Introduction to Vector Calculus

Gudfit

Contents

0

Line Integrals

We develop the theory of integration along curves in $\mathbb{R}^n$. This generalisation, known as the line integral, is fundamental to vector calculus, physics, and complex analysis.

We distinguish between two types of line integrals:

Scalar Line Integrals (Type I): Integration of a scalar field with respect to arc length. This measures cumulative quantities like the mass of a wire.

Vector Line Integrals (Type II): Integration of a vector field along a directed curve. This measures quantities like work done by a force.

We begin with the scalar line integral.

Curves and Rectifiability

Before defining the integral, we must formalise the notion of a curve and its length.

> **Definition 0.1.** ***Simple Curve.***
> A continuous mapping $\gamma : [a, b] \to \mathbb{R}^n$ is called a **curve**. The image set $\Gamma = \gamma([a, b])$ is the geometric locus of the curve.
> · The curve is **simple** (or Jordan) if γ is injective on (a, b).
> · The curve is **closed** if $\gamma(a) = \gamma(b)$.
> · The curve is **smooth** if γ is continuously differentiable and $\gamma'(t) \neq 0$ for all $t \in [a, b]$.
>
> 定義

To define the length of a curve, we approximate it using polygonal chains. Let $P = \{t_0, t_1, \ldots, t_k\}$ be a partition of $[a, b]$ such that $a = t_0 < t_1 < \cdots < t_k = b$. Let $A_i = \gamma(t_i)$. The polygonal chain connecting $A_0, \ldots, A_k$ has length:

$$L(\Gamma, P) = \sum_{i=1}^{k} |A_i - A_{i-1}| = \sum_{i=1}^{k} |\gamma(t_i) - \gamma(t_{i-1})|.$$

Definition 0.2. ***Rectifiable Curve.***
A curve Γ is **rectifiable** if the set of lengths of all inscribed polygonal chains is bounded. The **arc length** $l(\Gamma)$ is defined as the supremum of these lengths:

$$l(\Gamma) = \sup_P \{L(\Gamma, P)\}.$$

定義

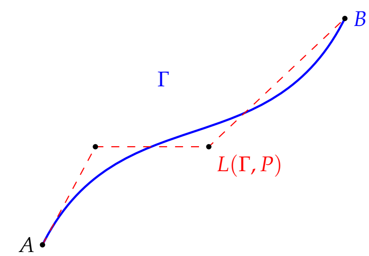


Figure 1: Approximation of a curve Γ by an inscribed polygonal chain. As the partition is refined, the polygon length approaches the arc length.

0.1 *The First Type of Line Integral*

Let Γ be a simple rectifiable curve in $\mathbb{R}^n$ with endpoints A and B. Let $f : \Gamma \to \mathbb{R}$ be a bounded function defined on the curve.
Consider a partition P that divides Γ into k arcs with lengths $\Delta s_1, \ldots, \Delta s_k$. Let $\lambda(P) = \max_i \Delta s_i$ be the mesh of the partition. Choose arbitrary sample points ξ_i on the i-th arc.

Definition 0.3. ***Scalar Line Integral.***
The **line integral of the first type** (or integral with respect to arc length) of f along Γ is defined as:

$$\int_\Gamma f(\mathbf{x})\, ds = \lim_{\lambda(P)\to 0} \sum_{i=1}^{k} f(\xi_i)\Delta s_i,$$

provided this limit exists and is independent of the choice of partitions and sample points.

定義

Note

Unlike the Riemann integral on an interval $[a, b]$ where dx represents a signed length, the differential ds represents the scalar arc length. Consequently, the first type line integral is **independent of orientation**. If Γ^- denotes the curve traversed in the opposite direction:

$$\int_\Gamma f(\mathbf{x})\, ds = \int_{\Gamma^-} f(\mathbf{x})\, ds.$$

Proposition 0.1. ***Orientation Independence.***
For any simple rectifiable curve Γ, the scalar line integral satisfies

$$\int_\Gamma f\, ds = \int_{\Gamma^-} f\, ds.$$

命題

Proof

Parametrise Γ by $\gamma : [a, b] \to \mathbb{R}^n$ and its reverse by $\tilde{\gamma}(t) = \gamma(a + b -$

$t)$. Then $|\tilde{\gamma}'(t)| = |\gamma'(a+b-t)|$, and

$$\int_{\Gamma^-} f\,ds = \int_a^b f(\tilde{\gamma}(t))\,|\tilde{\gamma}'(t)|\,dt = \int_a^b f(\gamma(a+b-t))\,|\gamma'(a+b-t)|\,dt.$$

The change of variables $u = a + b - t$ shows the two integrals coincide.

For a merely rectifiable curve one can approximate Γ uniformly by smooth parametrisations (or inscribed polygonal chains); the line integral is the uniform limit of the smooth cases, so the equality holds without assuming differentiability.

■

Proposition 0.2. ***Additivity Over Subarcs.***
If $\Gamma = \Gamma_1 \cup \Gamma_2$ with the common point their only intersection and a consistent orientation along the chain, then

$$\int_{\Gamma} f\,ds = \int_{\Gamma_1} f\,ds + \int_{\Gamma_2} f\,ds.$$

命题

Proof

Take a C^1 parametrisation of Γ that runs first along Γ_1 then Γ_2. The evaluation formula converts the line integral into the sum of the ordinary integrals over the two parameter intervals. Alternately, in the Riemann-sum definition choose partitions that respect the junction point; the sum splits accordingly, and limits add.

■

Proposition 0.3. ***Uniform Bound by Length.***
If $|f| \le M$ on a rectifiable curve Γ, then

$$\left|\int_{\Gamma} f\,ds\right| \le M\,l(\Gamma).$$

命题

Proof

For any partition P, $|\sum f(\xi_i)\Delta s_i| \le \sum |f(\xi_i)|\Delta s_i \le M \sum \Delta s_i = M\,L(\Gamma, P)$. Taking the supremum over partitions and then the limit $\lambda(P) \to 0$ yields the claim.

■

Evaluation of Line Integrals

The definition involves limits of sums, which are cumbersome for calculation. We reduce the evaluation to a standard Riemann integral via parametrisation.

Theorem 0.1. ***Evaluation Formula.***
Let Γ be a smooth curve parametrized by $\gamma : [a,b] \to \mathbb{R}^n$, where γ is continuously differentiable. If f is continuous on Γ, then:

$$\int_\Gamma f(\mathbf{x})\,ds = \int_a^b f(\gamma(t))|\gamma'(t)|\,dt.$$

In $\mathbb{R}^3$, with $\gamma(t) = (x(t), y(t), z(t))$, this becomes:

$$\int_\Gamma f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt.$$

定理

Proof

The arc length function from the starting point is given by $s(t) = \int_a^t |\gamma'(\tau)|\,d\tau$. By the Fundamental Theorem of Calculus, $ds/dt = |\gamma'(t)|$, or formally $ds = |\gamma'(t)|dt$. Substituting the change of variables from arc length s to parameter t in the integral definition yields the result.

■

Corollary 0.1. *Existence for Continuous f.* If Γ is smooth and f is continuous on Γ, the scalar line integral $\int_\Gamma f\,ds$ exists.

推論

Proof

The composition $f \circ \gamma$ is continuous on the compact interval $[a,b]$, hence Riemann integrable. The evaluation formula expresses $\int_\Gamma f\,ds$ as that Riemann integral, so the limit exists.

■

Proposition 0.4. ***Reparametrisation Invariance.***
Let $\gamma : [a,b] \to \Gamma$ be a smooth parametrisation with $|\gamma'| > 0$, and let $\phi : [c,d] \to [a,b]$ be a C^1 bijection with $\phi'(t) > 0$. Then

$$\int_\Gamma f\,ds = \int_c^d f(\gamma(\phi(t)))\,|\gamma'(\phi(t))|\,\phi'(t)\,dt.$$

If ϕ' is everywhere negative (orientation reversal), the right-hand side is unchanged because of the absolute value.

命題

Proof

Substitute $u = \phi(t)$ in the evaluation formula. For $\phi'(t) < 0$, the integral limits swap but the absolute value on γ' removes the sign, leaving the same value.

