Preludes to Dynamics

Gudfit

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0

Further Convergence Criteria

While the Comparison and Ratio Tests suffice for many series, their efficacy is limited by the need for explicit inequalities or the existence of a limit distinct from unity. To address simpler asymptotic behaviours and the delicate boundary between convergence and divergence, we introduce the Limit Comparison Test and the Root Test, followed by the refined Raabe and Gauss tests.

0.1 *The Limit Comparison Test*

The Direct Comparison Test requires a global inequality $a_n \leq b_n$. Often, two series exhibit identical asymptotic behaviour, yet satisfy no simple inequality. The Limit Comparison Test formalises the intuition that if $a_n \approx cb_n$ for large n, the series must behave identically.

Theorem 0.1. *Limit Comparison Test*.
Let $\sum a_n$ and $\sum b_n$ be series with strictly positive terms. Suppose that the limit exists:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

(i) If $0 < c < \infty$, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.
(ii) If $c = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
(iii) If $c = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

定理

Proof

We prove (i). Since $a_n/b_n \to c > 0$, for $\epsilon = c/2$, there exists N such that for all $n \geq N$:

$$\frac{c}{2} < \frac{a_n}{b_n} < \frac{3c}{2} \implies \frac{c}{2}b_n < a_n < \frac{3c}{2}b_n$$

The result follows immediately from the Direct Comparison Test. If $\sum b_n$ converges, $\sum a_n$ is bounded by a convergent multiple of $\sum b_n$. If $\sum b_n$ diverges, $\sum a_n$ is bounded below by a divergent multiple. Cases (ii) and (iii) follow similarly from the definitions of limits 0 and ∞.

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Example 0.1. Rational Functions. Consider the series $\sum_{n=1}^{\infty} \frac{n+1}{2n^3-1}$. For large n, the term behaves like $n/2n^3 = 1/(2n^2)$. We compare with $b_n = 1/n^2$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2(n+1)}{2n^3-1} = \frac{1}{2}$$

Since $0 < 1/2 < \infty$ and $\sum 1/n^2$ converges ($p = 2$), the original series converges.

範例

The Root Test

The Ratio Test estimates convergence by comparing a series to a geometric series locally (term-by-term). The Root Test achieves this globally. It is strictly stronger than the Ratio Test, as it does not require the limit of ratios to exist.

Theorem 0.2. ***Cauchy's Root Test.***
Let $\sum a_n$ be a series and let

$$L = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

(i) If $L < 1$, the series converges absolutely.
(ii) If $L > 1$, the series diverges.
(iii) If $L = 1$, the test is inconclusive.

定理

Case $L < 1$

Choose r such that $L < r < 1$. By the definition of limits, there exists N such that for all $n \geq N$, $\sqrt[n]{|a_n|} < r$, which implies $|a_n| < r^n$. The series is dominated by the convergent geometric series $\sum r^n$.
証明終

Case $L > 1$

There exists a subsequence where $\sqrt[n_k]{|a_{n_k}|} > 1$, implying $|a_{n_k}| > 1$. The terms do not tend to zero, so the series diverges.

証明終

Note

The Root Test can determine convergence where the Ratio Test fails or is inapplicable, such as for the rearranged geometric series $1/2 + 1 + 1/8 + 1/4 + \ldots$ where consecutive ratios oscillate but the root limit is constant.

Refined Ratio Tests

When the Ratio Test yields $L = 1$, the series falls into a "grey zone" of polynomial decay, typically behaving like $1/n^p$. To resolve these cases, we examine the rate at which the ratio approaches 1.

Theorem 0.3. ***Raabe's Test.***
Let $\sum a_n$ be a positive series. Suppose that:

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \rho$$

(i) If $\rho > 1$, the series converges.
(ii) If $\rho < 1$, the series diverges.

定理

Proof

If $\rho > 1$, choose p such that $1 < p < \rho$. For large n, we have $n(a_n/a_{n+1} - 1) > p$, which rearranges to:

$$\frac{a_{n+1}}{a_n} < 1 - \frac{p}{n}$$

Using the Taylor expansion $(1 - 1/n)^p = 1 - p/n + O(1/n^2)$, we can compare a_n with the sequence $b_n = 1/n^p$. Since $p > 1$, $\sum 1/n^p$ converges, implying $\sum a_n$ converges. The divergence case is analogous.

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For series involving products of arithmetic progressions (hypergeometric series), Gauss provided the definitive criterion.

Theorem 0.4. ***Gauss's Test.***
Let $\sum a_n$ be a positive series. Suppose the ratio admits the asymptotic expansion:

$$\frac{a_n}{a_{n+1}} = 1 + \frac{\mu}{n} + O\left(\frac{1}{n^{1+\epsilon}}\right) \quad (\epsilon > 0)$$

Then $\sum a_n$ converges if $\mu > 1$ and diverges if $\mu \leq 1$.

定理

Note

Gauss's Test resolves the case $\rho = 1$ in Raabe's Test, confirming divergence (like the harmonic series).

0.2 *Case Study: Liu Hui's Circle Division*

We conclude this section with a historical application that serves as an early precursor to modern series acceleration methods. In the 3rd

century AD, the Chinese mathematician Liu Hui sought to calculate π by inscribing regular polygons in a circle.
Let the circle have radius $R = 1$. Let S_n denote the area of a regular polygon with $N = 6 \cdot 2^{n-1}$ sides. Using basic geometry, the area is given by:

$$S_n = 3 \cdot 2^{n-1} \sin\left(\frac{\pi}{3 \cdot 2^{n-1}}\right)$$

As $n \to \infty$, $S_n \to \pi$. This generates a series of increments $a_n = S_n - S_{n-1}$. Using the Taylor expansion $\sin x \approx x - x^3/6$, we can analyze the error term $\Delta_n = \pi - S_n$. Let $x_n = \frac{\pi}{3 \cdot 2^{n-1}}$. Then $S_n = \frac{\pi}{x_n} \sin x_n \approx \frac{\pi}{x_n}(x_n - \frac{x_n^3}{6}) = \pi - \frac{\pi x_n^2}{6}$. Since $x_{n+1} = x_n/2$, the error scales predictably:

$$\pi - S_{n+1} \approx \frac{1}{4}(\pi - S_n)$$

This asymptotic relationship implies that the error reduces by a factor of 4 at each step (linear convergence with rate 1/4). Liu Hui empirically observed this regular loss and proposed an extrapolation to recover the limit.

$$\begin{aligned} \pi - S_{n+1} &\approx \frac{1}{4}(\pi - S_n) \\ 4\pi - 4S_{n+1} &\approx \pi - S_n \\ 3\pi &\approx 4S_{n+1} - S_n \\ \pi &\approx S_{n+1} + \frac{1}{3}(S_{n+1} - S_n) \end{aligned}$$

This formula, known as Richardson Extrapolation, allowed Liu Hui to obtain a high-precision value of π from a relatively coarse polygon.

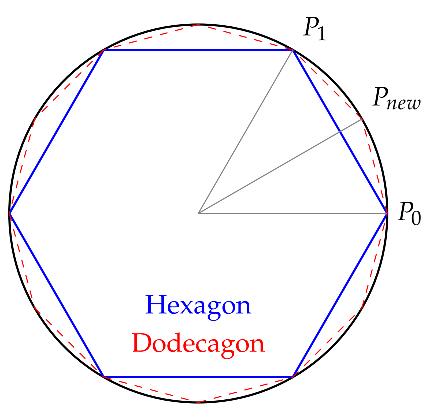

Figure 1: Liu Hui's method of circle division. The area of the inscribed polygon approaches the area of the circle as the number of sides doubles.

The corresponding infinite series is geometric in nature:

$$\pi = S_1 + \sum_{n=2}^{\infty}(S_n - S_{n-1})$$

Liu Hui's insight was that the tail of this series could be approximated by the tail of a geometric series with ratio $1/4$, significantly accelerating the convergence.

0.3 Kummer's Criterion and Universal Scales

While the Ratio, Raabe, and Gauss tests provide a hierarchy of criteria for convergence, they are but specific instances of a more general principle established by Kummer. This criterion unifies the comparison methods into a single powerful theorem.

Theorem 0.5. ***Kummer's Test.***
Let (a_n) be a sequence of positive terms.

(i) The series $\sum a_n$ converges if and only if there exists a sequence of positive numbers (b_n) and a constant $\delta > 0$ such that for all sufficiently large n:

$$K_n = b_n \frac{a_n}{a_{n+1}} - b_{n+1} \geq \delta$$

(ii) The series $\sum a_n$ diverges if and only if there exists a sequence of positive numbers (b_n) such that $\sum 1/b_n$ diverges and for all sufficiently large n:

$$b_n \frac{a_n}{a_{n+1}} - b_{n+1} \leq 0$$

定理

Sufficiency (Convergence)

Suppose the condition holds for $n \geq N$. We rearrange the inequality as:

$$\delta a_{n+1} \leq b_n a_n - b_{n+1} a_{n+1}$$

Summing from $n = N$ to M:

$$\sum_{n=N}^{M} \delta a_{n+1} \leq \sum_{n=N}^{M} (b_n a_n - b_{n+1} a_{n+1}) = b_N a_N - b_{M+1} a_{M+1} < b_N a_N$$

Since the partial sums are bounded above, $\sum a_n$ converges.

証明終

Necessity (Convergence)

If $\sum a_n$ converges, let $R_n = \sum_{k=n+1}^{\infty} a_k$ be the remainder. Set $b_n = R_n / a_n$. Then:

$$b_n \frac{a_n}{a_{n+1}} - b_{n+1} = \frac{R_n}{a_{n+1}} - \frac{R_{n+1}}{a_{n+1}} = \frac{a_{n+1}}{a_{n+1}} = 1$$

Choosing $\delta = 1$ satisfies the condition.

証明終

Sufficiency (Divergence)

The condition implies $b_{n+1}a_{n+1} \geq b_n a_n$. Thus, the sequence $(b_n a_n)$ is non-decreasing. For $n \geq N$, $b_n a_n \geq b_N a_N = C > 0$, so $a_n \geq C/b_n$. By the Comparison Test, since $\sum 1/b_n$ diverges, $\sum a_n$ diverges.

証明終

Necessity (Divergence)

If $\sum a_n$ diverges, let $S_n = \sum_{k=1}^{n} a_k$ be the partial sums. Set $b_n = S_n/a_n$. Then:

$$b_n \frac{a_n}{a_{n+1}} - b_{n+1} = \frac{S_n}{a_{n+1}} - \frac{S_{n+1}}{a_{n+1}} = -1 < 0$$

We must show $\sum 1/b_n = \sum a_n/S_n$ diverges. This follows from a standard divergence property: if $\sum a_n$ diverges, so does $\sum a_n/S_n$. (Consider the integral analogue $\int dx/x = \ln x \to \infty$).

証明終

Note

Kummer's Test generates the standard tests by specific choices of the auxiliary sequence (b_n):

- **Ratio Test:** Take $b_n = 1$. The condition becomes $a_n/a_{n+1} - 1 \geq \delta$, or $a_{n+1}/a_n \leq 1/(1+\delta) < 1$.
- **Raabe's Test:** Take $b_n = n$. The condition becomes $n(a_n/a_{n+1}) - (n+1) \geq \delta$, which rearranges to $n(a_n/a_{n+1} - 1) \geq 1 + \delta > 1$.

The Non-Existence of a Universal Scale

Given the hierarchy of tests, one might hope to find a "universal" comparison series that decides convergence for all series. Specifically, is there a series that converges "so slowly" that any series decaying slower must diverge, or a series that diverges "so slowly" that any series decaying faster must converge? The theorems of Du Bois-Reymond and Abel resolve this in the negative.

Theorem 0.6. ***Universal Comparison Theorems.***

(i) **Du Bois-Reymond:** For any convergent positive series $\sum a_n$, there exists a convergent positive series $\sum b_n$ such that $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$.

(ii) **Abel:** For any divergent positive series $\sum a_n$, there exists a divergent positive series $\sum b_n$ such that $\lim_{n\to\infty} \frac{b_n}{a_n} = 0$.

定理

Proof

(i) Let $R_n = \sum_{k=n+1}^{\infty} a_k$ be the remainder of the convergent series. Define $b_n = \sqrt{R_{n-1}} - \sqrt{R_n}$ (with R_0 the total sum). The series

$\sum b_n$ is a telescoping sum converging to $\sqrt{R_0}$. Checking the ratio:

$$\frac{a_n}{b_n} = \frac{R_{n-1} - R_n}{\sqrt{R_{n-1}} - \sqrt{R_n}} = \sqrt{R_{n-1}} + \sqrt{R_n}$$

Since $R_n \to 0$, the ratio tends to 0. Thus, $\sum b_n$ converges "much slower" than $\sum a_n$.

