_n$$

(See Appendix B for proofs.)

Example A.1.1. Oscillating Sequence. Let $a_n = (-1)^n(1+\frac{1}{n})$. The terms are roughly $-1, 1, -1, 1\dots$. The supremum of any tail is slightly larger than 1, converging to 1. The infimum of any tail is slightly smaller than -1, converging to -1. Thus, $\limsup a_n = 1$ and $\liminf a_n = -1$. Since they are unequal, the sequence diverges.

Application: Subadditive Sequences

A powerful application of these concepts is Fekete's Lemma, which guarantees the convergence of "subadditive" sequences.

Example A.1.2. Fekete's Subadditive Lemma. Let (a_n) be a sequence of non-negative real numbers satisfying the subadditivity condition:

$$a_{m+n} \le a_m + a_n \quad \text{for all } m, n \in \mathbb{N}$$

Prove that the sequence $(\frac{a_n}{n})$ converges.

Proof. Fix an arbitrary integer k. Any integer n can be written as $n = qk + r$, where $0 \le r < k$. Using the subadditivity property repeatedly:

$$a_n = a_{qk+r} \le \underbrace{a_k + \cdots + a_k}_{q \text{ times}} + a_r = qa_k + a_r$$

Dividing by n:

$$\frac{a_n}{n} \le \frac{qa_k}{n} + \frac{a_r}{n} = \frac{qa_k}{qk+r} + \frac{a_r}{n}$$

As $n \to \infty$, note that $\frac{q}{n} \approx \frac{1}{k}$ and the remainder term $\frac{a_r}{n} \to 0$ (since r is bounded by k). Taking the limit superior:

$$\limsup_{n\to\infty} \frac{a_n}{n} \le \frac{a_k}{k}$$

Since this inequality holds for *any* fixed k, we can take the limit inferior as $k \to \infty$:

$$\limsup_{n\to\infty} \frac{a_n}{n} \le \liminf_{k\to\infty} \frac{a_k}{k}$$

This implies the limit superior is less than or equal to the limit inferior. Since $\liminf \le \limsup$ always, equality must hold. Thus, the limit exists. ■

A.2 The Stolz-Cesàro Theorem

When calculating the limit of a ratio of sequences $\frac{a_n}{b_n}$, we often encounter the indeterminate form $\frac{\infty}{\infty}$. The Stolz-Cesàro theorem serves as a discrete analogue to L'Hôpital's Rule.

***Theorem* A.2.1. Stolz-Cesàro Theorem.** Let (b_n) be a strictly increasing sequence of positive real numbers such that $\lim_{n\to\infty} b_n = +\infty$. If the limit of the ratio of differences exists:

$$\lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = L$$

Then the limit of the sequence ratio also exists and is equal to L:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

(See Appendix B for proof.)

Example A.2.1. Cesàro Means (Arithmetic Mean). Let (x_n) be a sequence converging to x. Prove that the sequence of averages converges to x. Let $a_n = \sum_{k=1}^{n} x_k$ and $b_n = n$. Note that (b_n) is strictly increasing and tends to ∞. Consider the ratio of differences:

$$\frac{a_n - a_{n-1}}{b_n - b_{n-1}} = \frac{x_n}{n - (n-1)} = \frac{x_n}{1} = x_n$$

Since $\lim x_n = x$, by the Stolz-Cesàro theorem:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{x_1 + \cdots + x_n}{n} = x$$

A.3 Completeness Revisited

Throughout our study of sequences, we have relied on various properties to establish convergence. It is worth noting that these properties are deeply interlinked and stem from the nature of the real number system.

Remark. There is one further equivalent formulation of completeness known as the **Heine-Borel Theorem** (or Finite Covering Theorem): Every open cover of a closed, bounded interval $[a, b]$ has a finite subcover. While this is typically treated in topology, it completes the "Big Six" theorems of real analysis: MCT, Nested Intervals, Bolzano-Weierstrass, Cauchy Criterion, Supremum Principle, and Heine-Borel. We cover this in greater detail next notes.

Example A.3.1. Importance of Closed Intervals. The requirement that the nested intervals be *closed* is essential. Consider the sequence of nested *open* intervals $I_n = (0, 1/n)$.

$$(0, 1) \supset (0, 1/2) \supset (0, 1/3) \dots$$

Although the lengths $1/n \to 0$, the intersection $\bigcap_{n=1}^{\infty} (0, 1/n)$ is empty. There is no real number x that satisfies $0 < x < 1/n$ for all n, because by the Archimedean property, we can always find an n such that $1/n < x$. This failure of intersection demonstrates that open intervals lack the property of *compactness*.

Appendix B

Appendix: Proofs of Advanced Theorems

This appendix contains the rigorous proofs for the theorems presented in the chapter on Advanced Sequence Analysis.

B.1 Limit Superior Properties

Proof of Consistency (Limit Existence)

Theorem Statement: $\lim_{n\to\infty} a_n = L \iff \liminf a_n = \limsup a_n = L$.