■

Example 0.1. Line Integral on a Circle. Compute $I = \oint_C x^2\,ds$, where C is the circle defined by the intersection of the sphere $x^2 + y^2 + z^2 = R^2$ and the plane $x + y + z = 0$.

範例

Solution

This problem admits two approaches: a standard parametrisation and a symmetry argument.

Parametrisation. The intersection lies on a plane passing through the origin, so C is a great circle of radius R centred at the origin (hence its length is $2\pi R$, used again in the symmetry argument). To parametrise, we construct an orthonormal basis for the plane $x + y + z = 0$. The normal is $\mathbf{n} = (1, 1, 1)$. We choose two orthogonal unit vectors in the plane. Let $\mathbf{u} = \frac{1}{\sqrt{2}}(1, -1, 0)$ and $\mathbf{v} = \mathbf{n} \times \mathbf{u} / \|\mathbf{n} \times \mathbf{u}\|$. Computing the cross product:

$$(1, 1, 1) \times (1, -1, 0) = (1, 1, -2).$$

Normalising yields $\mathbf{v} = \frac{1}{\sqrt{6}}(1, 1, -2)$. The curve can be parametrised as $\gamma(t) = R(\cos t)\mathbf{u} + R(\sin t)\mathbf{v}$ for $t \in [0, 2\pi]$.

$$x(t) = \frac{R}{\sqrt{2}} \cos t + \frac{R}{\sqrt{6}} \sin t.$$

Since C is a circle of radius R, $|\gamma'(t)| = R$. Thus $ds = R\,dt$.

$$\begin{aligned} I &= \int_0^{2\pi} \left(\frac{R}{\sqrt{2}} \cos t + \frac{R}{\sqrt{6}} \sin t \right)^2 R\,dt \\ &= R^3 \int_0^{2\pi} \left(\frac{1}{2} \cos^2 t + \frac{1}{6} \sin^2 t + \frac{1}{\sqrt{3}} \sin t \cos t \right) dt. \end{aligned}$$

Using $\int_0^{2\pi} \cos^2 t\,dt = \int_0^{2\pi} \sin^2 t\,dt = \pi$ and $\int_0^{2\pi} \sin t \cos t\,dt = 0$:

$$I = R^3 \left(\frac{1}{2}\pi + \frac{1}{6}\pi \right) = \frac{2}{3}\pi R^3.$$

Symmetry. This method is far more elegant. By the symmetry of the sphere $x^2 + y^2 + z^2 = R^2$ and the plane $x + y + z = 0$ with respect to permuting variables, the integrals of x^2, y^2, and z^2 along C must be equal:

$$\oint_C x^2\,ds = \oint_C y^2\,ds = \oint_C z^2\,ds.$$

Summing them:

$$3I = \oint_C (x^2 + y^2 + z^2)\,ds.$$

On the curve C, $x^2 + y^2 + z^2 = R^2$ (a constant). Thus:

$$3I = \oint_C R^2\, ds = R^2 \oint_C ds = R^2 \cdot l(C).$$

Since C is a great circle of radius R, its length is $l(C) = 2\pi R$.

$$3I = R^2(2\pi R) \implies I = \frac{2}{3}\pi R^3.$$

■

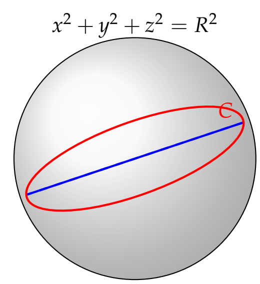


Figure 2: The intersection curve C. The symmetry argument exploits the fact that x, y, z play identical roles in the definitions of the sphere and the plane.

Applications of Type I Line Integrals

The first type line integral allows us to calculate geometric and physical properties of curved objects.

Mass and Centroids

Consider a wire represented by a curve Γ. If the wire has a linear mass density $\rho(x, y, z)$ at point (x, y, z), the total mass M is:

$$M = \int_\Gamma \rho(x, y, z)\, ds.$$

The coordinates of the **centroid** (or centre of mass) $(\bar{x}, \bar{y}, \bar{z})$ are given by the first moments normalised by the mass:

$$\bar{x} = \frac{1}{M}\int_\Gamma x\rho\, ds, \quad \bar{y} = \frac{1}{M}\int_\Gamma y\rho\, ds, \quad \bar{z} = \frac{1}{M}\int_\Gamma z\rho\, ds.$$

If the density is uniform ($\rho \equiv 1$), this yields the geometric centroid.

Example 0.2. Centroid of a Spherical Arc. Find the centroid of the curve Γ forming the boundary of the sphere octant $x^2 + y^2 + z^2 = a^2, x \geq 0, y \geq 0, z \geq 0$.

範例

Solution

The boundary Γ consists of three circular arcs:

- Γ_1 in the xy-plane ($z = 0$): quarter circle from $(a, 0, 0)$ to $(0, a, 0)$.
- Γ_2 in the yz-plane ($x = 0$): quarter circle from $(0, a, 0)$ to $(0, 0, a)$.
- Γ_3 in the zx-plane ($y = 0$): quarter circle from $(0, 0, a)$ to $(a, 0, 0)$.

The total length is $L = 3 \times (\frac{1}{4} \cdot 2\pi a) = \frac{3\pi a}{2}$. Due to symmetry, $\bar{x} = \bar{y} = \bar{z}$. We compute $\bar{x} = \frac{1}{L}\int_\Gamma x\, ds$.

$$\int_\Gamma x\, ds = \int_{\Gamma_1} x\, ds + \int_{\Gamma_2} x\, ds + \int_{\Gamma_3} x\, ds.$$

1. On Γ_2, $x = 0$, so $\int_{\Gamma_2} x\, ds = 0$.

2. On Γ_1, use polar coordinates: $x = a\cos\theta, y = a\sin\theta, ds = a\,d\theta$ for $\theta \in [0, \pi/2]$.

$$\int_{\Gamma_1} x\,ds = \int_0^{\pi/2} (a\cos\theta)a\,d\theta = a^2[\sin\theta]_0^{\pi/2} = a^2.$$

3. On Γ_3, similarly $x = a\cos\phi, z = a\sin\phi$.

$$\int_{\Gamma_3} x\,ds = a^2.$$

Thus, $\int_\Gamma x\,ds = 2a^2$.

$$\bar{x} = \frac{2a^2}{3\pi a/2} = \frac{4a}{3\pi}.$$

The centroid is $\left(\frac{4a}{3\pi}, \frac{4a}{3\pi}, \frac{4a}{3\pi}\right)$.

■

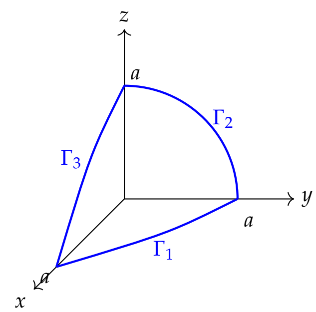


Figure 3: The boundary of the first octant of the sphere. The curve Γ is the union of three quarter-circles.

Change of Variables in Arc Length

Occasionally, algebraic curves require clever coordinate transformations to evaluate arc length.

Example 0.3. Curve Rectification. Find the length of the arc of the curve defined by $(x-y)^2 = a(x+y)$ and $x^2 - y^2 = \frac{9}{8}z^2$ from the origin to a point $A(x_0, y_0, z_0)$.