(ii) Let S_n be the partial sums of the divergent series. Define $b_n = a_n / S_n$. By *Theorem 0.5* proof (necessity of divergence), $\sum b_n$ diverges. Checking the ratio:

$$\frac{b_n}{a_n} = \frac{1}{S_n}$$

Since $S_n \to \infty$, the ratio tends to 0. Thus, $\sum b_n$ diverges "much slower" than $\sum a_n$.

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This result implies that there is no "boundary" between convergence and divergence; the scale is infinitely refutable.

0.4 *Tests for Non-Absolute Convergence*

In the previous notes, we established the Alternating Series Test for (well you guessed it) alternating series. We now generalise this to series of the form $\sum a_n b_n$, where (a_n) provides the sign oscillation and (b_n) provides the decay. These tests rely on a discrete analogue of integration by parts.

Lemma 0.1. ***Abel's Summation Formula.***
Let (a_n) and (b_n) be sequences. Let $A_n = \sum_{k=1}^{n} a_k$ be the partial sums of (a_n). Then:

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} + \sum_{k=1}^{n} A_k (b_k - b_{k+1})$$

引理

Proof

Write $a_k = A_k - A_{k-1}$ (with $A_0 = 0$).

$$\begin{aligned}\sum_{k=1}^{n} (A_k - A_{k-1}) b_k &= \sum_{k=1}^{n} A_k b_k - \sum_{k=1}^{n} A_{k-1} b_k \\ &= \sum_{k=1}^{n} A_k b_k - \sum_{j=0}^{n-1} A_j b_{j+1} \quad \text{(shift index } j = k-1) \\ &= \sum_{k=1}^{n} A_k b_k - \sum_{k=1}^{n-1} A_k b_{k+1} \quad \text{(since } A_0 = 0)\end{aligned}$$

$$= A_n b_n + \sum_{k=1}^{n-1} A_k(b_k - b_{k+1})$$

$$= A_n b_{n+1} + A_n(b_n - b_{n+1}) + \sum_{k=1}^{n-1} A_k(b_k - b_{k+1}).$$

Combining terms yields the result.

This is structurally identical to $\int u dv = uv - \int v du$.

■

Theorem 0.7. ***Dirichlet's Test.***
Let $\sum a_n$ be a series with bounded partial sums (i.e., $|\sum_{k=1}^{n} a_k| \leq M$ for all n). Let (b_n) be a sequence that is monotonic and converges to 0. Then the series $\sum a_n b_n$ converges.

定理

Proof

We apply *Lemma 0.1*. Since $b_n \to 0$ and A_n is bounded, $A_n b_{n+1} \to 0$. It remains to show that $\sum A_k(b_k - b_{k+1})$ converges. Since (b_n) is monotonic, the terms $(b_k - b_{k+1})$ have constant sign. Thus:

$$\sum_{k=1}^{n} |A_k(b_k - b_{k+1})| \leq M \sum_{k=1}^{n} |b_k - b_{k+1}| = M|b_1 - b_{n+1}|$$

Since $b_n \to 0$, this sum is bounded. The series converges absolutely, implying the convergence of the original sum.

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Theorem 0.8. ***Abel's Test.***
Let $\sum a_n$ be a convergent series. Let (b_n) be a monotonic and bounded sequence. Then the series $\sum a_n b_n$ converges.

定理

Note

Unlike Dirichlet's Test, Abel's Test requires $\sum a_n$ to converge, but relaxes the condition on (b_n) to mere boundedness rather than vanishing.

Example 0.2. Trigonometric Series. Consider the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ for $x \in \mathbb{R}$. Let $a_n = \sin nx$ and $b_n = 1/n$. The sequence (b_n) is monotonic and tends to 0. To apply Dirichlet's Test, we require the partial sums of $\sin nx$ to be bounded. Using the identity $2\sin(x/2)\sum_{k=1}^{n} \sin kx = \cos(x/2) - \cos((n+1/2)x)$, we have:

$$\left|\sum_{k=1}^{n} \sin kx\right| \leq \frac{2}{|2\sin(x/2)|} = \frac{1}{|\sin(x/2)|}$$

Provided $x \neq 2k\pi$, the partial sums are bounded. Thus, the series converges for all x not a multiple of 2π.

範例

0.5 Inequalities and Further Examples

Carleman's Inequality

A profound result in the theory of series is Carleman's Inequality, which asserts that the geometric mean of the terms of a convergent series decays sufficiently fast to form a convergent series itself.

Theorem 0.9. ***Carleman's Inequality.***
Let $\sum a_n$ be a convergent series of positive terms. Then:

$$\sum_{n=1}^{\infty}(a_1a_2\dots a_n)^{1/n} \leq e\sum_{n=1}^{\infty} a_n$$

The constant e is sharp.

定理

Proof

We use the AM-GM inequality with weighted terms. Recall that for the sequence $c_k = \frac{(k+1)^k}{k^{k-1}}$, we have $(c_1 \dots c_n)^{1/n} = n+1$. Write the geometric mean as:

$$G_n = (a_1\dots a_n)^{1/n} = \frac{(a_1c_1 \cdot a_2c_2 \dots a_nc_n)^{1/n}}{(c_1\dots c_n)^{1/n}} = \frac{(a_1c_1\dots a_nc_n)^{1/n}}{n+1}$$

By AM-GM:

$$G_n \leq \frac{1}{n(n+1)}\sum_{k=1}^{n} a_kc_k$$

Summing over n:

$$\sum_{n=1}^{\infty} G_n \leq \sum_{n=1}^{\infty}\frac{1}{n(n+1)}\sum_{k=1}^{n} a_kc_k = \sum_{k=1}^{\infty} a_kc_k\sum_{n=k}^{\infty}\frac{1}{n(n+1)}$$

The inner sum is telescoping: $\sum_{n=k}^{\infty}(\frac{1}{n} - \frac{1}{n+1}) = \frac{1}{k}$. Thus:

$$\sum_{n=1}^{\infty} G_n \leq \sum_{k=1}^{\infty} a_k\frac{c_k}{k} = \sum_{k=1}^{\infty} a_k\frac{(k+1)^k}{k^k} = \sum_{k=1}^{\infty} a_k\left(1+\frac{1}{k}\right)^k$$

Since $(1+1/k)^k$ increases to e, we have $\sum G_n \leq e\sum a_n$.

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Quantitative Rearrangements

We previously established Riemann's Rearrangement Theorem, which guarantees that a conditionally convergent series can be rearranged to sum to any value. We now provide a specific calculation for the alternating harmonic series.

Example 0.3. Rearranging the Harmonic Series. Let $\sum a_n$ be a rearrangement of $\sum \frac{(-1)^{n-1}}{n}$. Suppose we construct the rearrangement by taking p positive terms followed by q negative terms, repeating this pattern indefinitely. Let S_N be the partial sum. Using the asymptotic expansion $H_n = \ln n + \gamma + o(1)$, one can derive (see Riemann's rearrangement theorem context):

$$\sum_{n=1}^{\infty} a_n = \ln 2 + \frac{1}{2} \ln \left(\frac{p}{q} \right)$$

For the standard series ($p = 1, q = 1$), the sum is $\ln 2$. If we take two positive terms for every one negative term ($p = 2, q = 1$), the sum shifts to $\ln 2 + \frac{1}{2} \ln 2 = \frac{3}{2} \ln 2$. This explicitly demonstrates how the "density" of positive terms shifts the limit.

範例

0.6 *Infinite Products*

Just as the study of the difference between consecutive terms of a sequence $a_n - a_{n-1}$ leads to the theory of infinite series, the study of the ratio a_n / a_{n-1} leads naturally to the theory of infinite products. While less ubiquitous than series in elementary calculus, infinite products are indispensable in complex analysis and analytic number theory, particularly in the study of the Riemann Zeta function and the Gamma function.

Convergence and Properties

Definition 0.1. ***Infinite Product.***
Let $(p_n)_{n=1}^{\infty}$ be a sequence of non-zero real numbers. The infinite product is denoted by:

$$\prod_{n=1}^{\infty} p_n = p_1 p_2 p_3 \dots$$

Let $P_n = \prod_{k=1}^{n} p_k$ be the sequence of partial products.

1. We say the infinite product **converges** if the limit $P = \lim_{n \to \infty} P_n$ exists and is **non-zero**. In this case, we write $\prod_{n=1}^{\infty} p_n = P$.
2. If the limit zero, or does not exist, or is infinite, the product is said to **diverge**. Specifically, if $P_n \to 0$, we say the product *diverges to zero*.

定義

Note

The exclusion of zero from the definition of convergence is to maintain the analogy with sums. For a sum, "convergence to infinity" is divergence; for a product, "convergence to zero" is similarly classified as divergence. This ensures that a convergent product has a well-defined multiplicative inverse.

Proposition 0.1. ***Necessary Condition.***
If $\prod_{n=1}^{\infty} p_n$ converges, then $\lim_{n\to\infty} p_n = 1$.

命題

Proof

Let $P = \lim P_n \neq 0$. Then:

$$p_n = \frac{P_n}{P_{n-1}} \to \frac{P}{P} = 1$$

■

Consequently, terms in a convergent product are typically written in the form $p_n = 1 + a_n$, where $a_n \to 0$. The study of $\prod(1 + a_n)$ is inextricably linked to the series $\sum \ln(1 + a_n)$.

Theorem 0.10. ***Logarithmic Criterion.***
Let $a_n > -1$. The infinite product $\prod_{n=1}^{\infty}(1 + a_n)$ converges if and only if the series $\sum_{n=1}^{\infty} \ln(1 + a_n)$ converges.

定理

Proof

Let $P_n = \prod_{k=1}^{n}(1 + a_k)$. Then $\ln P_n = \sum_{k=1}^{n} \ln(1 + a_k)$. The sequence (P_n) converges to a non-zero limit P if and only if $(\ln P_n)$ converges to $\ln P$.

■

If $\sum \ln(1 + a_n)$ diverges to $-\infty$, then $P_n \to 0$, consistent with our definition of divergence to zero.

This criterion allows us to translate results from series directly to products.

Theorem 0.11. ***Convergence Tests for Products.***

(i) **Positive Terms:** If $a_n \geq 0$, then $\prod(1 + a_n)$ converges if and only if $\sum a_n$ converges.

(ii) **Absolute Convergence:** We say $\prod(1 + a_n)$ converges *absolutely* if $\prod(1 + |a_n|)$ converges. Absolute convergence implies ordinary convergence.

(iii) **Conditional Convergence:** If $\sum a_n$ converges but $\sum a_n^2$ diverges (and terms alternate), the product may diverge. Specifically, using $\ln(1 + x) \approx x - x^2/2$, one can show that if $\sum a_n$ converges, then $\prod(1 + a_n)$ converges if and only if $\sum a_n^2$ converges.

定理

Euler's Product Formula for the Sine Function

The most celebrated infinite product is Euler's factorisation of the sine function. Just as a polynomial is determined by its roots $P(x) = C\prod(x - r_i)$, Euler reasoned that $\sin x$, having roots at $n\pi$, should behave like an infinite polynomial $x\prod(1 - \frac{x^2}{n^2\pi^2})$.

Theorem 0.12. ***Euler's Sine Product.***
For all $x \in \mathbb{R}$:

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right)$$

定理

Setting $x = \pi/2$ recovers Wallis's Product:

$$1 = \frac{\pi}{2} \prod_{n=1}^{\infty} \left(1 - \frac{1}{4n^2}\right) \implies \frac{\pi}{2} = \prod_{n=1}^{\infty} \left(\frac{2n}{2n-1} \cdot \frac{2n}{2n+1}\right)$$

Proof

We employ a limit argument involving the factorisation of Chebyshev polynomials. Recall that $\sin(2n + 1)\varphi$ can be expressed as a polynomial of degree $2n + 1$ in $\sin\varphi$. Roots occur when $(2n+1)\varphi = k\pi$, i.e., $\sin^2\varphi = \sin^2(\frac{k\pi}{2n+1})$. This leads to the identity:

$$\sin(2n+1)\varphi = (2n+1)\sin\varphi \prod_{k=1}^{n} \left(1 - \frac{\sin^2\varphi}{\sin^2\frac{k\pi}{2n+1}}\right)$$

Let $x \in \mathbb{R}$ be fixed. Set $\varphi = \frac{x}{2n+1}$. Substituting into the identity:

$$\sin x = (2n+1)\sin\left(\frac{x}{2n+1}\right) \prod_{k=1}^{n} \left(1 - \frac{\sin^2\frac{x}{2n+1}}{\sin^2\frac{k\pi}{2n+1}}\right)$$

We split the product into a "head" (fixed m) and a "tail". Let $m < n$:

$$\sin x = \underbrace{(2n+1)\sin\frac{x}{2n+1}}_{\to x} \cdot \underbrace{\prod_{k=1}^{m} \left(1 - \frac{\sin^2\frac{x}{2n+1}}{\sin^2\frac{k\pi}{2n+1}}\right)}_{U_{n,m}} \cdot \underbrace{\prod_{k=m+1}^{n} (\dots)}_{V_{n,m}}$$

As $n \to \infty$, for fixed k, $\frac{\sin^2\frac{x}{2n+1}}{\sin^2\frac{k\pi}{2n+1}} \to \frac{(x/(2n+1))^2}{(k\pi/(2n+1))^2} = \frac{x^2}{k^2\pi^2}$. Thus, the first product converges to the desired infinite product partial sum:

$$\lim_{n\to\infty} U_{n,m} = \prod_{k=1}^{m} \left(1 - \frac{x^2}{k^2\pi^2}\right)$$

We must show the tail $V_{n,m}$ tends to 1 as $m \to \infty$. Using the inequality $\sin y < y$ and Jordan's inequality $\sin y \geq \frac{2}{\pi}y$ on $[0, \pi/2]$, we can bound the terms. For sufficiently large n, the terms in the tail satisfy:

$$1 \geq \text{Term}_k \geq 1 - \frac{x^2}{4k^2}$$

The product of these lower bounds converges to 1 as $m \to \infty$ (since $\sum 1/k^2$ converges). By the Squeeze Theorem, the formula holds.