Proof. Let $\alpha_n = \inf_{k\geq n} a_k$ and $\beta_n = \sup_{k\geq n} a_k$. Note that by definition, for all n:

$$\alpha_n \leq a_n \leq \beta_n$$

($\Rightarrow$) Suppose $\lim a_n = L$. For any $\epsilon > 0$, there exists N such that for $k \geq N$, $L - \epsilon < a_k < L + \epsilon$. It follows that for $n \geq N$:

$$L - \epsilon \leq \inf_{k\geq n} a_k = \alpha_n \quad \text{and} \quad \beta_n = \sup_{k\geq n} a_k \leq L + \epsilon$$

Thus $\lim \alpha_n = L$ and $\lim \beta_n = L$. ($\Leftarrow$) Suppose $\lim \alpha_n = L$ and $\lim \beta_n = L$. By the Squeeze Theorem applied to the inequality $\alpha_n \leq a_n \leq \beta_n$, it follows immediately that $\lim a_n = L$. ∎

Proof of Sub-additivity

Theorem Statement: $\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$.

Proof. Let $k \geq n$. By the definition of supremum:

$$a_k \leq \sup_{j\geq n} a_j \quad \text{and} \quad b_k \leq \sup_{j\geq n} b_j$$

Summing these inequalities:

$$a_k + b_k \leq \sup_{j\geq n} a_j + \sup_{j\geq n} b_j$$

Since this holds for all $k \geq n$, the Right Hand Side (RHS) is an upper bound for the set $\{a_k + b_k : k \geq n\}$. The supremum is the *least* upper bound, so:

$$\sup_{k\geq n}(a_k + b_k) \leq \sup_{j\geq n} a_j + \sup_{j\geq n} b_j$$

Taking the limit as $n \to \infty$ preserves the inequality (Order Preservation Law):

$$\lim_{n\to\infty} \sup_{k\geq n}(a_k + b_k) \leq \lim_{n\to\infty} \sup_{j\geq n} a_j + \lim_{n\to\infty} \sup_{j\geq n} b_j$$

Which is exactly $\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$. ∎

B.2 Proof of the Stolz-Cesàro Theorem

Theorem Statement: Let (b_n) be strictly increasing with $b_n \to \infty$. If $\lim \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = A$, then $\lim \frac{a_n}{b_n} = A$.

Proof. Let $x_n = a_n - a_{n-1}$ and $y_n = b_n - b_{n-1}$. We are given $\lim_{n\to\infty} \frac{x_n}{y_n} = A$. Let $\epsilon > 0$. Since $\frac{x_n}{y_n} \to A$, there exists N_0 such that for all $n > N_0$:

$$A - \frac{\epsilon}{2} < \frac{a_n - a_{n-1}}{b_n - b_{n-1}} < A + \frac{\epsilon}{2}$$

Since b_n is strictly increasing, $b_n - b_{n-1} > 0$. We can multiply through without flipping inequalities:

$$(A - \frac{\epsilon}{2})(b_n - b_{n-1}) < a_n - a_{n-1} < (A + \frac{\epsilon}{2})(b_n - b_{n-1})$$

We now sum these inequalities from $k = N_0 + 1$ to n:

$$(A - \frac{\epsilon}{2}) \sum_{k=N_0+1}^{n} (b_k - b_{k-1}) < \sum_{k=N_0+1}^{n} (a_k - a_{k-1}) < (A + \frac{\epsilon}{2}) \sum_{k=N_0+1}^{n} (b_k - b_{k-1})$$

These are telescoping sums. They simplify to:

$$(A - \frac{\epsilon}{2})(b_n - b_{N_0}) < a_n - a_{N_0} < (A + \frac{\epsilon}{2})(b_n - b_{N_0})$$

Divide the entire inequality by b_n (which is positive for large n):

$$(A - \frac{\epsilon}{2}) \left(1 - \frac{b_{N_0}}{b_n}\right) < \frac{a_n}{b_n} - \frac{a_{N_0}}{b_n} < (A + \frac{\epsilon}{2}) \left(1 - \frac{b_{N_0}}{b_n}\right)$$

Rearranging for a_n/b_n:

$$\frac{a_{N_0}}{b_n} + (A - \frac{\epsilon}{2}) \left(1 - \frac{b_{N_0}}{b_n}\right) < \frac{a_n}{b_n} < \frac{a_{N_0}}{b_n} + (A + \frac{\epsilon}{2}) \left(1 - \frac{b_{N_0}}{b_n}\right)$$

Now take the limit as $n \to \infty$. Since $b_n \to \infty$, terms with b_n in the denominator vanish ($b_{N_0}/b_n \to 0$ and $a_{N_0}/b_n \to 0$). The Lower Bound approaches $0 + (A - \epsilon/2)(1) = A - \epsilon/2$. The Upper Bound approaches $0 + (A + \epsilon/2)(1) = A + \epsilon/2$. Thus, for sufficiently large n:

$$A - \epsilon < \frac{a_n}{b_n} < A + \epsilon$$

Since ϵ is arbitrary, $\lim_{n\to\infty} \frac{a_n}{b_n} = A$. **Extension to Infinite Limits:** If $L = +\infty$, the logic follows similarly. Since $\frac{a_n - a_{n-1}}{b_n - b_{n-1}} \to \infty$, for any large M, eventually $a_n - a_{n-1} > M(b_n - b_{n-1})$. Summing this yields $a_n > Mb_n$ (ignoring small small constants from the head of the sequence), implying $a_n/b_n \to \infty$. ∎