範例

Solution

This system is difficult to parametrise directly in Cartesian coordinates. We perform a change of variables to simplify the equations. Let $u = x - y$ and $v = x + y$. Note that $x^2 - y^2 = uv$. The equations become:

$$u^2 = av, \quad uv = \frac{9}{8}z^2.$$

Substitute $v = u^2/a$ into the second equation:

$$u(u^2/a) = \frac{9}{8}z^2 \implies u^3 = \frac{9a}{8}z^2 \implies u = \frac{1}{2}(9a)^{1/3}z^{2/3}.$$

Consequently, $v = \frac{u^2}{a} = \frac{1}{a}\frac{(9a)^{2/3}}{4}z^{4/3}$. We can express x and y in terms of z:

$$x = \frac{v+u}{2}, \quad y = \frac{v-u}{2}.$$

The differential arc length is $ds^2 = dx^2 + dy^2 + dz^2$. Note that $dx^2 + dy^2 = \frac{1}{2}(du^2 + dv^2)$, and

$$\frac{du}{dz} = \frac{1}{3}(9a)^{1/3}z^{-1/3}, \qquad \frac{dv}{dz} = \frac{(9a)^{2/3}}{3a}z^{1/3}.$$

Hence

$$\left(\frac{ds}{dz}\right)^2 = 1+\frac{1}{2}\left[\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2\right] = 1 + Az^{-2/3} + Bz^{2/3}, \quad A = \frac{(9a)^{2/3}}{18},\ B = \frac{(9a)^{4/3}}{18a^2}.$$

Since $AB = \frac{1}{4}$, the quadratic in $z^{\pm 1/3}$ is a perfect square:

$$1 + Az^{-2/3} + Bz^{2/3} = \left(pz^{-1/3} + qz^{1/3}\right)^2, \quad pq = \tfrac{1}{2},\ p^2 = A,\ q^2 = B.$$

Therefore $\dfrac{ds}{dz} = pz^{-1/3} + qz^{1/3}$ and

$$s(z) = \int_0^z \left(pt^{-1/3} + qt^{1/3}\right) dt = \frac{3p}{2} z^{2/3} + \frac{3q}{4} z^{4/3}.$$

Substituting $u = \frac{1}{2}(9a)^{1/3} z^{2/3}$ into $x = \frac{1}{2}(u + u^2/a)$ shows that the bracket equals $2x/\sqrt{2}$, giving

$$s(z) = \sqrt{2}\, x.$$

Thus the arc length from the origin to $x = x_0$ on the branch where x increases from 0 is

$$l = \sqrt{2}\, x_0.$$

■

0.2 *Vector Line Integrals*

The physical motivation for the vector line integral is the calculation of work done by a force field on a moving particle. Consequently, the direction of motion is significant. We consider simple rectifiable curves equipped with an orientation, referred to as **directed curves**.

Definition 0.4. ***Vector Line Integral.***
Let $\Gamma = \overline{AB}$ be a simple rectifiable directed curve in $\mathbb{R}^3$. Let

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

be a vector field defined on Γ. For any partition T of Γ given by $A = A_0, A_1, \ldots, A_m = B$ consistent with the orientation, let $\Delta\mathbf{r}_i = (\Delta x_i, \Delta y_i, \Delta z_i)$ be the displacement vector from A_{i-1} to A_i. Let $d(T)$ be the maximum arc length of the segments. If the limit

$$\lim_{d(T)\to 0} \sum_{i=1}^{m} \mathbf{F}(\xi_i, \eta_i, \zeta_i) \cdot \Delta\mathbf{r}_i$$

exists and is independent of the partition and the choice of sample points (ξ_i, η_i, ζ_i), it is called the **vector line integral** (or line integral of the second type) of $\mathbf{F}$ along Γ. It is denoted by:

$$\int_\Gamma \mathbf{F} \cdot d\mathbf{r} = \int_\Gamma P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz.$$

定義

This integral is also referred to as the integral with respect to coordinates.

Proposition 0.5. ***Orientation Change.***
If Γ^- denotes Γ with reversed direction, then

$$\int_{\Gamma^-} \mathbf{F} \cdot d\mathbf{r} = -\int_\Gamma \mathbf{F} \cdot d\mathbf{r}.$$

命題

Proof

Parametrise Γ by $\gamma : [a,b] \to \mathbb{R}^3$; then Γ^- is $\tilde{\gamma}(t) = \gamma(a+b-t)$ with $\tilde{\gamma}' = -\gamma'(a+b-t)$. The evaluation formula yields the sign change.

■

Proposition 0.6. ***Additivity Over Directed Subarcs.***
If a directed curve Γ is decomposed as the concatenation of directed subarcs Γ_1, Γ_2 with matching orientations, then

$$\int_\Gamma \mathbf{F} \cdot d\mathbf{r} = \int_{\Gamma_1} \mathbf{F} \cdot d\mathbf{r} + \int_{\Gamma_2} \mathbf{F} \cdot d\mathbf{r}.$$

命題

Proof

Use a parametrisation that runs along Γ_1 then Γ_2; the evaluation formula breaks the integral into the sum over the two parameter intervals.

■

Theorem 0.2. ***Evaluation Formula.***
Let Γ be a piecewise smooth directed curve with parametric representation

$$x = x(t), \quad y = y(t), \quad z = z(t), \quad a \le t \le b,$$

where the parameter t increasing from a to b corresponds to the orientation of Γ. If P, Q, R are continuous on Γ, then:

$$\int_\Gamma P\,dx + Q\,dy + R\,dz = \int_a^b \left[P(x(t),y(t),z(t))x'(t) + Q(\dots)y'(t) + R(\dots)z'(t)\right]\,dt.$$

定理

Corollary 0.2. *Reparametrisation.* If $\phi : [c, d] \to [a, b]$ is a C^1 bijection with $\phi'(t) > 0$, then for the reparametrised curve $\tilde{\gamma}(t) = \gamma(\phi(t))$,

$$\int_{\tilde{\gamma}} \mathbf{F} \cdot d\mathbf{r} = \int_{\Gamma} \mathbf{F} \cdot d\mathbf{r}.$$

If $\phi' < 0$, the value changes sign (orientation reversal).

推論

Proof

Substitute $u = \phi(t)$ in the evaluation formula. A negative ϕ' reverses the limits, introducing the sign flip.

■

Proposition 0.7. ***Gradient Fundamental Theorem.***
If $\mathbf{F} = \nabla\phi$ with $\phi \in C^1$ on an open set containing Γ, then for endpoints A, B of Γ,

$$\int_{\Gamma} \mathbf{F} \cdot d\mathbf{r} = \phi(B) - \phi(A),$$

so the integral depends only on the endpoints.

命題

Proof

Parametrise Γ by $\gamma(t)$, $t \in [a, b]$. The evaluation formula gives

$$\int_a^b \nabla\phi(\gamma(t)) \cdot \gamma'(t)\, dt = \int_a^b \frac{d}{dt}\left[\phi(\gamma(t))\right] dt = \phi(\gamma(b)) - \phi(\gamma(a)).$$

■

Example 0.4. Line Integral on an Ellipse. Compute

$$I = \int_C (x^2 + 2xy)\, dy,$$

where C is the upper half of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ traversed counterclockwise.

範例

Solution

We use the standard parametric equations for the ellipse:

$$x = a\cos t, \quad y = b\sin t.$$

For the upper half traversed counterclockwise, t varies from 0 to π.

Substituting $dy = b\cos t\,dt$:

$$\begin{aligned} I &= \int_0^{\pi} (a^2\cos^2 t + 2ab\cos t\sin t)\, b\cos t\,dt \\ &= a^2 b\int_0^{\pi} \cos^3 t\,dt + 2ab^2\int_0^{\pi} \cos^2 t\sin t\,dt. \end{aligned}$$

The first integral vanishes (odd symmetry of $\cos^3 t$ about $\pi/2$ or direct evaluation). The second integral is evaluated by substitution $u = \cos t$:

$$I = 0 + 2ab^2\left[-\frac{\cos^3 t}{3}\right]_0^{\pi} = \frac{4}{3}ab^2.$$

■

Example 0.5. Viviani's Window. Find

$$I = \int_{\Gamma} y^2\,dx + z^2\,dy + x^2\,dz,$$

where Γ is the curve defined by the intersection of the sphere $x^2 + y^2 + z^2 = a^2$ and the cylinder $x^2 + y^2 = ax$ $(a > 0)$ in the region $z \geq 0$. The curve is oriented counterclockwise when viewed from the positive x-axis.