■

The Gamma Function

The Gamma function, $\Gamma(x)$, extends the factorial to real (and complex) arguments. While often defined by an integral $\int_0^\infty t^{x-1}e^{-t}dt$, its definition via infinite products is more fundamental for establishing its functional properties.

Definition 0.2. ***Gamma Function (Euler Form).***
For $x \neq 0, -1, -2, \ldots$, we define:

$$\Gamma(x) = \frac{1}{x}\prod_{n=1}^{\infty}\frac{(1+1/n)^x}{1+x/n}$$

定義

The general term behaves like $1 + \frac{x(x-1)}{2n^2}$, so the product converges absolutely. Expanding the partial product:

$$P_n = \frac{1}{x}\prod_{k=1}^{n}\frac{(\frac{k+1}{k})^x}{\frac{x+k}{k}} = \frac{1}{x}\frac{(n+1)^x}{1}\frac{n!}{\prod_{k=1}^{n}(x+k)} = \frac{n!(n+1)^x}{x(x+1)\ldots(x+n)}$$

This yields the celebrated **Euler-Gauss Limit**:

$$\Gamma(x) = \lim_{n\to\infty}\frac{n!n^x}{x(x+1)\ldots(x+n)}$$

(Replacing $(n+1)^x$ with n^x does not change the limit).

Proposition 0.2. ***Properties of*** $\Gamma(x)$**.**

1. **Functional Equation:** $\Gamma(x+1) = x\Gamma(x)$.

   Proof

   Using the limit form:

   $$\frac{\Gamma(x+1)}{\Gamma(x)} = \lim\frac{n!n^{x+1}\cdot x(x+1)\ldots(x+n)}{(x+1)\ldots(x+n+1)\cdot n!n^x} = \lim\frac{nx}{x+n+1} = x$$

   ■

2. **Factorial Generalisation:** Since $\Gamma(1) = \lim\frac{n!n}{1\cdot 2\ldots(n+1)} = 1$, induction gives $\Gamma(n+1) = n!$.
3. **Weierstrass Form:** Using the Euler-Mascheroni constant $\gamma = \lim(\sum_{k=1}^{n}\frac{1}{k} - \ln n)$, one can rewrite the product as:

   $$\frac{1}{\Gamma(x)} = xe^{\gamma x}\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)e^{-x/n}$$

4. **Reflection Formula:**

   $$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin\pi x}$$

Proof

From the definition, $\Gamma(1-x) = -x\Gamma(-x)$. We compute the product $\Gamma(x)\Gamma(-x)$:

$$\Gamma(x)\Gamma(-x) = \frac{1}{-x^2}\prod_{n=1}^{\infty}\frac{(1+1/n)^0}{(1+x/n)(1-x/n)} = \frac{-1}{x^2}\prod\frac{1}{1-x^2/n^2}$$

Comparing this with Euler's Sine Product *Theorem 0.12* (with argument πx):

$$\frac{\sin\pi x}{\pi x} = \prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2}\right)$$

Thus $\Gamma(x)\Gamma(-x) = \frac{-\pi}{x\sin\pi x}$. Multiplying by $-x$ gives the result. ■

命題

Example 0.4. Value at 1/2. Using the reflection formula at $x = 1/2$:

$$\Gamma(1/2)^2 = \frac{\pi}{\sin(\pi/2)} = \pi \implies \Gamma(1/2) = \sqrt{\pi}$$

This is equivalent to the Gaussian integral $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$.

範例

Asymptotic Analysis of Products

Infinite products provide a powerful tool for analyzing the asymptotic behaviour of sequences.

Example 0.5. Stirling's Approximation (Weak Form). We revisit the sequence $a_n = \left|\binom{\alpha}{n}\right|$. Using the Gamma function limit:

$$\left|\binom{\alpha}{n}\right| = \left|\frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}\right| \sim \frac{1}{|\Gamma(-\alpha)|n^{1+\alpha}}$$

For $\alpha = -1/2$, this yields $\binom{-1/2}{n} \sim \frac{1}{\sqrt{\pi n}}$, consistent with our earlier Wallis product derivations.

範例

This concludes our development of infinite processes.

0.7 *Exercises*

1. **Convergence of Infinite Products.** Determine the convergence or divergence of the following infinite products.

   (a) $\prod_{n=1}^{\infty}\sqrt[n]{\frac{n+1}{n+2}}$

(b) $\prod_{n=2}^{\infty}\left(1+(-1)^n\frac{1}{n}\right)$

(c) $\prod_{n=1}^{\infty}\left(\frac{1}{e}\left(1+\frac{1}{n}\right)^n\right)$

Recall the asymptotic expansion of $(1+1/n)^n$ involving e.

2. **Series and Products.** Suppose each term of the sequence (a_n) satisfies $0 < a_n < \pi/2$. Prove that the series $\sum_{n=1}^{\infty} a_n^2$ converges if and only if the infinite product $\prod_{n=1}^{\infty} \cos a_n$ converges.

Utilise the Logarithmic Criterion and the Taylor expansion of $\ln(\cos x)$.

3. **Raabe's Limit and Absolute Convergence.** Suppose a positive series $\sum a_n$ satisfies the limit condition of Raabe's Test:

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right) = r > 0.$$

(a) Prove that the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ converges.

(b) Prove that the squared series $\sum_{n=1}^{\infty} a_n^2$ converges.

4. **Hypergeometric Products.** Let α, β, γ be real parameters such that none of the factors below are zero. Discuss the convergence of the infinite product:

$$\prod_{n=1}^{\infty}\frac{(\alpha+n)(\beta+n)}{(1+n)(\gamma+n)}.$$

Consider the asymptotic behaviour of the general term p_n. Under what condition does $p_n = 1 + O(1/n)$ versus $1 + \mu/n + O(1/n^2)$? This generalises the convergence criteria for the Gamma function.

5. **Euler's Partition Identity.** For $|x| < 1$, prove the following identity connecting infinite products:

$$(1+x)(1+x^2)(1+x^3)\cdots = \frac{1}{(1-x)(1-x^3)(1-x^5)\cdots}.$$

Multiply the left-hand side by $(1-x)$ and observe the telescoping structure of the terms $(1-x)(1+x) = (1-x^2)$, etc.

6. **The Boundary of Gauss's Test.** Suppose a positive sequence (a_n) satisfies the ratio condition:

$$\frac{a_n}{a_{n+1}} = 1 + \frac{1}{n} + O(b_n),$$

where the series $\sum b_n$ converges absolutely. Prove that the series $\sum a_n$ diverges.

This resolves the $p = 1$ case in Gauss's Test, confirming the divergence of series behaving like the harmonic series $\sum 1/n$.

7. **Stirling's Formula via Products.** By analysing the infinite product expansion of the Gamma function or otherwise, prove that the sequence

$$u_n = \frac{n!e^n}{n^{n+\frac{1}{2}}}$$

converges to a non-zero limit.

Consider the ratio u_n/u_{n+1} and use asymptotic expansions for logarithms.

8. **Product Growth.** Prove using two different methods that for any positive sequence (a_n) :

$$\lim_{n\to\infty}\frac{a_n}{(1+a_1)(1+a_2)\cdots(1+a_n)} = 0.$$

Method 1: Consider the convergence of the product. Method 2: Treat the expression as terms of a telescoping series.

9. **Monotonicity and Square Convergence.** Let $\sum a_n$ be a positive series with monotonically decreasing terms. Prove that if the weighted series $\sum \frac{a_n}{\sqrt{n}}$ converges, then the series $\sum a_n^2$ converges.

10. **Discrete Abel Summation.** Let (a_n) be a positive, monotonically decreasing sequence converging to 0.
    (a) Prove that the series $\sum_{n=1}^{\infty} a_n$ and the series $\sum_{n=1}^{\infty} n(a_n - a_{n+1})$ converge or diverge together, and if they converge, they sum to the same value.
    (b) Provide a counter-example to show that if monotonicity is dropped, the conclusion no longer holds.

11. **Sum of Zeta Functions.** For integer $m \geq 2$, let $\zeta(m) = \sum_{n=1}^{\infty} n^{-m}$. Prove that:
$$\sum_{m=2}^{\infty} (\zeta(m) - 1) = 1.$$

This requires exchanging the order of summation for a double series of positive terms.

12. **Decay of Tail Sums.** Let $\sum a_n b_n$ be a convergent series, where (b_n) is a monotonically decreasing sequence converging to 0. Let $S_n = \sum_{k=1}^{n} a_k$. Prove that $\lim_{n\to\infty} S_n b_n = 0$.

This result is a partial converse to Abel's Summation Lemma conditions.

13. **Advanced Convergence Testing.** Determine the absolute or conditional convergence of the following series.
    (a) $\displaystyle\sum_{n=2}^{\infty} \frac{\sin(n\pi/12)}{\ln n}$
    (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n (n^{1/n} - 1)$
    (c) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin nx}{n} \left(1 + \frac{1}{n}\right)^n$ (for $x \in \mathbb{R}$)
    (d) $\displaystyle\sum_{n=1}^{\infty} \ln\left(1 + \frac{(-1)^n}{n^p}\right)$ (for $p > 0$)

14. **Generalised Rearrangements.** Consider the alternating harmonic series $\sum \frac{(-1)^{n-1}}{n}$. Construct a rearrangement such that every block of p positive terms is followed by a block of q negative terms, preserving their internal order. Prove that this rearranged series converges to:
$$\ln 2 + \frac{1}{2} \ln\left(\frac{p}{q}\right).$$
Deduce that the rearrangement converges to the original sum $\ln 2$ if and only if $p = q$.

15. **Averaged Alternating Series.** Let (a_n) be a strictly decreasing sequence converging to 0. Prove the convergence of the series formed by the alternating averages:
$$\sum_{n=1}^{\infty} (-1)^n \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right).$$

Let A_n be the average. Is A_n necessarily monotonic? Does Dirichlet's Test apply?

1

Complex Numbers and Series

This chapter is basically a review of my previous notes but with a new section on complex series and convergence.

1.1 The Field of Complex Numbers

We postulate the existence of an imaginary unit i satisfying the property $i^2 = -1$. The set of complex numbers $\mathbb{C}$ is constructed as a formal extension of $\mathbb{R}$.

> **Definition 1.1.** ***Complex Numbers.***
> The set of complex numbers is defined as:
>
> $$\mathbb{C} := \{x + iy : x, y \in \mathbb{R}\}$$
>
> For a complex number $z = x + iy$, we define:
> · The *real part*: $\Re(z) := x$.
> · The *imaginary part*: $\Im(z) := y$.
> Equality is defined component-wise: $z_1 = z_2$ if and only if $\Re(z_1) = \Re(z_2)$ and $\Im(z_1) = \Im(z_2)$.
>
> 定義

Algebraic Structure

We define addition and multiplication on $\mathbb{C}$ to be consistent with the arithmetic of real polynomials evaluated at i, subject to the reduction $i^2 = -1$. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

$$z_1 + z_2 := (x_1 + x_2) + i(y_1 + y_2)$$
$$z_1 z_2 := (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$$

Under these operations, $\mathbb{C}$ forms a field. The additive identity is $0 = 0 + 0i$, and the multiplicative identity is $1 = 1 + 0i$.

> **Definition 1.2.** ***Conjugate and Modulus.***
> Let $z = x + iy$.
> 1. The *complex conjugate* of z is $\bar{z} := x - iy$.