範例

Solution

We parametrise the curve using cylindrical coordinates. From $x^2 + y^2 = ax$, we have $r = a\cos\theta$. Thus:

$$x = a\cos^2\theta, \quad y = a\cos\theta\sin\theta, \quad z = \sqrt{a^2 - r^2} = a|\sin\theta|.$$

For the loop $z \geq 0$, take two smooth pieces: $\theta \in \left[-\frac{\pi}{2}, 0\right]$ with $z = -a\sin\theta$ and $\theta \in \left[0, \frac{\pi}{2}\right]$ with $z = a\sin\theta$. Viewed from the $+x$-axis, counterclockwise traversal corresponds to θ increasing from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$; splitting the interval removes the cusp at $\theta = 0$ while keeping that orientation intact. On each piece the integral is

$$I = \sum_{\text{pieces}} \int \left[y^2 x'(\theta) + z^2 y'(\theta) + x^2 z'(\theta)\right] d\theta.$$

Substituting the functions:

- $x'(\theta) = -2a\cos\theta\sin\theta$. The first term involves $y^2 x' \propto (\cos^2\sin^2)(\cos\sin)$. This is an odd function of θ.
- $z(\theta) = a|\sin\theta|$ is even. Thus $z'(\theta)$ is odd. The third term $x^2 z'(\theta) \propto (\cos^4)(\text{odd})$ is an odd function.
- $y'(\theta) = a(\cos^2\theta - \sin^2\theta)$ is even. The second term $z^2 y'(\theta)$ is the product of even functions, hence even.

The integrals of the odd terms vanish. We remain with:

$$I = \int_{-\pi/2}^{\pi/2} a^2 \sin^2\theta \cdot a(\cos^2\theta - \sin^2\theta)\,d\theta = a^3 \int_{-\pi/2}^{\pi/2} \sin^2\theta(\cos^2\theta - \sin^2\theta)\,d\theta.$$

Using symmetry on $[-\pi/2, \pi/2]$ and the identity $\cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$:

$$I = 2a^3 \int_0^{\pi/2} (\sin^2\theta - 2\sin^4\theta)\,d\theta.$$

Using Wallis' integrals $\int_0^{\pi/2} \sin^2\theta\,d\theta = \frac{\pi}{4}$ and $\int_0^{\pi/2} \sin^4\theta\,d\theta = \frac{3\pi}{16}$:

$$I = 2a^3\left(\frac{\pi}{4} - 2\cdot\frac{3\pi}{16}\right) = 2a^3\left(\frac{\pi}{4} - \frac{3\pi}{8}\right) = -\frac{\pi}{4}a^3.$$

■

If a space curve lies on a surface $z = f(x, y)$, we can project the integral onto the plane.

Proposition 0.8. ***Reduction to Plane Integral.***
Suppose a piecewise smooth curve Γ lies on a smooth surface $z = f(x, y)$, and its projection onto the xy-plane is γ. If $P(x, y, z)$ is continuous on Γ, then:

$$\oint_\Gamma P(x, y, z)\,dx = \oint_\gamma P(x, y, f(x, y))\,dx.$$

命题

Proof

Let γ be parametrized by $x = \varphi(t), y = \psi(t)$ for $t \in [a, b]$. Then Γ is given by $x = \varphi(t), y = \psi(t), z = f(\varphi(t), \psi(t))$. Substituting into the definition of the line integral:

$$\oint_\Gamma P(x, y, z)\,dx = \int_a^b P(\varphi(t), \psi(t), f(\varphi(t), \psi(t)))\varphi'(t)\,dt.$$

The right-hand side is precisely $\oint_\gamma P(x, y, f(x, y))\,dx$.

■

Relationship Between the Two Types of Line Integrals

The coordinate differential vector $d\mathbf{r}$ is related to the arc length differential ds by the unit tangent vector $\boldsymbol{\tau}$:

$$d\mathbf{r} = \boldsymbol{\tau}\,ds.$$

Thus, the vector line integral can be expressed as a scalar line integral of the tangential component of the field:

$$\int_\Gamma \mathbf{F}\cdot d\mathbf{r} = \int_\Gamma (\mathbf{F}\cdot\boldsymbol{\tau})\,ds.$$

If $\cos\alpha, \cos\beta, \cos\gamma$ are the direction cosines of $\boldsymbol{\tau}$, then:

$$\int_\Gamma P\,dx + Q\,dy + R\,dz = \int_\Gamma (P\cos\alpha + Q\cos\beta + R\cos\gamma)\,ds.$$

For a plane curve Γ, let $\mathbf{n}$ be the unit normal vector such that the angle from $\mathbf{n}$ to $\boldsymbol{\tau}$ is $\pi/2$. The direction cosines of $\mathbf{n}$ and $\boldsymbol{\tau}$ are related by a rotation: If the tangent makes an angle θ with the positive x-axis, then $d\mathbf{r} = (\cos\theta, \sin\theta)\,ds$ and

$$\int_\Gamma P\,dx + Q\,dy = \int_\Gamma \big[-P\sin\theta + Q\cos\theta\big]\,ds.$$

Example 0.6. Inverse Square Field. Consider a planar force field pointing towards the origin with magnitude inversely proportional to the square of the distance r:

$$\mathbf{F} = -\frac{\mu}{r^2}\hat{\mathbf{r}}.$$

Calculate the work done by this field on a particle of mass $m = 1$ moving from point A to B (assume the path avoids the origin so $r > 0$).

範例

Solution

The force components are $F_x = -\mu\frac{x}{r^3}$ and $F_y = -\mu\frac{y}{r^3}$. The work is given by:

$$W = \int_{AB} -\mu\frac{x\,dx + y\,dy}{r^3}.$$

Observe that $x\,dx + y\,dy = \frac{1}{2}d(x^2+y^2) = r\,dr$. Thus:

$$-\frac{x\,dx + y\,dy}{r^3} = -\frac{r\,dr}{r^3} = -\frac{dr}{r^2} = d\left(\frac{1}{r}\right).$$

Alternatively, using parametrisation $x = \varphi(t), y = \psi(t)$, this relationship holds.

$$W = \mu\int_A^B d\left(\frac{1}{r}\right) = \mu\left(\frac{1}{r_B} - \frac{1}{r_A}\right).$$

■

Definition 0.5. ***Gradient Curve.***
Let f be a C^1 function on a domain $D \subset \mathbb{R}^3$ with $\nabla f \neq \mathbf{0}$. A curve Γ is a **gradient curve** of f if the tangent direction at every point coincides with the direction of ∇f. If Γ is parametrized by $\mathbf{r}(t)$, it satisfies the system:

$$\frac{d\mathbf{r}}{dt} = \frac{\nabla f}{|\nabla f|}.$$

Along such a curve, $ds = dt$.

定義

We use this concept to solve a classical problem from the Putnam Competition.

Example 0.7. Bound on Gradient Magnitude. Let $f(x, y)$ be continuously differentiable on the unit disk $D = \{x^2 + y^2 \leq 1\}$ and satisfy $|f(x, y)| \leq 1$. Prove that there exists a point $(x_0, y_0) \in \text{int}D$ such that

$$|\nabla f(x_0, y_0)| \leq 2.$$

i.e., $(f_x^2 + f_y^2)_{(x_0, y_0)} \leq 4$.

範例

Proof

If ∇f vanishes anywhere, the inequality holds trivially. Assume instead that $|\nabla f| > 2$ everywhere on D. Consider the gradient curve Γ starting at the origin, defined by $\mathbf{r}'(t) = \nabla f(\mathbf{r}(t))/|\nabla f(\mathbf{r}(t))|$; the vector field is continuous and nonzero on the compact disk, so this flow exists (and is unique) until it hits the boundary. Along Γ,

$$\frac{d}{dt} f(\mathbf{r}(t)) = |\nabla f(\mathbf{r}(t))| > 2,$$

so f increases strictly and $f(0, 0) < 1$ (an interior point with $f = 1$ would force $\nabla f = 0$ by Fermat's lemma). Because $\nabla f \neq 0$ inside D, f cannot attain its maximum value 1 in the interior, hence Γ must reach ∂D exactly when f hits 1. The time (and length) to do so satisfies

$$1 - f(0, 0) = \int_0^T \frac{d}{dt} f(\mathbf{r}(t))\, dt > 2T \quad \Rightarrow \quad T < \frac{1 - f(0, 0)}{2} \leq 1.$$

But any path from the origin to ∂D has length at least 1 (the Euclidean distance), so Γ cannot reach the boundary in length $T < 1$ — a contradiction. Therefore $|\nabla f|$ cannot exceed 2 everywhere, and some point $(x_0, y_0) \in \text{int}\, D$ must satisfy $|\nabla f(x_0, y_0)| \leq 2$.

■

Remark.

This method can be generalised to prove a mean value theorem in n-dimensions. For a ball of radius r, there exists a point $\mathbf{p}_0$ such that the oscillation of the function is related to the gradient by:

$$\max f - \min f = |\nabla f(\mathbf{p}_0)| \cdot 2r.$$

Using this sharper mean-value estimate one can improve the constant 2 in the previous example to 1; a short proof follows from applying the identity with $r = 1$ to f on the unit disk (details omitted here for brevity).

0.3 Exercises

1. **Calculating Scalar Integrals.** Compute the following line integrals of the first type:
   (a) $\int_C (x^{4/3} + y^{4/3})\, ds$, where C is the astroid $x^{2/3} + y^{2/3} = a^{2/3}$.
   (b) $\int_C e^{\sqrt{x^2+y^2}}\, ds$, where C is the boundary of the circular sector consisting of the two radial segments $\varphi = 0$, $\varphi = \frac{\pi}{4}$ and the circular arc $r = a$ joining them (orientation of C is arbitrary).
   (c) $\int_C |y|\, ds$, where C is the lemniscate $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.
   (d) $\int_C \frac{ds}{y^2}$, where C is the catenary $y = a \cosh \frac{x}{a}$.
   (e) $\int_C z\, ds$, where C is the space curve defined by the intersection of $x^2 + y^2 = z^2$ and $y^2 = ax$ from $(0, 0, 0)$ to $(a, a, \sqrt{2}a)$.
2. **Arc Length in Space.** Calculate the arc lengths of the following curves from the origin (or specified starting point) to a generic point (x_0, y_0, z_0) or (x, y, z):
   (a) The curve given by $y = a \arcsin \frac{x}{a}$ and $z = \frac{a}{4} \ln \frac{a-x}{a+x}$.
   (b) The intersection of $x^2 + y^2 + z^2 = a^2$ and $\sqrt{x^2 + y^2} \cosh\big(\arctan \frac{y}{x}\big) = a$, starting from $(a, 0, 0)$. (Note: cosh is intentional; since $\cosh u \geq 1$, the relation forces $x > 0$ so the angle $\arctan \frac{y}{x}$ is single-valued. If a different surface was intended, replace cosh with the desired function.)
3. **Centroid of a Catenary.** Find the coordinates of the centroid of the arc of the homogeneous catenary $y = a \cosh \frac{x}{a}$ between the points $(0, a)$ and (b, h).
4. **Intrinsic Definition.** Let $\Gamma = \overline{AB}$ be a simple rectifiable curve with length L. For $s \in [0, L]$, let $\mathbf{x}(s)$ be the unique point on Γ such that the arc length from A to $\mathbf{x}(s)$ is s. Prove that for any function f defined on Γ for which the line integral exists:

$$\int_\Gamma f(\mathbf{x})\, ds = \int_0^L f(\mathbf{x}(s))\, ds.$$

> *Remark.*
> This confirms that the scalar line integral is equivalent to a standard Riemann integral over the arc length parameter.

5. **Path Dependence.** Compute the integral

$$\int_L xy\,dx + (y-x)\,dy$$

where L is the directed path from $A(1,1)$ to $B(2,3)$ along:

(a) The straight line segment AB.

(b) The parabolic arc $y = 2(x-1)^2 + 1$.

(c) The broken line segment ADB, where $D = (2,1)$.

6. **Cycloid Integral.** Evaluate $\int_C \frac{x}{y}\,dx + \frac{1}{y-a}\,dy$, where C is the arc of the cycloid $x = a(t - \sin t), y = a(1-\cos t)$ for $t \in [\pi/6, \pi/3]$.

7. **Closed Loop Integration.** Calculate $\oint_C (x+y)^2\,dx + (x^2 - y^2)\,dy$, where C is the triangle with vertices $(1,1), (3,2), (3,1)$ traversed clockwise.

8. **Parabolic Work.** Find $\int_C 4xy^2\,dx - 3x^4\,dy$ along the parabola $y = \frac{1}{2}x^2$ from $(0,0)$ to $(2,2)$.

9. **Space Curve Intersection.** Compute

$$\oint_C (y^2+z^2)\,dx + (z^2+x^2)\,dy + (x^2+y^2)\,dz,$$

where C is the intersection of the sphere $x^2+y^2+z^2 = 2Rx$ and the cylinder $x^2+y^2 = 2ax$ $(0 < a < R, z > 0)$, oriented counterclockwise as viewed from the positive z-axis.

10. **Spherical Curves.** Evaluate $\int_C y\,dx + z\,dy + x\,dz$ along curves on the sphere of radius R:

$$x = R\sin\varphi\cos\theta, \quad y = R\sin\varphi\sin\theta, \quad z = R\cos\varphi.$$

Consider the cases:

(a) A latitude circle (R, φ constant, θ varies from 0 to 2π).

(b) A longitude semi-circle (R, θ constant, φ varies from 0 to π).

11. **Helix vs Line.** Calculate the integral

$$I = \int_C (x^2+5y+3yz)\,dx + (5x+3xy-2)\,dy + (3xy-4z)\,dz$$

along two different paths from $A(a,0,0)$ to $B(a,0,b)$:

(a) The helical segment $x = a\cos t, y = a\sin t, z = \frac{bt}{2\pi}$ for $t \in [0, 2\pi]$.

(b) The straight line segment connecting A and B.

12. Work Done by a Field. Given the force field

$$\mathbf{F} = y\mathbf{i} - x\mathbf{j} + (x + y + z)\mathbf{k},$$

find the work done moving a particle along one turn of the helix $x = a\cos t, y = a\sin t, z = \frac{b}{2\pi}t$ starting from $(a, 0, 0)$.

1

Green's Formula and Conservative Fields

We now establish the connection between double integrals over a planar region and line integrals along its boundary. This result, Green's Formula, provides powerful methods for evaluating integrals and leads to the conditions for path independence of line integrals.

1.1 Green's Formula

Let D be a bounded closed region in $\mathbb{R}^2$, whose boundary ∂D consists of smooth or piecewise smooth curves. Let $P(x, y)$ and $Q(x, y)$ be functions with continuous partial derivatives on D.

> **Theorem 1.1. *Green's Formula*.**
> If the boundary ∂D is traversed in the positive direction with respect to D (keeping the region on the left), then:
>
> $$\iint_D \left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} \right) dx\, dy = \oint_{\partial D} P\, dx + Q\, dy.$$

定理

From the relationship between the two types of line integrals on the plane, Green's Formula can be expressed using the unit outward normal vector $\mathbf{n}$. Recall that $dx = -\cos(y, \mathbf{n})\, ds$ and $dy = \cos(x, \mathbf{n})\, ds$. However, using the geometric identity $\angle(x, \mathbf{n}) = \angle(x, y) + \angle(y, \mathbf{n}) = \frac{\pi}{2} + \angle(y, \mathbf{n})$, the formula transforms as follows:

$$\iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dx\, dy = \oint_{\partial D} -Q\, dx + P\, dy$$
$$= \oint_{\partial D} [Q \sin(x, \mathbf{n}) + P \cos(x, \mathbf{n})]\, ds.$$

Using the directional cosines:

$$\iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dx\, dy = \oint_{\partial D} [P \cos(x, \mathbf{n}) + Q \cos(y, \mathbf{n})]\, ds.$$

These formulas allow us to convert line integrals into double integrals. This is particularly useful even when the curve C is not closed, by employing the method of adding "auxiliary lines" to form a closed loop.

Example 1.1. Auxiliary Lines. Let C be the arc of the parabola $2x = \pi y^2$ from $O(0,0)$ to $B(\frac{\pi}{2},1)$. Compute

$$I = \int_C (2xy^3 - y^2\cos x)\,dx + (1 - 2y\sin x + 3x^2y^2)\,dy.$$

範例

Solution

Let $P(x,y) = 2xy^3 - y^2\cos x$ and $Q(x,y) = 1 - 2y\sin x + 3x^2y^2$. Calculating the partial derivatives:

$$\frac{\partial Q}{\partial x} = -2y\cos x + 6xy^2, \quad \frac{\partial P}{\partial y} = 6xy^2 - 2y\cos x.$$

Thus, $-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} = 0$. To apply Green's Formula, we add auxiliary lines BA and AO to close the curve, where $A = (\frac{\pi}{2},0)$. Let D be the region enclosed by $C \cup BA \cup AO$.

$$I + \int_{BA} + \int_{AO} = \iint_D 0\,dx\,dy = 0.$$

Therefore, $I = -\left(\int_{BA} + \int_{AO}\right) = \int_{AB} + \int_{OA}$.