2. The *modulus* (or norm) of z is $|z| := \sqrt{x^2 + y^2}$.

定義

The conjugate provides an immediate mechanism for division. Observe that $z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$. Thus, for any non-zero $z \in \mathbb{C}$, the multiplicative inverse is given by:

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$$

Proposition 1.1. ***Properties of Conjugation and Modulus.***
For all $z, w \in \mathbb{C}$:
1. $\overline{z + w} = \bar{z} + \bar{w}$ and $\overline{zw} = \bar{z}\bar{w}$.
2. $|zw| = |z||w|$.
3. $|z| = 0 \iff z = 0$.
4. $\Re(z) = \frac{z+\bar{z}}{2}$ and $\Im(z) = \frac{z-\bar{z}}{2i}$.

命題

Proof

These follow directly from the definitions. For (ii), observe $|zw|^2 = (zw)\overline{(zw)} = zw\bar{z}\bar{w} = (z\bar{z})(w\bar{w}) = |z|^2|w|^2$.

■

1.2 *Geometric Interpretation*

We identify the complex number $z = x + iy$ with the vector (x, y) in the Euclidean plane $\mathbb{R}^2$, often termed the Argand plane. Addition of complex numbers corresponds to vector addition, obeying the parallelogram law. Conjugation corresponds to reflection across the real axis (x-axis).
The modulus $|z|$ represents the Euclidean distance from the origin to z. This allows us to define the distance between two complex numbers z, w as $|z - w|$.

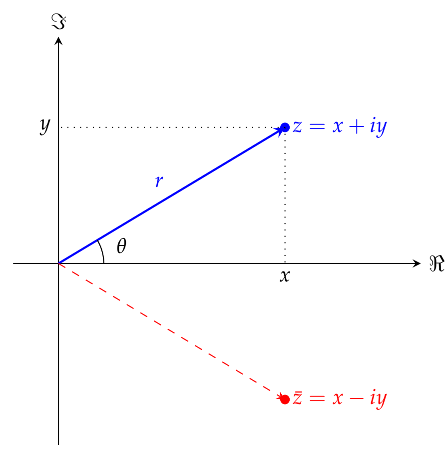

Figure 1.1: The geometric representation of a complex number z and its conjugate $\bar{z}$.

Proposition 1.2. ***Triangle Inequality.***
For any $z_1, z_2 \in \mathbb{C}$:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

with equality if and only if one is a non-negative real multiple of the other.

命題

Proof

Geometrically, this states that the length of one side of a triangle is less than the sum of the other two. Algebraically, let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Squaring both sides, the inequality is equivalent to $|z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2$.

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) = |z_1|^2 + z_1\bar{z}_2 + \bar{z}_1 z_2 + |z_2|^2 \\ &= |z_1|^2 + 2\Re(z_1\bar{z}_2) + |z_2|^2 \end{aligned}$$

The right hand side is $|z_1|^2 + 2|z_1||z_2| + |z_2|^2$. Thus, we must show $\Re(z_1\bar{z}_2) \leq |z_1\bar{z}_2|$. This is true since for any complex number w, $\Re(w) \leq |w|$ (as $x \leq \sqrt{x^2 + y^2}$).

■

Polar Representation

Using polar coordinates (r, θ) in the plane, we can write $x = r\cos\theta$ and $y = r\sin\theta$, where $r = |z|$. Thus:

$$z = r(\cos\theta + i\sin\theta)$$

The angle θ is called the *argument* of z, denoted $\arg z$. It is defined modulo 2π. Polar form is particularly powerful for multiplication. Let $z_k = r_k(\cos\theta_k + i\sin\theta_k)$ for $k = 1, 2$.

$$\begin{aligned} z_1 z_2 &= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \\ &= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \end{aligned}$$

This yields the fundamental geometric insight: multiplication by a complex number scales the modulus by r and rotates the argument by θ.

Theorem 1.1. ***De Moivre's Theorem.***
For any $\theta \in \mathbb{R}$ and $n \in \mathbb{N}$:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

定理

Proof

This follows immediately from the iterative application of the multiplication rule derived above.

■

This theorem provides an elegant method for finding roots of unity.

Example 1.1. Roots of Unity. We solve $z^n = 1$. Let $z = \cos\theta + i\sin\theta$ (since $|z|$ must be 1). By De Moivre, $\cos(n\theta) + i\sin(n\theta) = 1$. This requires $\cos(n\theta) = 1$ and $\sin(n\theta) = 0$, implying $n\theta = 2\pi k$ for $k \in \mathbb{Z}$. The distinct solutions are given by $\theta_k = \frac{2\pi k}{n}$ for $k = 0, 1, \ldots, n-1$.

$$\zeta_k = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right)$$

These points form the vertices of a regular n-gon inscribed in the unit circle.

範例

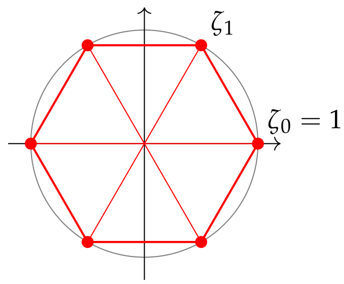

Figure 1.2: The 6-th roots of unity forming a regular hexagon.

1.3 *Analysis of Complex Sequences*

The modulus induces a metric on $\mathbb{C}$, defined by $d(z, w) = |z - w|$. This allows us to import the machinery of limits and convergence from real analysis directly into the complex domain.

Definition 1.3. ***Convergence.***
A sequence of complex numbers $(z_n)_{n=1}^{\infty}$ is said to converge to $z^* \in \mathbb{C}$ if the sequence of real distances $|z_n - z^*|$ converges to 0. Formally:

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } n > N \implies |z_n - z^*| < \epsilon$$

定義

Convergence in $\mathbb{C}$ is equivalent to component-wise convergence in $\mathbb{R}$.

Proposition 1.3. ***Component-wise Convergence.***
Let $z_n = x_n + iy_n$ and $z^* = x^* + iy^*$. The sequence (z_n) converges to z^* if and only if (x_n) converges to x^* and (y_n) converges to y^*.

命題

Proof

We rely on the inequalities relating the modulus to the components:

$$\max(|x|, |y|) \leq \sqrt{x^2 + y^2} \leq |x| + |y|$$

Let $\epsilon > 0$.
$(\implies)$ If $|z_n - z^*| \to 0$, then $|x_n - x^*| \leq |z_n - z^*| \to 0$ and similarly for y_n.
$(\impliedby)$ If $|x_n - x^*| \to 0$ and $|y_n - y^*| \to 0$, then $|z_n - z^*| \leq |x_n - x^*| + |y_n - y^*| \to 0$.

■

Corollary 1.1. *Boundedness*. Every convergent sequence of complex numbers is bounded. This follows directly from the real case applied to the sequence of norms.

推論

Cauchy Sequences

Definition 1.4. ***Cauchy Sequence.***
A sequence (z_n) is called a Cauchy sequence if for every $\epsilon > 0$, there exists N such that $n, m > N$ implies $|z_n - z_m| < \epsilon$.

定義

Since $\mathbb{R}$ is complete (every Cauchy sequence converges), the component-wise equivalence (*Proposition 1.2*) implies that $\mathbb{C}$ is also complete. This is a fundamental result: we can verify convergence without knowing the limit.

1.4 *Complex Series*

Definition 1.5. ***Infinite Series.***
Given a sequence (z_n), we define the infinite series $\sum_{n=0}^{\infty} z_n$ as the limit of the sequence of partial sums $S_N = \sum_{n=0}^{N} z_n$. If the limit $S = \lim_{N\to\infty} S_N$ exists, we say the series converges to S.

定義

Proposition 1.4. ***Vanishing Condition.***
If $\sum z_n$ converges, then $\lim_{n\to\infty} z_n = 0$.

命題

Proof

Let S_N be the partial sums converging to S. Then $z_N = S_N - S_{N-1} \to S - S = 0$.

■

Absolute Convergence

Analogous to the real case, we distinguish between absolute and conditional convergence.

Definition 1.6. ***Absolute Convergence.***
The series $\sum z_n$ is absolutely convergent if the series of real numbers $\sum |z_n|$ is convergent.

定義

Theorem 1.2. ***Completeness of*** $\mathbb{C}$**.**
Every absolutely convergent series is convergent.

定理

Proof

Let S_N be the partial sums of $\sum z_n$ and T_N be the partial sums of $\sum |z_n|$. For $M > N$:

$$|S_M - S_N| = \left| \sum_{k=N+1}^{M} z_k \right| \leq \sum_{k=N+1}^{M} |z_k| = |T_M - T_N|$$

If $\sum |z_n|$ converges, (T_N) is a Cauchy sequence in $\mathbb{R}$. The inequality implies (S_N) is a Cauchy sequence in $\mathbb{C}$. By the completeness of $\mathbb{C}$, the series converges.

■

The Geometric Series

The geometric series serves as our primary benchmark for convergence.

Example 1.2. Complex Geometric Series. Consider $\sum_{n=0}^{\infty} z^n$. The partial sum is $S_N = \frac{1-z^{N+1}}{1-z}$ for $z \neq 1$.

- If $|z| < 1$, then $|z|^{N+1} \to 0$, so $z^{N+1} \to 0$. The series converges to $\frac{1}{1-z}$.
- If $|z| \geq 1$, then $|z^n| \geq 1$, so terms do not vanish. The series diverges.

範例

This leads naturally to the Ratio Test for complex series, derived from the real series of moduli.

Theorem 1.3. ***Ratio Test*****.**
Suppose $L = \lim_{n\to\infty} \left| \frac{z_{n+1}}{z_n} \right|$ exists.

(i) If $L < 1$, the series $\sum z_n$ converges absolutely.
(ii) If $L > 1$, the series diverges.

定理

Proof

If $L < 1$, then $\sum |z_n|$ converges by the real Ratio Test, implying absolute convergence. If $L > 1$, $|z_n|$ grows indefinitely, so $z_n \not\to 0$.

■

This framework of complex analysis allows us to define functions like the complex exponential, sine, and cosine via power series, unifying the disparate trigonometric identities seen in real analysis into a cohesive algebraic structure.

1.5 Complex Power Series

We now generalise the concept of power series to the complex domain. A complex power series is a series of the form

$$S(z) = \sum_{n=0}^{\infty} a_n z^n$$

where the coefficients $(a_n)_{n=0}^{\infty}$ and the variable z are complex numbers.

Radius of Convergence

The convergence properties of complex power series mirrors that of the real case, with intervals replaced by disks.

Theorem 1.4. *Radius of Convergence.*
For any power series $\sum a_n z^n$, there exists a unique $R \in [0, \infty]$, called the radius of convergence, such that:
1. The series converges absolutely for all z in the open disk $D_R(0) = \{z \in \mathbb{C} : |z| < R\}$.
2. The series diverges for all $|z| > R$.

The radius R is given by the Cauchy-Hadamard formula (derived from the Root Test):

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

定理

Proof

Let $L = \limsup \sqrt[n]{|a_n|}$. If $|z| < 1/L$, then $\limsup \sqrt[n]{|a_n z^n|} = |z|L < 1$. By the Root Test for real series applied to $\sum |a_n z^n|$, the series converges absolutely. If $|z| > 1/L$, then $\limsup \sqrt[n]{|a_n z^n|} > 1$, so the terms $a_n z^n$ do not converge to 0. The series diverges.

■

Example 1.3. A Geometric Series. Consider the series $\sum_{n=0}^{\infty}(-2z)^n$. The coefficients are $a_n = (-2)^n$. We compute the radius of convergence:

$$\sqrt[n]{|a_n|} = |-2| = 2 \implies R = \frac{1}{2}$$

Thus, the series converges absolutely on the open disk $|z| < 1/2$ and represents the function $1/(1+2z)$.

範例

1.6 *The Complex Exponential*

We define the exponential function for complex arguments via its Maclaurin series.

Definition 1.7. ***Complex Exponential.***
For any $z \in \mathbb{C}$, we define:

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

定義

Applying the Ratio Test, we find $\lim |\frac{z^{n+1}/(n+1)!}{z^n/n!}| = \lim \frac{|z|}{n+1} = 0$ for all z. Thus, the series has an infinite radius of convergence ($R = \infty$) and defines an entire function on $\mathbb{C}$.
The defining algebraic property of the exponential function is preserved in the complex domain.

Theorem 1.5. ***Exponential Addition Theorem.***
For any $z, w \in \mathbb{C}$:

$$e^{z+w} = e^z e^w$$

定理

Proof

Since the series for e^z and e^w are absolutely convergent, Mertens' Theorem (from our analysis of infinite sums) implies that their Cauchy product converges to the product of their sums. The n-th term of the Cauchy product is:

$$c_n = \sum_{k=0}^{n} \frac{z^k}{k!} \frac{w^{n-k}}{(n-k)!}$$

Multiplying and dividing by $n!$, we recognise the binomial expansion:

$$c_n = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k} = \frac{1}{n!}(z+w)^n$$

Summing these terms yields $\sum_{n=0}^{\infty} c_n = e^{z+w}$.

■

Euler's Formula

This series definition illuminates the profound connection between the exponential function and trigonometry. Restricting the exponential to the imaginary axis $z = it$ (where $t \in \mathbb{R}$), we obtain:

$$e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} = 1 + it - \frac{t^2}{2!} - i\frac{t^3}{3!} + \frac{t^4}{4!} + \dots$$

Separating the real and imaginary parts (which is permissible due to absolute convergence):

$$e^{it} = \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots\right) + i\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots\right)$$

We recognise the series in the parentheses as the Maclaurin expansions for cosine and sine respectively.

Theorem 1.6. ***Euler's Formula.***
For any real number t:

$$e^{it} = \cos t + i \sin t$$

定理

This identity allows us to define the trigonometric functions for complex arguments:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Evaluating Euler's formula at $t = \pi$, and recalling that $\sin \pi = 0$ and $\cos \pi = -1$, we arrive at the celebrated identity uniting the five fundamental constants of analysis.

Corollary 1.2. *Euler's Identity.*

$$e^{i\pi} + 1 = 0$$

推論

1.7 *Exercises*

10.1. The Parallelogram Law. In the study of Euclidean geometry, the lengths of the diagonals of a parallelogram are related to the lengths of its sides.

(a) Prove the identity:

$$|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2)$$

for any $z, w \in \mathbb{C}$. Interpret this geometrically.

(b) A normed vector space is an inner product space if and only if the norm satisfies the parallelogram law. Show that the space $\mathbb{R}^2$ equipped with the "taxicab norm" $|(x, y)|_1 = |x| + |y|$ cannot be induced by an inner product by showing it fails the parallelogram law.

10.2. Geometry of Roots of Unity. Let $n \geq 2$ be an integer. The roots of the equation $z^n - 1 = 0$ are given by $1, \omega, \omega^2, \ldots, \omega^{n-1}$ where $\omega = e^{2\pi i/n}$.

(a) Prove that $1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0$. Explain why this implies that the centroid of a regular n-gon centered at the origin is the origin itself.

(b) Consider the polynomial $P(z) = z^n - 1$. Factor $P(z)$ into linear terms involving the roots of unity. By computing the limit $\lim\limits_{z \to 1} \frac{z^n - 1}{z - 1}$, prove that:

$$(1 - \omega)(1 - \omega^2) \cdots (1 - \omega^{n-1}) = n.$$

(c) Interpret result (b) geometrically as a statement about the product of the lengths of the chords from one vertex of a regular n-gon to all other vertices.

10.3. Lagrange's Trigonometric Identity. While real variable techniques for summing trigonometric series can be cumbersome, complex exponentials simplify the process significantly.

(a) Consider the geometric series $\sum_{k=0}^{n} z^k$. By substituting $z = e^{i\theta}$ (where θ is not an integer multiple of 2π), prove that:

$$\sum_{k=0}^{n} \cos(k\theta) = \frac{1}{2} + \frac{\sin((n + \frac{1}{2})\theta)}{2 \sin(\frac{\theta}{2})}.$$

(b) Deduce a similar closed-form expression for $\sum_{k=1}^{n} \sin(k\theta)$.

10.4. Differentiation of Power Series. Let $\sum a_n z^n$ be a power series with radius of convergence $R > 0$.

(a) Prove that the "derived series" $\sum n a_n z^{n-1}$ also has radius of convergence R. Recall that $\lim\limits_{n \to \infty} \sqrt[n]{n} = 1$.

(b) Let $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. Use part (a) to show that $f'(z) = f(z)$ for all $z \in \mathbb{C}$ (in the sense of term-wise differentiation).

10.5. Complex Trigonometry. The behaviour of sine and cosine in the complex plane differs markedly from the real line.

(a) Using the definition $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$, prove that:

$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y,$$

where $\cosh y = \frac{e^y + e^{-y}}{2}$ and $\sinh y = \frac{e^y - e^{-y}}{2}$.

(b) Prove that $|\sin(x + iy)|^2 = \sin^2 x + \sinh^2 y$.

(c) Conclude that the function $\sin z$ is unbounded on $\mathbb{C}$, unlike its real restriction. Find a sequence of points z_n such that $|\sin z_n| \to \infty$.

2

Ordinary Differential Equations Introduction

Having established the foundations of integration and infinite series in previous notes, we now apply these tools to the study of differential equations. These equations, which relate functions to their derivatives, are the fundamental language of dynamical systems in physics, engineering, and geometry. Our goal is to classify these equations and establish systematic methods for their solution, beginning with the theory of first-order ordinary differential equations.

2.1 *Fundamental Concepts*

A differential equation is a mathematical relation linking an independent variable, a dependent variable, and the derivatives of the dependent variable with respect to the independent one.

> **Definition 2.1. *Ordinary Differential Equation (ODE).***
> An ordinary differential equation is an equation involving an unknown function y of a single independent variable x, and its derivatives $y', y'', \ldots, y^{(n)}$. It can be expressed generally as:
>
> $$F(x, y, y', \ldots, y^{(n)}) = 0$$
>
> The *order* of the differential equation is the order of the highest derivative appearing in the equation.
>
> 定義

> *Note*
>
> If the unknown function depends on multiple independent variables (e.g., position x, y, z and time t) and the equation involves partial derivatives, it is termed a *Partial Differential Equation* (PDE). Famous examples include Maxwell's equations for electromagnetism or the Heat Equation. In this course, we restrict our attention to ODEs.

We classify ODEs based on their structure, as this determines the methods available for their solution.

Definition 2.2. ***Linearity and Homogeneity.***
An n-th order ODE is said to be *linear* if it can be written in the form:

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x)$$

where $a_i(x)$ and $g(x)$ are functions of x alone.
· If $g(x) = 0$ for all x, the equation is *homogeneous*.
· If $g(x) \neq 0$, the equation is *non-homogeneous*.
· If the coefficients $a_i(x)$ are constants, the equation is said to have *constant coefficients*.

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Example 2.1. Classification.
1. $\frac{d^2y}{dx^2} + 5y = 0$: Second-order, linear, constant coefficient, homogeneous.
2. $y''' + x^2y = \sin x$: Third-order, linear, non-homogeneous.
3. $\frac{dy}{dx} + y^2 = x$: First-order, non-linear (due to the y^2 term).

範例

2.2 *Solutions to Differential Equations*

Unlike algebraic equations, where solutions are numbers, the solution to a differential equation is a function (or a family of functions). We distinguish between three types of solutions.

Definition 2.3. ***Types of Solutions.***
1. An **explicit solution** is a function $y = \phi(x)$ which, when substituted into the differential equation, satisfies the identity for all x in an interval.
2. An **implicit solution** is a relation $G(x, y) = 0$ which defines y as a function of x (locally) such that the function satisfies the differential equation.
3. The **general solution** is a family of functions containing arbitrary constants (parameters). For an n-th order equation, the general solution typically contains n independent constants.

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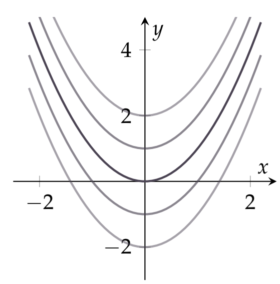

Figure 2.1: Family of parabolas $y = x^2 + C$.

Example 2.2. Families of Curves. Consider the first-order equation $\frac{dy}{dx} = 2x$. Integration yields $y = x^2 + C$. This is the general solution, representing a family of parabolas.
· If we specify an initial condition, say $y(0) = 1$, we determine $C = 1$, yielding the particular explicit solution $y = x^2 + 1$.
· Consider the equation $\frac{dy}{dx} = -\frac{x}{y}$. Separating terms gives $ydy = -xdx$. Integration yields $\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C_0$, or $x^2 + y^2 = C$. This

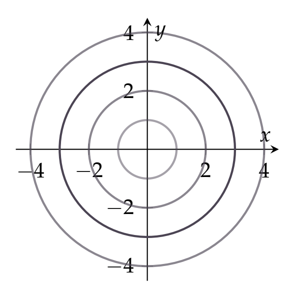

Figure 2.2: Family of circles $x^2 + y^2 = C$.

is an implicit solution describing a family of circles.

範例

2.3 Separation of Variables

The most elementary technique for solving first-order ODEs is the method of separation of variables. This applies to non-linear equations where the dependence on x and y can be factored.

Theorem 2.1. ***Separable Equations.***
A first-order differential equation is *separable* if it can be written in the form:

$$\frac{dy}{dx} = g(x)h(y)$$

If $h(y) \neq 0$, the general solution is given implicitly by:

$$\int \frac{1}{h(y)}\,dy = \int g(x)\,dx + C$$

定理

Proof

We treat y as a function of x. Rearranging the equation gives:

$$\frac{1}{h(y(x))} \cdot \frac{dy}{dx} = g(x)$$

We integrate both sides with respect to x:

$$\int \frac{1}{h(y(x))}\frac{dy}{dx}\,dx = \int g(x)\,dx$$

On the left-hand side, we apply the substitution rule. Let $u = y(x)$, then $du = y'(x)dx$. The integral becomes:

$$\int \frac{1}{h(u)}\,du = \int g(x)\,dx$$

This yields the required relation.

■

Remark.

One must be careful with division. The roots of $h(y) = 0$ correspond to constant solutions $y(x) = c$, known as *equilibrium solutions*. These are often lost during the separation process and must be checked separately.

Examples and Applications

We illustrate the method with a series of examples ranging from elementary geometry to mechanics.

Example 2.3. Exponential Growth and Decay. Consider the linear equation $\frac{dy}{dx} = ky$. Separating variables (assuming $y \neq 0$):

$$\int \frac{1}{y}\, dy = \int k\, dx$$

$$\ln |y| = kx + C_1$$

Exponentiating both sides yields $|y| = e^{C_1} e^{kx}$. Letting $C = \pm e^{C_1}$, we obtain $y = Ce^{kx}$. Note that $y = 0$ is an equilibrium solution, which is recovered if we allow $C = 0$.

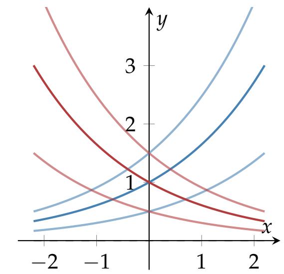

Figure 2.3: Exponential solutions $y = Ce^{kx}$: growth ($k > 0$, blue) and decay ($k < 0$, red).

範例

Example 2.4. Implicit Solutions. Solve $\frac{dy}{dx} = -\frac{x}{y}$ with initial condition $y(4) = -3$.

$$\int y\, dy = \int -x\, dx$$

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C_1$$

$$x^2 + y^2 = C$$

Using the initial condition $(4, -3)$: $4^2 + (-3)^2 = 16 + 9 = 25$, so $C = 25$. The implicit solution is $x^2 + y^2 = 25$. Solving for y gives two branches: $y = \pm\sqrt{25 - x^2}$. Since $y(4) = -3$, we must select the negative branch:

$$y = -\sqrt{25 - x^2}$$

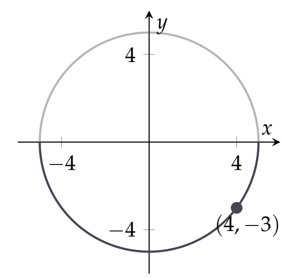

Figure 2.4: Circle $x^2 + y^2 = 25$ with particular solution $y = -\sqrt{25 - x^2}$ highlighted.

範例

Example 2.5. Complex Algebraic Separation. Find the general solution to $\frac{dy}{dx} = \frac{xy^3}{\sqrt{1+x^2}}$. Separating variables:

$$\int y^{-3}\, dy = \int x(1 + x^2)^{-1/2}\, dx$$

For the right-hand integral, let $u = 1 + x^2$, so $du = 2xdx$.

$$\frac{y^{-2}}{-2} = \frac{1}{2}\int u^{-1/2}\, du = \frac{1}{2}(2u^{1/2}) + C_0$$

$$-\frac{1}{2y^2} = \sqrt{1 + x^2} + C_0$$

Rearranging for y:

$$y^2 = \frac{-1}{2(\sqrt{1 + x^2} + C_0)} \implies y = \pm\sqrt{\frac{-1}{2\sqrt{1 + x^2} + C}}$$

This solution is valid only where the term under the square root is non-negative.