1. Along OA: $y = 0$, so $dy = 0$ and $P(x,0) = 0$. The integral is 0.

2. Along AB: $x = \frac{\pi}{2}$, so $dx = 0$. y ranges from 0 to 1.

$$\int_{AB} = \int_0^1 \left[1 - 2y\sin\frac{\pi}{2} + 3\left(\frac{\pi}{2}\right)^2 y^2\right] dy = \int_0^1 \left(1 - 2y + \frac{3\pi^2}{4}y^2\right) dy.$$

Evaluating this:

$$I = \left[y - y^2 + \frac{\pi^2}{4}y^3\right]_0^1 = 1 - 1 + \frac{\pi^2}{4} = \frac{\pi^2}{4}.$$

■

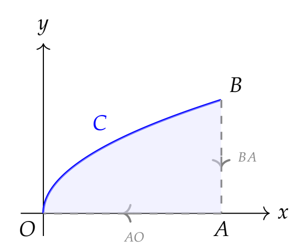


Figure 1.1: The path of integration involving the parabolic arc C (where $2x = \pi y^2$) and auxiliary lines BA and AO.

Example 1.2. Normal Derivative Integral. Compute the integral

$$I = \oint_C \frac{\cos(\mathbf{r},\mathbf{n})}{r}\,ds,$$

where C is a piecewise smooth simple closed curve, $\mathbf{r} = (x,y)$, $r = |\mathbf{r}|$, and $\mathbf{n}$ is the unit outward normal vector on C.

範例

Solution

Using the identity $\cos(\mathbf{r}, \mathbf{n}) = \frac{\mathbf{r}\cdot\mathbf{n}}{r} = \frac{1}{r}(x\cos(\mathbf{n}, x) + y\cos(\mathbf{n}, y))$, we write:

$$I = \oint_C \left(\frac{x}{r^2}\cos(\mathbf{n}, x) + \frac{y}{r^2}\cos(\mathbf{n}, y) \right) ds.$$

We consider the position of the origin relative to C:

Case 1: $(0, 0)$ ***is outside*** C**.** Using Green's Formula in the form involving directional cosines:

$$I = \iint_D \left[\frac{\partial}{\partial x}\left(\frac{x}{r^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^2}\right) \right] dx\, dy.$$

Computing the derivatives shows the integrand is identically zero. Thus $I = 0$.

Case 2: $(0, 0)$ ***is inside*** C**.** We excise the singularity by drawing a circle C_ε centred at the origin with small radius ε. Let D_ε be the region between C and C_ε.

$$I + \oint_{C_\varepsilon} \left(\frac{x}{r^2}\cos(\mathbf{n}, x) + \frac{y}{r^2}\cos(\mathbf{n}, y) \right) ds = \iint_{D_\varepsilon} 0 = 0.$$

In Green's Formula the inner boundary is taken with *clockwise* orientation; rewriting that integral with the usual counterclockwise orientation (equivalently flipping the normal) changes its sign. Hence:

$$I = \oint_{C_\varepsilon} \left(\frac{x}{r^2}\cos(\mathbf{n}, x) + \frac{y}{r^2}\cos(\mathbf{n}, y) \right) ds.$$

On C_ε, $r = \varepsilon$, $\cos(\mathbf{n}, x) = x/\varepsilon$, and $\cos(\mathbf{n}, y) = y/\varepsilon$.

$$I = \oint_{C_\varepsilon} \left(\frac{x^2}{\varepsilon^3} + \frac{y^2}{\varepsilon^3} \right) ds = \oint_{C_\varepsilon} \frac{\varepsilon^2}{\varepsilon^3}\, ds = \frac{1}{\varepsilon}(2\pi\varepsilon) = 2\pi.$$

Case 3: $(0, 0)$ ***lies on*** C**.** Draw tangents OA and OB to C at the origin. Let θ be the angle between them inside the region. We consider the limit as we excise the origin with a small arc C_ε.

$$I = \lim_{\varepsilon\to 0} \oint_{C_\varepsilon} \frac{1}{\varepsilon}\, ds = \lim_{\varepsilon\to 0} \theta_\varepsilon = \theta.$$

If C is smooth at the origin, $\theta = \pi$.

■

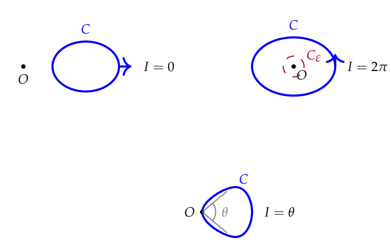


Figure 1.2: Three cases for $\oint_C \frac{\cos(\mathbf{r},\mathbf{n})}{r}\, ds$: origin outside ($I = 0$), inside ($I = 2\pi$), or on the curve ($I = \theta$).

Note

It is crucial to verify the differentiability of P, Q. Many errors arise from applying Green's Formula when the origin (a singularity) is inside the domain.

Example 1.3. Singularity Handling. Compute

$$I = \oint_C \frac{e^y}{x^2+y^2}\left[(x\sin x + y\cos x)dx + (y\sin x - x\cos x)dy\right],$$

where C is the circle $x^2+y^2=1$ traversed counterclockwise.

範例

Solution

Let P, Q be the components. We find that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ outside the origin. Since $(0,0)$ is inside C, we introduce a small circle $C_\epsilon : x^2 + y^2 = \epsilon^2$. By Green's Formula on the annulus, the outer boundary C is counter-clockwise while the inner boundary C_ϵ is clockwise:

$$\oint_C + \oint_{C_\epsilon}^{(\text{cw})} = 0, \qquad \text{so} \qquad \oint_C = -\oint_{C_\epsilon}^{(\text{cw})} = \oint_{C_\epsilon}^{(\text{ccw})}.$$

On C_ϵ:

$$I = \oint_{C_\epsilon} \frac{e^y}{\epsilon^2}[\dots] = \frac{1}{\epsilon^2}\oint_{C_\epsilon} e^y[(x\sin x + y\cos x)dx + (y\sin x - x\cos x)dy].$$

Now, let D_ϵ be the disk enclosed by C_ϵ. We apply Green's Formula *again* to this new integral (which is now over a loop enclosing a region D_ϵ where the integrand is defined everywhere, as we pulled $1/\epsilon^2$ out). The new integrand in the double integral is:

$$\frac{\partial}{\partial x}(e^y(y\sin x - x\cos x)) - \frac{\partial}{\partial y}(e^y(x\sin x + y\cos x)).$$

Simplifying yields $-2e^y\cos x$. Thus:

$$I = \frac{1}{\epsilon^2}\iint_{D_\epsilon} -2e^y\cos x\,dx\,dy.$$

By the Mean Value Theorem for integrals, there exists $(\xi,\eta) \in D_\epsilon$ such that the integral equals $\text{Area}(D_\epsilon)\times(-2e^\eta\cos\xi)$.

$$I = \frac{1}{\epsilon^2}(\pi\epsilon^2)(-2e^\eta\cos\xi) = -2\pi e^\eta\cos\xi.$$

Letting $\epsilon\to 0$, $(\xi,\eta)\to(0,0)$.

$$I = -2\pi e^0\cos 0 = -2\pi.$$

■

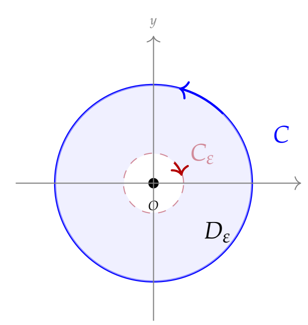


Figure 1.3: Excising the singularity at the origin: the annulus D_ε between the outer curve C (ccw) and inner circle C_ε (cw).

Corollary 1.1. *Area by line integral.* The area S of a region D bounded by a piecewise smooth simple curve C is

$$S = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C (x\,dy - y\,dx).$$

推論

Proof

Apply Green's Formula with $(P, Q) = (0, x)$:

$$\iint_D \left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right) dx\,dy = \iint_D 1\,dx\,dy = S = \oint_C x\,dy.$$

Similarly, with $(P, Q) = (-y, 0)$ we obtain $S = -\oint_C y\,dx$. Averaging the two equal expressions yields the symmetric form $\frac{1}{2}\oint_C (x\,dy - y\,dx)$.

■

Example 1.4. Area of a Lemniscate. Calculate the area enclosed by the lemniscate $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

範例

Solution

Method 1: Line Integral. By symmetry, we consider the first and fourth quadrants where $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$. Parametrising using polar coordinates: $x = r\cos\theta, y = r\sin\theta$. The equation becomes $r^2 = a^2\cos 2\theta$, so $r = a\sqrt{\cos 2\theta}$.

$$x(\theta) = a\cos\theta\sqrt{\cos 2\theta}, \quad y(\theta) = a\sin\theta\sqrt{\cos 2\theta}.$$

Calculating the differential form:

$$xy' - yx' = x\frac{dy}{d\theta} - y\frac{dx}{d\theta}.$$

Substituting the derivatives yields $xy' - yx' = a^2\cos 2\theta$.

$$S = 2 \times \frac{1}{2}\int_{-\pi/4}^{\pi/4} a^2\cos 2\theta\,d\theta = a^2[\frac{1}{2}\sin 2\theta]_{-\pi/4}^{\pi/4} = a^2.$$

(The prefactor 2 doubles the area of the one lobe covered by $\theta \in [-\pi/4, \pi/4]$.)

Method 2: Polar Area. Using the standard polar area formula:

$$S = 4 \times \frac{1}{2}\int_0^{\pi/4} r^2\,d\theta = 2\int_0^{\pi/4} a^2\cos 2\theta\,d\theta = a^2.$$

■

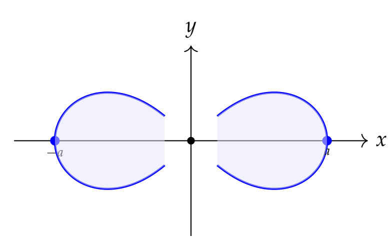


Figure 1.4: The lemniscate $(x^2 + y^2)^2 = a^2(x^2 - y^2)$. Total area $= a^2$.

1.2 Conditions for Path Independence

We investigate the conditions under which a line integral depends only on the endpoints, a property characterising conservative fields. Let Ω be a region in $\mathbb{R}^2$. If for any points $A, B \in \Omega$, the integral $\int_L P\,dx + Q\,dy$ yields the same value for all piecewise smooth paths $L \subset \Omega$ connecting A to B, the integral is called **path-independent**.

Theorem 1.2. ***Equivalence of Conditions.***
Let D be a **simply connected region** (any simple closed curve encloses a region entirely within D). Let P, Q have continuous partial derivatives. The following are equivalent:

1. For any closed curve $C \subset D$, $\oint_C P\,dx + Q\,dy = 0$.
2. The integral is path-independent.
3. $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ everywhere in D.
4. $P\,dx + Q\,dy$ is an **exact differential**; i.e., there exists $\varphi(x, y)$ such that $d\varphi = P\,dx + Q\,dy$. φ is called the potential function.

If these conditions hold, the integral can be evaluated as:

$$\int_{(x_0,y_0)}^{(x,y)} P\,dx + Q\,dy = \varphi(x, y) - \varphi(x_0, y_0).$$

定理

Proof
We show the cycle of implications.

(3)⇒(1) For any simple closed curve $C \subset D$, Green's Formula gives

$$\oint_C P\,dx + Q\,dy = \iint_{D_C} \left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right) dx\,dy = 0.$$

Here D_C is the region enclosed by C (possible since D is simply connected).

(1)⇒(2) If L_1, L_2 join A to B, then $L_1 \cup \overline{L_2}$ is a closed curve (traverse L_2 backwards). By (1), its integral is 0, so the integrals along L_1 and L_2 coincide; the integral is path-independent.

(2)⇒(4) Fix a base point (x_0, y_0) and define

$$\varphi(x, y) = \int_{(x_0,y_0)}^{(x,y)} P\,dx + Q\,dy,$$

which is well-defined by path independence. Differentiating along horizontal and vertical segments yields $\varphi_x = P$, $\varphi_y = Q$, so $d\varphi = P\,dx + Q\,dy$.

(4)⇒(3) From $\varphi_x = P$, $\varphi_y = Q$ and continuity of mixed partials,

$$\frac{\partial P}{\partial y} = \varphi_{xy} = \varphi_{yx} = \frac{\partial Q}{\partial x}.$$

This closes the equivalence.

■

Example 1.5. Proof of Condition Equivalence. Prove that if P, Q have continuous partial derivatives, the condition $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ is necessary and sufficient for the relation

$$\int_{(x_0,y_0)}^{(x,y)} P\,dx + Q\,dy = \int_{x_0}^{x} P(x, y_0)\,dx + \int_{y_0}^{y} Q(x, y)\,dy$$

to hold (implying path independence).

範例

Sufficiency.

Assume the hypothesis $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ holds on D. Define

$$\varphi(x, y) = \int_{x_0}^{x} P(x, y_0)\,dx + \int_{y_0}^{y} Q(x, y)\,dy.$$

Differentiating with respect to y:

$$\frac{\partial \varphi}{\partial y} = 0 + Q(x, y) = Q(x, y).$$

Now differentiate with respect to x. Note that the first term depends on x in the limit and integrand, and the second on x in the integrand. Differentiating the RHS with respect to y yields $Q(x, y)$. Differentiating the RHS with respect to x:

$$\frac{\partial}{\partial x}\left(\int_{x_0}^{x} P(t, y_0)\,dt + \int_{y_0}^{y} Q(x, t)\,dt\right) = P(x, y_0) + \int_{y_0}^{y} \frac{\partial Q}{\partial x}(x, t)\,dt.$$

If

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y},$$

then

$$\int_{y_0}^{y} \frac{\partial P}{\partial y}(x, t)dt = P(x, y) - P(x, y_0).$$

The expression becomes $P(x, y_0) + P(x, y) - P(x, y_0) = P(x, y)$. Thus $d\varphi = Pdx + Qdy$.

証明終

Necessity.

If the integral is path independent, then $\varphi(x, y)$ is a potential. $\frac{\partial \varphi}{\partial x} = P$ and $\frac{\partial \varphi}{\partial y} = Q$. By continuity of mixed partial derivatives:

$$\frac{\partial P}{\partial y} = \frac{\partial^2 \varphi}{\partial y \partial x} = \frac{\partial^2 \varphi}{\partial x \partial y} = \frac{\partial Q}{\partial x}.$$

証明終

Example 1.6. Elliptic Circulation. Let a, b, c be constants satisfying $ac - b^2 > 0$. Consider the form

$$\omega = \frac{x\,dy - y\,dx}{ax^2 + 2bxy + cy^2}.$$

Find the circulation $\oint_C \omega$ where C is any simple closed curve enclosing the origin.

範例

Solution

Let P and Q be the components of ω. Direct calculation verifies that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ everywhere except at $(0, 0)$. Since $ac - b^2 > 0$, the quadratic form is definite, so the denominator vanishes only at the origin. The value of the integral is independent of the specific shape of C as long as it encloses the origin. Using the result related to the area of the ellipse $ax^2 + 2bxy + cy^2 \leq 1$, or by transforming coordinates to diagonalise the form, the integral evaluates to:

$$\oint_C \omega = \frac{2\pi}{\sqrt{ac - b^2}}.$$

■

The Isoperimetric Inequality

The isoperimetric problem poses a classic geometric question: among all simple closed curves of a fixed perimeter, which one encloses the maximal area? While the answer — the circle — was intuited by ancient Greek mathematicians such as Pappus (c. 300–350 AD), rigorous proofs were not developed until the 19th century by Steiner and others. We present here an elegant analytic proof for piecewise smooth curves using Green's Formula, provided by E. Schmidt in 1939.

Theorem 1.3. ***Isoperimetric Inequality.***

Let Γ be a piecewise smooth simple closed curve of length L enclosing a region of area A. Then:

$$4\pi A \leq L^2.$$

Equality holds if and only if Γ is a circle.