範例

Example 2.6. Trigonometric Substitution. Solve $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$. Separating variables implies:

$$\int \frac{dy}{\sqrt{1-y^2}} = \int \frac{dx}{\sqrt{1-x^2}}$$

Recognising the standard integrals (inverse sine):

$$\arcsin y = \arcsin x + C$$

Taking the sine of both sides yields the explicit form $y = \sin(\arcsin x + C)$. Using the addition formula $\sin(A + B) = \sin A \cos B + \cos A \sin B$:

$$y = x \cos C + \sqrt{1-x^2} \sin C$$

範例

2.4 *Applications to Dynamics*

Differential equations naturally arise in physics, particularly in mechanics where Newton's Second Law relates force to acceleration (the second derivative of position).

Theorem 2.2. ***Newton's Second Law.***
The motion of a particle of mass m subject to a force F is governed by:

$$F = ma = m\frac{d^2x}{dt^2} = m\frac{dv}{dt}$$

where v is velocity and x is position.

定理

When the force depends only on velocity (e.g., air resistance), we can solve for velocity as a function of time. Often, however, we wish to find velocity as a function of *position*. We employ the chain rule to transform the derivative:

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

This substitution reduces the second-order equation in time to a first-order separable equation in space.

Example 2.7. Velocity-Dependent Force. Consider a particle of mass m subject to a drag force $F = -\mu v^3$. We wish to find the velocity

v as a function of position x, given an initial velocity v_0 at $x = 0$. Using $F = ma$:

$$-\mu v^3 = mv\frac{dv}{dx}$$

Assuming $v \neq 0$, we cancel one factor of v:

$$-\mu v^2 = m\frac{dv}{dx}$$

Separating variables:

$$-\frac{\mu}{m}\,dx = v^{-2}\,dv$$

Integrating both sides:

$$-\frac{\mu}{m}x + C = -v^{-1}$$

$$\frac{1}{v} = \frac{\mu}{m}x + C$$

At $x = 0$, $v = v_0$, so $C = 1/v_0$.

$$\frac{1}{v} = \frac{\mu x}{m} + \frac{1}{v_0} = \frac{\mu v_0 x + m}{m v_0}$$

Inverting gives the explicit solution:

$$v(x) = \frac{m v_0}{\mu v_0 x + m}$$

範例

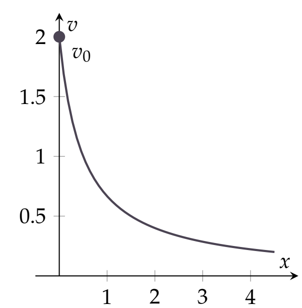

Figure 2.5: Velocity decay $v(x) = \frac{mv_0}{\mu v_0 x + m}$ under cubic drag.

2.5 *Modelling with First-Order Equations*

Having developed the method of separation of variables, we now apply it to model dynamic systems. We examine three archetypal classes of problems: natural growth and decay, the logistic constraints of populations, and the geometric problem of orthogonal trajectories.

Exponential Growth and Decay

The simplest dynamic model assumes that the rate of change of a quantity $y(t)$ is proportional to its current size. This assumption governs phenomena ranging from radioactive decay to compound interest and unrestricted biological reproduction.

Definition 2.4. ***The Law of Natural Growth.***
A quantity $y(t)$ obeys the law of natural growth (or decay) if it satis-

fies the linear differential equation:

$$\frac{dy}{dt} = ky$$

where k is a constant. If $k > 0$, it is a *growth constant*; if $k < 0$, it is a *decay constant*.

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The solution is immediate via separation of variables. Assuming $y \neq 0$:

$$\int \frac{1}{y}\,dy = \int k\,dt \implies \ln|y| = kt + C$$

Exponentiating yields $y(t) = y_0 e^{kt}$, where $y_0 = y(0)$ is the initial quantity.

Parameters of the Model

For decay models ($k < 0$), it is customary to characterize the substance by its *half-life* τ, the time required for the quantity to reduce to half its initial value.

$$\frac{1}{2}y_0 = y_0 e^{k\tau} \implies k\tau = \ln(1/2) = -\ln 2 \implies \tau = -\frac{\ln 2}{k}$$

Conversely, for growth models, the *doubling time* is $\tau_{double} = \frac{\ln 2}{k}$.

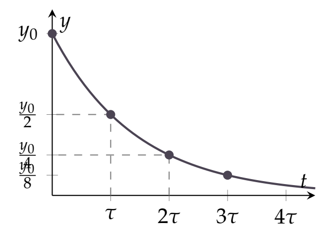

Figure 2.6: Exponential decay showing successive half-lives.

Example 2.8. Radioactive Decay. Let $m(t)$ denote the mass of a radioactive isotope. The decay is governed by $m'(t) = km(t)$. Suppose a sample of "Balonium" has a half-life of 1 year. We wish to determine the remaining percentage after 0.1 years. First, we determine k:

$$k = -\frac{\ln 2}{1} \approx -0.693$$

The mass function is $m(t) = m_0 e^{-0.693t}$. At $t = 0.1$:

$$\frac{m(0.1)}{m_0} = e^{-0.0693} \approx 0.933$$

Thus, approximately 93.3% of the substance remains.

範例

Remark.

Model Limitations: The exponential model for population growth $P(t) = P_0 e^{kt}$ implies that $P \to \infty$ as $t \to \infty$. In a finite universe, this is physically impossible. Exponential models are thus valid only over short time intervals where resources are effectively infinite. For long-term predictions, we require a model that accounts for environmental constraints.

The Logistic Equation

To address the limitations of exponential growth, we introduce the *Logistic Model*. This model assumes that the per-capita growth rate $\frac{1}{P}\frac{dP}{dt}$ is not constant, but decreases linearly as the population approaches a limiting value K, known as the *carrying capacity*.

> **Definition 2.5.** ***Logistic Differential Equation.***
> The logistic equation is given by:
>
> $$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$$
>
> where $k > 0$ is the intrinsic growth rate and $K > 0$ is the carrying capacity.
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Qualitative Analysis

Before solving the equation analytically, we can deduce the global behaviour of solutions by examining the phase line (the sign of P').

- **Equilibria:** $P' = 0$ when $P = 0$ or $P = K$.
- **Growth:** If $0 < P < K$, then $P' > 0$, so the population increases towards K.
- **Decay:** If $P > K$, then $P' < 0$, so the population decreases towards K.

To determine the concavity of the solution curves, we differentiate the ODE with respect to t using the Chain Rule:

$$\begin{aligned} \frac{d^2P}{dt^2} &= \frac{d}{dt}\left[kP - \frac{k}{K}P^2\right] = k\frac{dP}{dt} - \frac{2k}{K}P\frac{dP}{dt} \\ &= k\frac{dP}{dt}\left(1 - \frac{2P}{K}\right) \end{aligned}$$

Since $P' > 0$ for $0 < P < K$, the sign of the second derivative is determined by the term $(1 - 2P/K)$.

- If $0 < P < K/2$, then $P'' > 0$ (concave up). The growth accelerates.
- If $K/2 < P < K$, then $P'' < 0$ (concave down). The growth decelerates.
- The point $P = K/2$ is an *inflection point* where the growth rate is maximal.

Analytic Solution

We solve the equation by separating variables:

$$\int \frac{K}{P(K-P)}\,dP = \int k\,dt$$

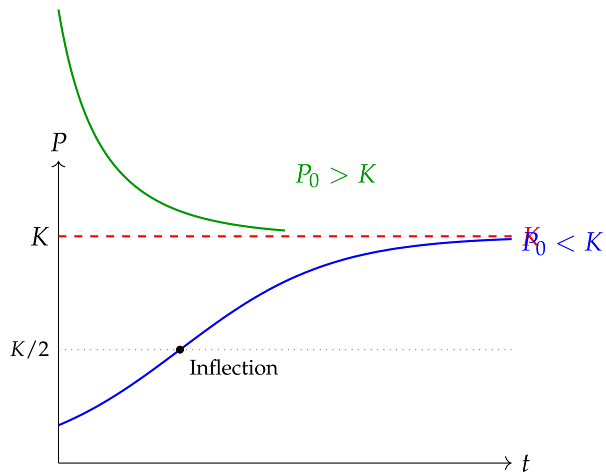

Figure 2.7: Solutions to the Logistic Equation. Trajectories approach the carrying capacity K asymptotically. The growth is fastest at $P = K/2$.

The integrand on the left admits a partial fraction decomposition:

$$\frac{K}{P(K-P)} = \frac{1}{P} + \frac{1}{K-P}$$

Integrating term by term:

$$\ln|P| - \ln|K-P| = kt + C$$
$$\ln\left|\frac{P}{K-P}\right| = kt + C$$

Exponentiating and solving for P (assuming $0 < P < K$ for the moment):

$$\frac{P}{K-P} = Ae^{kt} \implies P(t) = \frac{K}{1+Ae^{-kt}}$$

where A is a constant determined by the initial population P_0. Specifically, at $t = 0$, $A = (K - P_0)/P_0$.

Example 2.9. Population Prediction. Suppose a population obeys logistic growth with carrying capacity $K = 1000$ (in millions). If $P(1990) = 250$ and $P(2000) = 275$, we predict the population in 2100. Let $t = 0$ correspond to 1990. Thus $P_0 = 250$.

$$A = \frac{1000-250}{250} = 3$$

The solution is $P(t) = \frac{1000}{1+3e^{-kt}}$. We use the data point at $t = 10$ (year 2000) to find k:

$$275 = \frac{1000}{1+3e^{-10k}} \implies 1+3e^{-10k} = \frac{1000}{275} \approx 3.636$$

$$3e^{-10k} \approx 2.636 \implies e^{-10k} \approx 0.8788 \implies k \approx -\frac{\ln(0.8788)}{10} \approx 0.0129$$

For the year 2100, $t = 110$:

$$P(110) = \frac{1000}{1 + 3e^{-0.0129 \times 110}} \approx \frac{1000}{1 + 3(0.242)} \approx 579 \text{ million}$$

範例

Orthogonal Trajectories

In geometric optics and electrostatics, one often encounters two families of curves that intersect at right angles. For instance, equipotential lines are orthogonal to electric field lines.

Definition 2.6. ***Orthogonal Trajectories.***
Given a family of curves defined by the differential equation $y' = f(x, y)$, the *orthogonal trajectories* are the solution curves to the differential equation:

$$\frac{dy}{dx} = -\frac{1}{f(x, y)}$$

Geometrically, the product of the slopes of orthogonal curves is -1.

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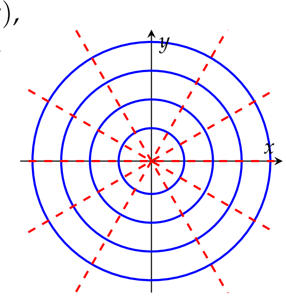

Figure 2.8: Circles $x^2 + y^2 = R^2$ (blue) and their orthogonal trajectories $y = mx$ (dashed red).

Example 2.10. Circles and Lines. Consider the family of circles centred at the origin, $x^2 + y^2 = R^2$. We wish to find their orthogonal trajectories. Differentiating the equation of the circles implicitly with respect to x:

$$2x + 2y\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{x}{y}$$

This gives the slope of the tangent to the circle at (x, y). The slope of the orthogonal trajectory must be the negative reciprocal:

$$\left(\frac{dy}{dx}\right)_{\text{orth}} = -\frac{1}{-x/y} = \frac{y}{x}$$

We solve this new separable differential equation:

$$\frac{dy}{dx} = \frac{y}{x} \implies \int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$$

$$\ln|y| = \ln|x| + C \implies y = mx$$

Thus, the orthogonal trajectories to the family of concentric circles are lines passing through the origin, consistent with geometric intuition.

範例

Example 2.11. Hyperbolas. Find the orthogonal trajectories to the family of hyperbolas $xy = C$. Differentiating $xy = C$ yields $y + xy' = 0$, so $y' = -y/x$. The differential equation for the orthogonal trajectories is:

$$\frac{dy}{dx} = -\frac{1}{-y/x} = \frac{x}{y}$$

Separating variables:

$$\int y\,dy = \int x\,dx \implies \frac{1}{2}y^2 = \frac{1}{2}x^2 + K$$

Rearranging gives $y^2 - x^2 = 2K$. This represents a family of hyperbolas rotated by 45° relative to the original family.

範例

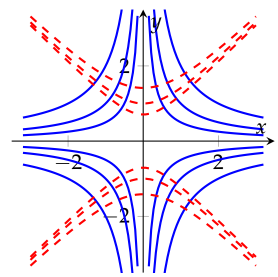

Figure 2.9: Orthogonal trajectories: $xy = c$ (blue) and $y^2 - x^2 = k$ (dashed red).

Mixing Problems

A classic application of first-order linear equations involves the mixing of fluids in a tank. The governing physical principle is the conservation of mass (or amount of substance).

Proposition 2.1. ***The Mixing Rate Law.***
Let $Y(t)$ be the amount of a substance in a tank at time t. The rate of change of Y is given by:

$$\frac{dY}{dt} = \text{Rate In} - \text{Rate Out}$$

where each rate is calculated as (Flow Rate) × (Concentration).

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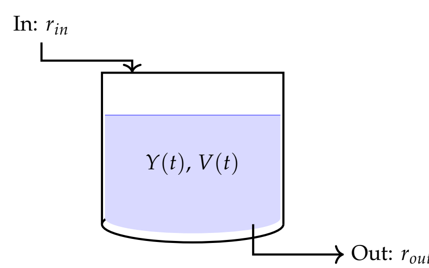

Figure 2.10: Mixing tank: fluid enters at rate r_{in}, exits at rate r_{out}.

Example 2.12. Saline Tank. Consider a tank containing 1000 L of water in which 15 kg of salt is dissolved. Pure water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate of 10 L/min. We determine the amount of salt $Y(t)$ remaining after 20 minutes.

1. **Rate In:** Since pure water enters, the concentration of salt is 0.

$$\text{Rate In} = 10 \text{ L/min} \times 0 \text{ kg/L} = 0 \text{ kg/min}$$

2. **Rate Out:** The volume of fluid in the tank remains constant at $V = 1000$ L (since flow in equals flow out). The concentration at time t is $Y(t)/1000$ kg/L.

$$\text{Rate Out} = 10 \text{ L/min} \times \frac{Y(t)}{1000} \text{ kg/L} = \frac{Y(t)}{100} \text{ kg/min}$$

The differential equation is:

$$\frac{dY}{dt} = 0 - \frac{Y}{100} = -\frac{1}{100}Y$$

This is a standard exponential decay equation. With initial condition $Y(0) = 15$:

$$Y(t) = 15e^{-t/100}$$

At $t = 20$ minutes:

$$Y(20) = 15e^{-20/100} = 15e^{-0.2} \approx 12.28 \text{ kg}$$

範例

2.6 *The Integrating Factor Method*

We have seen that the method of separation of variables applies only to a restricted class of differential equations. We now turn our attention to the general first-order *linear* differential equation. Linearity allows us to develop a systematic algorithm for constructing the general solution, relying on a clever transformation that reduces the differential equation to a standard integration problem.

Definition 2.7. ***Standard Form.***
A first-order linear ordinary differential equation is an equation that can be written in the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where P and Q are continuous functions on a given interval.

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Derivation of the Method

Consider the equation in standard form. Our goal is to transform the left-hand side into the derivative of a product. Recall the product rule for differentiation:

$$\frac{d}{dx}[\mu(x)y] = \mu(x)\frac{dy}{dx} + \frac{d\mu}{dx}y$$

Comparing this with our equation $\frac{dy}{dx} + P(x)y = Q(x)$, we see that if we multiply the entire equation by a non-zero function $\mu(x)$, we obtain:

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)Q(x)$$

For the left-hand side to be the exact derivative $\frac{d}{dx}[\mu y]$, the second term must satisfy:

$$\frac{d\mu}{dx} = \mu(x)P(x)$$

This is a separable differential equation for the auxiliary function $\mu(x)$, which we call the *integrating factor*. Separating variables:

$$\frac{1}{\mu}\,d\mu = P(x)\,dx$$

Integrating yields $\ln|\mu| = \int P(x)\,dx$. Exponentiating, we choose the simplest particular solution (setting the integration constant to zero):

$$\mu(x) = \exp\left(\int P(x)\,dx\right)$$

With this choice of $\mu(x)$, the differential equation becomes:

$$\frac{d}{dx}[\mu(x)y] = \mu(x)Q(x)$$

Integrating both sides with respect to x (by the Fundamental Theorem of Calculus):

$$\mu(x)y = \int \mu(x)Q(x)\,dx + C$$

Solving for y gives the general solution.

Theorem 2.3. ***General Solution of Linear First-Order ODEs.***
The general solution to the equation $y' + P(x)y = Q(x)$ is given by:

$$y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)Q(x)\,dx + C\right]$$

where $\mu(x) = e^{\int P(x)\,dx}$.

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Note

While the formula exists, it is pedagogically superior to perform the steps (multiply, collapse derivative, integrate) for each specific problem. This provides a natural check: if the left-hand side does not collapse into a perfect derivative, an error has occurred in calculating $\mu(x)$.

Examples

We illustrate the robustness of this method with several examples, including cases where variable substitution is required.

Example 2.13. Polynomial Coefficients. Consider the equation:

$$x\frac{dy}{dx} + 2y = 4x^3, \quad x > 0$$

First, we normalize the equation to standard form by dividing by x:

$$\frac{dy}{dx} + \frac{2}{x}y = 4x^2$$

Here $P(x) = 2/x$. We calculate the integrating factor:

$$\mu(x) = \exp\left(\int \frac{2}{x}\,dx\right) = \exp(2\ln x) = \exp(\ln x^2) = x^2$$

Multiplying the standard form by $\mu(x) = x^2$:

$$x^2\frac{dy}{dx} + 2xy = 4x^4$$

We observe that the LHS is indeed $\frac{d}{dx}[x^2y]$. Thus:

$$\frac{d}{dx}[x^2y] = 4x^4$$

Integrating both sides:

$$x^2y = \int 4x^4\,dx = \frac{4}{5}x^5 + C$$

Solving for y:

$$y(x) = \frac{4}{5}x^3 + \frac{C}{x^2}$$

範例

Example 2.14. Transcendental Functions. Solve the initial value problem:

$$xy' + (1+x)y = e^{-x}\sin(2x), \quad y(\pi) = 1$$

Divide by x to standardise (assuming $x \neq 0$):

$$y' + \left(\frac{1+x}{x}\right)y = \frac{e^{-x}\sin(2x)}{x}$$
$$y' + \left(\frac{1}{x}+1\right)y = \frac{e^{-x}\sin(2x)}{x}$$

The integrating factor is:

$$\mu(x) = \exp\left(\int \left(\frac{1}{x}+1\right)dx\right) = \exp(\ln|x| + x) = |x|e^x$$

Restricting to $x > 0$ for the initial condition at π, we use $\mu(x) = xe^x$. Multiplying the standardised equation by xe^x:

$$xe^xy' + xe^x\left(\frac{1}{x}+1\right)y = xe^x\frac{e^{-x}\sin(2x)}{x}$$

Simplifying the coefficients:

$$xe^x y' + (e^x + xe^x)y = \sin(2x)$$

Recognising the reverse product rule $(xe^x)' = e^x + xe^x$:

$$\frac{d}{dx}[xe^x y] = \sin(2x)$$

Integrate:

$$xe^x y = \int \sin(2x)\,dx = -\frac{1}{2}\cos(2x) + C$$

Applying the initial condition $y(\pi) = 1$:

$$\pi e^\pi (1) = -\frac{1}{2}\cos(2\pi) + C \implies \pi e^\pi = -\frac{1}{2} + C \implies C = \pi e^\pi + \frac{1}{2}$$

The specific solution is:

$$y(x) = \frac{\pi e^\pi + \frac{1}{2} - \frac{1}{2}\cos(2x)}{xe^x}$$

範例

Example 2.15. Swapping Variables. Occasionally, a non-linear equation in $y(x)$ becomes linear if we regard x as the dependent variable and y as independent. Consider:

$$y\,dx + (2xy - e^{-2y})\,dy = 0$$

Rearranging terms:

$$y\frac{dx}{dy} + 2xy = e^{-2y}$$

Dividing by y (standard form for $x(y)$):

$$\frac{dx}{dy} + 2x = \frac{e^{-2y}}{y}$$

Here the independent variable is y. The integrating factor is $\mu(y) = e^{\int 2\,dy} = e^{2y}$. Multiplying the equation by e^{2y}:

$$e^{2y}\frac{dx}{dy} + 2e^{2y}x = e^{2y}\frac{e^{-2y}}{y} = \frac{1}{y}$$

Recognising the derivative:

$$\frac{d}{dy}[xe^{2y}] = \frac{1}{y}$$

Integrating with respect to y:

$$xe^{2y} = \ln|y| + C$$

$$x(y) = e^{-2y}(\ln|y| + C)$$

範例

Existence and Uniqueness

The explicit construction of the solution leads to a fundamental result in the theory of linear ODEs. Unlike non-linear equations (e.g., $y' = y^{2/3}$), which may fail to have unique solutions at certain points, linear equations behave predictably.

> **Theorem 2.4.** ***Existence and Uniqueness for Linear ODEs.***
> If $P(x)$ and $Q(x)$ are continuous on an interval (a, b) containing x_0, then for any $y_0 \in \mathbb{R}$, there exists a unique solution $y(x)$ to the initial value problem
> $$y' + P(x)y = Q(x), \quad y(x_0) = y_0$$
> defined on the entire interval (a, b).

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> *Proof*
> The formula derived in *theorem* 2.3 provides an explicit candidate for the solution. Since P and Q are continuous, $\mu(x)$ is continuously differentiable and non-zero. The integral of μQ exists by the Fundamental Theorem of Calculus. Thus, existence is guaranteed. Uniqueness follows from the fact that each step in the derivation (multiplication by non-zero μ, integration) is reversible.
>
> ■

2.7 *Second-Order Linear Homogeneous Equations*

We conclude this chapter by examining a specific yet ubiquitous class of differential equations: second-order linear homogeneous equations with constant coefficients. These equations govern the dynamics of mechanical vibrations, electrical circuits, and quantum mechanical wavefunctions.

> **Definition 2.8.** ***Constant Coefficient Homogeneous Equation.***
> A second-order linear homogeneous differential equation with constant coefficients is of the form:
> $$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$
> where $a, b, c \in \mathbb{R}$ are constants and $a \neq 0$.

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The Characteristic Equation

To find the general solution, we exploit the property that the derivative of an exponential function is a multiple of itself. We propose a trial solution of the form $y = e^{\lambda x}$, where λ is a constant to be determined. Substituting derivatives $y' = \lambda e^{\lambda x}$ and $y'' = \lambda^2 e^{\lambda x}$ into the differential equation yields:

$$a(\lambda^2 e^{\lambda x}) + b(\lambda e^{\lambda x}) + c(e^{\lambda x}) = 0$$

Factoring out the non-zero term $e^{\lambda x}$:

$$e^{\lambda x}(a\lambda^2 + b\lambda + c) = 0$$

Since $e^{\lambda x} \neq 0$, the characteristic parameter λ must satisfy the algebraic equation:

$$a\lambda^2 + b\lambda + c = 0$$

This quadratic equation is called the *characteristic equation* (or auxiliary equation). Its roots determine the nature of the solution.

Classification of Solutions

The roots of the characteristic equation are given by the quadratic formula:

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We distinguish three cases based on the discriminant $\Delta = b^2 - 4ac$.

Case I: Distinct Real Roots ($\Delta > 0$)

If $\Delta > 0$, there are two distinct real roots λ_1 and λ_2. These generate two fundamental solutions $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$. Since $\lambda_1 \neq \lambda_2$, these functions are linearly independent (their Wronskian is non-zero). By the principle of superposition for linear equations, the general solution is:

$$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

> **Example 2.16.** Real Roots. Solve $y'' - 3y' + 2y = 0$. The characteristic equation is $\lambda^2 - 3\lambda + 2 = 0$, which factors as $(\lambda - 1)(\lambda - 2) = 0$. The roots are $\lambda_1 = 1$ and $\lambda_2 = 2$. The general solution is $y(x) = c_1 e^x + c_2 e^{2x}$.
>
> 範例

Case II: Repeated Real Roots ($\Delta = 0$)

If $\Delta = 0$, there is a single repeated root $\lambda = -b/2a$. This yields only one exponential solution $y_1 = e^{\lambda x}$. To form the general solution, we require a second linearly independent solution. We verify that $y_2 = xe^{\lambda x}$ is a solution. Differentiating y_2:

$$y_2' = e^{\lambda x} + \lambda x e^{\lambda x}$$
$$y_2'' = \lambda e^{\lambda x} + \lambda e^{\lambda x} + \lambda^2 x e^{\lambda x} = 2\lambda e^{\lambda x} + \lambda^2 x e^{\lambda x}$$

Substituting into $ay'' + by' + cy = 0$ (and noting $b = -2a\lambda$ and $c = a\lambda^2$):

$$a(2\lambda + \lambda^2 x)e^{\lambda x} - 2a\lambda(1 + \lambda x)e^{\lambda x} + a\lambda^2(x)e^{\lambda x} = 0$$

Terms involving x sum to $a\lambda^2 - 2a\lambda^2 + a\lambda^2 = 0$. Constant terms sum to $2a\lambda - 2a\lambda = 0$. Thus, y_2 is a solution. The general solution is:

$$y(x) = c_1 e^{\lambda x} + c_2 x e^{\lambda x} = e^{\lambda x}(c_1 + c_2 x)$$

> **Example 2.17.** Critical Damping. Solve $y'' + 4y' + 4y = 0$. Characteristic equation: $\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0$. Root: $\lambda = -2$ (repeated). General solution: $y(x) = c_1 e^{-2x} + c_2 x e^{-2x}$.
>
> 範例

Case III: Complex Conjugate Roots ($\Delta < 0$)

If $\Delta < 0$, the roots are complex conjugates $\lambda = \alpha \pm i\beta$, where $\alpha = -b/2a$ and $\beta = \sqrt{4ac - b^2}/2a$. The formal solution is $y = K_1 e^{(\alpha+i\beta)x} + K_2 e^{(\alpha-i\beta)x}$. Using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, we can extract real-valued solutions.

$$e^{(\alpha+i\beta)x} = e^{\alpha x}(\cos\beta x + i\sin\beta x)$$
$$e^{(\alpha-i\beta)x} = e^{\alpha x}(\cos\beta x - i\sin\beta x)$$

By taking linear combinations, we isolate the real and imaginary parts:

$$y_1 = e^{\alpha x}\cos\beta x, \quad y_2 = e^{\alpha x}\sin\beta x$$

The general real-valued solution is:

$$y(x) = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$$

> **Example 2.18.** Simple Harmonic Motion. Solve $y'' + \omega^2 y = 0$. Characteristic equation: $\lambda^2 + \omega^2 = 0 \implies \lambda = \pm i\omega$. Here $\alpha = 0$ and $\beta = \omega$. General solution: $y(t) = c_1\cos\omega t + c_2\sin\omega t$. This describes undamped oscillations with angular frequency ω.
>
> 範例

> **Example 2.19.** Damped Oscillations. Solve the initial value problem $y'' + 2y' + 5y = 0, \quad y(0) = 1, y'(0) = 3$. Characteristic equation: $\lambda^2 + 2\lambda + 5 = 0$. Roots: $\lambda = \frac{-2\pm\sqrt{4-20}}{2} = -1 \pm 2i$. Thus $\alpha = -1, \beta = 2$. General solution: $y(x) = e^{-x}(c_1\cos 2x + c_2\sin 2x)$. Applying initial conditions:
>
> $$y(0) = 1(c_1 \cdot 1 + c_2 \cdot 0) = c_1 \implies c_1 = 1$$

Differentiating $y(x)$:

$$y'(x) = -e^{-x}(\cos 2x + c_2 \sin 2x) + e^{-x}(-2\sin 2x + 2c_2 \cos 2x)$$

$$y'(0) = -1(1) + 1(2c_2) = -1 + 2c_2$$

Setting $y'(0) = 3$:

$$-1 + 2c_2 = 3 \implies 2c_2 = 4 \implies c_2 = 2$$

Solution: $y(x) = e^{-x}(\cos 2x + 2\sin 2x)$.

範例

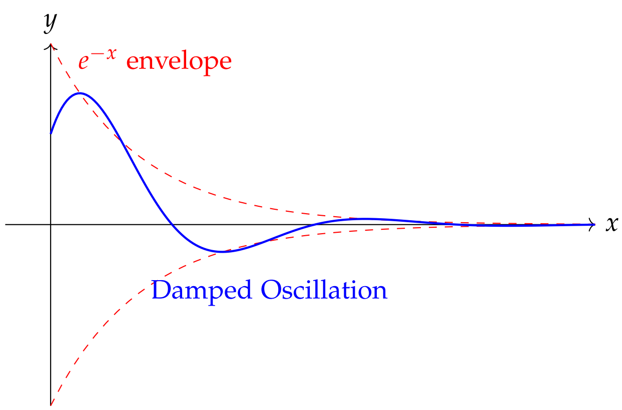

Figure 2.11: Solution to a damped harmonic oscillator. The amplitude decays exponentially while the frequency remains constant.

2.8 *Summary of Methods*

Equation Type	**General Solution**
Separable $y' = g(x)h(y)$	$\int \frac{dy}{h(y)} = \int g(x)dx + C$
Linear $y' + P(x)y = Q(x)$	$y = \frac{1}{\mu(x)}[\int \mu(x)Q(x)dx + C], \quad \mu = e^{\int P dx}$
$ay'' + by' + cy = 0$ $(\Delta > 0)$	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
$ay'' + by' + cy = 0$ $(\Delta = 0)$	$y = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$
$ay'' + by' + cy = 0$ $(\Delta < 0)$	$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$

Table 2.1: Classification of solutions for standard differential equations.

This concludes our introductory study of differential equations. The techniques established here form the bedrock for analyzing more complex systems, including forced oscillations ($ay'' + by' + cy = F(x)$) and systems of coupled equations, which will be explored in future analysis courses.

2.9 Exercises

1. **Classification and Verification.** Classify the following differential equations by order, linearity, and homogeneity. Verify the indicated solution.
   (a) $y'' + y = \tan x; \quad y(x) = -\cos x \ln(\sec x + \tan x)$.
   (b) $x^2 y'' - xy' + y = 0; \quad y(x) = x \ln x$ (for $x > 0$).
   (c) $y' = \frac{y}{x} + \tan\left(\frac{y}{x}\right); \quad \sin(y/x) = Cx$.

2. **Separation of Variables.** Find the general solution to the following equations. Express the solution explicitly where possible.
   (a) $(1 + e^x) yy' = e^x$.
   (b) $\frac{dy}{dx} = \frac{x^2+xy+y^2}{x^2}$.

   > *Remark.*
   >
   > Hint: This is a homogeneous polar equation. Substitute $y = vx$.

   (c) $e^y \sin(2x) dx + \cos x (e^{2y} - y) dy = 0$.

3. **Linear First-Order and Integrating Factors.** Solve the following initial value problems.
   (a) $\frac{dy}{dx} - \frac{y}{x} = xe^x, \quad y(1) = e - 1$.
   (b) $(x^2 + 1)\frac{dy}{dx} + 3xy = 6x, \quad y(0) = 2$.
   (c) $y' + y \cos x = \frac{1}{2}\sin(2x), \quad y(0) = 1$.

4. **Second-Order Constants.** Determine the general solution for the following second-order equations.
   (a) $y'' - 4y' + 13y = 0$.
   (b) $y'' - 6y' + 9y = 0$.
   (c) $y'' + (\omega^2 - \epsilon)y = 0$, where $\epsilon \ll \omega^2$. Use the approximation $\sqrt{1-x} \approx 1 - x/2$ to describe the behaviour of the frequency.

5. **Population Dynamics.** A population $P(t)$ obeys the logistic equation with carrying capacity K and intrinsic rate r.
   (a) Prove that the rate of population growth is maximised when $P = K/2$.
   (b) If the population starts at $P_0 = K/3$, how long does it take to reach $2K/3$? Express your answer in terms of r.

6. **Bernoulli's Equation.** A differential equation of the form

$$y' + P(x)y = Q(x)y^n, \quad n \neq 0, 1$$

   is non-linear but can be reduced to a linear form.
   (a) Make the substitution $u = y^{1-n}$. Show that the equation

transforms into the linear equation:

$$\frac{1}{1-n}u' + P(x)u = Q(x).$$

(b) Use this method to solve the logistic equation $P' = kP - \frac{k}{K}P^2$ purely as a Bernoulli equation (identifying $n = 2$).

(c) Solve $xy' + y = x^2y^2$ with $y(1) = 1$.

7. **Reduction of Order.** Suppose we know one non-trivial solution $y_1(x)$ to the second-order linear homogeneous equation:

$$y'' + P(x)y' + Q(x)y = 0.$$

(a) Let $y(x) = v(x)y_1(x)$. Differentiate and substitute this into the ODE to show that $v(x)$ satisfies the separable equation:

$$y_1v'' + (2y_1' + Py_1)v' = 0.$$

(b) By solving for v', derive the formula for the second linearly independent solution:

$$y_2(x) = y_1(x) \int \frac{\exp\left(-\int P(x)dx\right)}{y_1(x)^2}\, dx.$$

(c) Use this method to find the general solution to $x^2y'' + xy' - y = 0$ given that $y_1 = x$ is a solution.

8. **The Riccati Equation.** The non-linear equation $y' = q_0(x) + q_1(x)y + q_2(x)y^2$ is known as a Riccati equation.

(a) Show that the substitution $y = -\frac{u'}{q_2u}$ transforms the Riccati equation into the second-order linear equation:

$$u'' - \left(q_1 + \frac{q_2'}{q_2}\right)u' + q_2q_0u = 0.$$

(b) Solve the equation $y' = 1 + x^2 + y^2$ is generally difficult, but solve the simpler case $y' = 1 + y^2$ using this substitution method and verify it matches the tangent solution.

9. **Orthogonal Trajectories in Polar Coordinates.**

(a) If a curve is given in polar coordinates by $r = r(\theta)$, show that the angle ψ between the tangent line and the radial vector satisfies $\tan\psi = r\frac{d\theta}{dr}$.

(b) Deduce that the differential equation for the orthogonal trajectories to the family $F(r, \theta, c) = 0$ is found by replacing $\frac{dr}{d\theta}$ with $-r^2\frac{d\theta}{dr}$.

(c) Find the orthogonal trajectories to the family of cardioids $r = a(1 + \cos\theta)$.

10. **The Catenary Problem.** A flexible cable of uniform density hangs between two poles. Let $y(x)$ denote the shape of the cable. Balancing horizontal and vertical tensions leads to the differential equation:

$$\frac{d^2y}{dx^2} = k\sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

where k is a constant depending on density and tension.

(a) Let $p = y'$. Solve the resulting first-order separable equation for $p(x)$.

(b) Integrate $p(x)$ to show that the cable hangs in the shape of a hyperbolic cosine (catenary): $y(x) = \frac{1}{k}\cosh(kx + C_1) + C_2$.

11. **Variable Mass Systems (The Rocket Equation).** A rocket of mass $m(t)$ moves with velocity $v(t)$. It expels fuel at a constant speed u relative to the rocket. By conserving momentum over a time interval Δt, the equation of motion (ignoring gravity) is:

$$m\frac{dv}{dt} = -u\frac{dm}{dt}.$$

(a) Solve this differential equation to derive Tsiolkovsky's rocket equation:

$$v(t) = v_0 + u\ln\left(\frac{m_0}{m(t)}\right).$$

(b) Now include gravity g. The equation becomes $m\frac{dv}{dt} = -u\frac{dm}{dt} - mg$. Solve for $v(t)$ assuming the burn rate $\frac{dm}{dt} = -\alpha$ is constant.

12. **Cauchy-Euler Equations.** An equation of the form $ax^2y'' + bxy' + cy = 0$ is a Cauchy-Euler equation.

(a) Use the substitution $x = e^t$ to transform the equation into a constant coefficient equation in terms of the independent variable t.

> *Remark.*
> Show that $x\frac{dy}{dx} = \frac{dy}{dt}$ and $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$.

(b) Solve $x^2y'' - 2xy' + 2y = 0$ using this method.

(c) Generalise the method to solve $x^2y'' - 3xy' + 4y = 0$ (Repeated roots case).

13. **Pursuit Curves.** A rabbit runs up the y-axis with constant speed v. A dog starts at $(L, 0)$ and chases the rabbit with speed kv, always running directly towards the rabbit's current position.

(a) Let (x, y) be the dog's position. Explain why the line of sight condition implies $\frac{dy}{dx} = \frac{y - vt}{x}$, where t is time.

(b) Use the chain rule $t = \int ds/(kv)$ and the arc length formula to derive the second-order equation:

$$x\frac{d^2y}{dx^2} = k\sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

(c) Using the substitution $p = y'$, solve the equation. Show that if $k = 1$, the dog never catches the rabbit.

14. Torricelli's Law and Uniqueness. Water drains from a tank through a hole in the bottom. The depth $h(t)$ satisfies $h' = -k\sqrt{h}$.

(a) Solve the equation given $h(0) = H$. At what time T does the tank empty?

(b) Consider the solution $h(t) \equiv 0$ for all t. Show that at the point where the tank empties, the solution is not unique by splicing the non-zero solution with the zero solution.

(c) Relate this failure of uniqueness to the condition on $\frac{\partial f}{\partial h}$ in the Existence and Uniqueness Theorem near $h = 0$.

15. Integral Equations. Consider the Volterra integral equation:

$$y(x) = 3 + \int_0^x (t - x)y(t)\,dt.$$

(a) Differentiate the equation with respect to x twice to convert it into a second-order initial value problem. (Use Leibniz's Integral Rule).

(b) Solve the resulting ODE to find the function $y(x)$.