定理

Proof

Let the curve Γ be enclosed between two parallel vertical tangent lines l_1 and l_2. We construct a circle S of radius r that is also tangent to these lines; thus the distance between l_1 and l_2 is $2r$. We establish a coordinate system with the origin at the centre of S and the x-axis perpendicular to the tangents. Consequently, the x-coordinates on Γ satisfy $-r \leq x \leq r$.
Let Γ be parametrised by arc length s, denoted by $(x(s), y(s))$ for $0 \leq s \leq L$, traversed in the counter-clockwise (positive) direction. Let $s = 0$ and $s = s_1$ correspond to the points where Γ touches l_1 and l_2 (where x achieves its minimum $-r$ and maximum r).
Using the area corollary of Green's Formula (*theorem* 1.1), the area enclosed by Γ is:

$$A = \int_0^L x(s)y'(s)\,ds.$$

We define a comparison function $\tilde{y}(s)$ representing the y-coordinates of the circle S corresponding to the x-coordinate $x(s)$:

$$\tilde{y}(s) = \begin{cases} \sqrt{r^2 - x(s)^2} & \text{if } x(s) \text{ is on the upper arc,} \\ -\sqrt{r^2 - x(s)^2} & \text{if } x(s) \text{ is on the lower arc.} \end{cases}$$

Because l_1 and l_2 are the only vertical tangents, $x'(s)$ keeps one sign between them: $x(s)$ increases strictly from $-r$ to r along the upper arc and decreases strictly from r back to $-r$ along the lower arc. Thus $(x(s), \tilde{y}(s))$ traces the upper semicircle once (counter-clockwise) and the lower semicircle once (counter-clockwise), i.e. the full circle once in the *clockwise* orientation. Hence the signed area is that of the circle with a negative sign:

$$\int_0^L \tilde{y}(s)x'(s)\,ds = -\pi r^2.$$

We sum the area expressions:

$$A + \pi r^2 = \int_0^L \left(x(s)y'(s) - \tilde{y}(s)x'(s)\right)\,ds.$$

We treat the integrand as a dot product of vectors $\mathbf{u} = (x, -\tilde{y})$ and $\mathbf{v} = (y', x')$. By the Cauchy-Schwarz inequality:

$$|xy' - \tilde{y}x'| \leq \sqrt{x^2 + \tilde{y}^2}\sqrt{(y')^2 + (x')^2}.$$

From our construction, $x(s)^2 + \tilde{y}(s)^2 = r^2$ (the point is on the circle). Since s is the arc length parameter, $(x')^2 + (y')^2 = 1$. Thus:

$$|xy' - \tilde{y}x'| \leq r \cdot 1 = r.$$

Integrating the absolute value and using $|\int f| \leq \int |f|$ gives

$$|A + \pi r^2| \leq \int_0^L |xy' - \tilde{y}x'|\, ds \leq \int_0^L r\, ds = Lr.$$

Both A and πr^2 are nonnegative, so $A + \pi r^2 = |A + \pi r^2|$, yielding

$$A + \pi r^2 \leq Lr.$$

We now apply the Arithmetic Mean-Geometric Mean (AM-GM) inequality to the terms A and πr^2:

$$\sqrt{A \cdot \pi r^2} \leq \frac{A + \pi r^2}{2} \leq \frac{Lr}{2}.$$

Simplifying $\sqrt{\pi A} r \leq \frac{Lr}{2}$ yields $2\sqrt{\pi A} \leq L$. Squaring both sides gives the isoperimetric inequality:

$$4\pi A \leq L^2.$$

Equality Condition: For equality to hold, all intermediate inequalities must be equalities.

1. $A = \pi r^2$ (from AM-GM), implying Γ has the same area as the circle S.

2. The vectors **u** and **v** must be parallel (from Cauchy-Schwarz) *and* $xy' - \tilde{y}x' \geq 0$ everywhere so that the absolute values can be removed. That is, $(-\tilde{y}, x) = c(s)(x', y')$. Taking magnitudes implies $|c(s)| = r$. By continuity, $c(s) = r$ (assuming consistent orientation).

The condition $(-\tilde{y}, x) = r(x', y')$ implies:

$$\frac{dx}{ds} = -\frac{\tilde{y}}{r} = -\frac{\sqrt{r^2 - x^2}}{r} \quad \text{and} \quad \frac{dy}{ds} = \frac{x}{r}.$$

This system of differential equations characterizes a circle of radius r. Thus, Γ must be a circle.

■

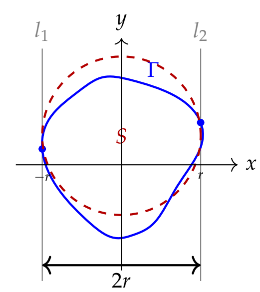


Figure 1.5: The curve Γ enclosed by vertical tangent lines l_1, l_2 separated by distance $2r$, and the comparison circle S of radius r.

1.3 *Rotation Degree of Continuous Vector Fields*

The rotation degree of a continuous vector field is an important topological invariant defined via line integrals. Let $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous mapping, which we refer to as a continuous vector field. For a piecewise smooth oriented closed curve $C \subset \mathbb{R}^2$, if $\mathbf{F}(\mathbf{x}) \neq \mathbf{0}$ for all $\mathbf{x} \in C$, we say $\mathbf{F}$ is **non-degenerate** on C.

> **Definition 1.1.** ***Rotation Degree.***
> Define the unit direction vector field
>
> $$\mathbf{T}(x, y) = \frac{\mathbf{F}(x, y)}{\|\mathbf{F}(x, y)\|}.$$
>
> This maps C to the unit circle S^1. As (x, y) traverses C once in the positive (counter-clockwise) direction, the vector $\mathbf{T}(x, y)$ winds around S^1. The algebraic sum of the number of counter-clockwise winds is called the **rotation degree** of $\mathbf{F}$ along C, denoted by $\gamma(\mathbf{F}, C)$.
> If $\mathbf{F}(x, y) = (u(x, y), v(x, y))$ is continuously differentiable (C^1), an orientation–invariant formula is
>
> $$\gamma(\mathbf{F}, C) = \frac{1}{2\pi} \oint_C \frac{u\,dv - v\,du}{u^2 + v^2}.$$
>
> This equals $\frac{1}{2\pi} \oint_C d(\arg(u + iv))$ whenever a continuous branch of the argument can be chosen along C; the differential form above avoids branch issues when u changes sign.
>
> 定義

If D is a region with boundary $\partial D = \bigcup_{i=1}^{n} \partial D_i$, we define $\gamma(\mathbf{F}, \partial D) = \sum_{i=1}^{n} \gamma(\mathbf{F}, \partial D_i)$, assuming the interior normal lies to the left of the positive direction.

Properties of Rotation Degree

The rotation degree satisfies the following fundamental properties.

> **Proposition 1.1.** ***Additivity.***
> If $D = D_1 \cup D_2$ where D_1, D_2 are closed regions with disjoint interiors, then:
>
> $$\gamma(\mathbf{F}, \partial D) = \gamma(\mathbf{F}, \partial D_1) + \gamma(\mathbf{F}, \partial D_2).$$
>
> 命題

> *Proof*
> Orient ∂D_1 and ∂D_2 so that each keeps its region on the left. The common boundary arc (if any) is then traversed once in each direction, so its contributions to the line integrals defining the degrees cancel. What remains is exactly the integral along ∂D, proving the

sum rule.

■

Proposition 1.2. ***Boundary Degree of Non-degenerate Fields.***
If $\mathbf{F}$ is non-degenerate on a bounded closed connected region D (i.e., $\mathbf{F} \neq \mathbf{0}$ everywhere in D), then:

$$\gamma(\mathbf{F}, \partial D) = 0.$$

命題

Assume D is simply connected with boundary $\mathcal{L}$.

Smooth Case.

Because $\mathbf{F} \neq 0$ on *all of D*, the unit vector field $\mathbf{T} = \mathbf{F}/\|\mathbf{F}\|$ is defined and continuous on D, hence its restriction to $\mathcal{L}$ is homotopic (within S^1) to the constant map $e_1 = (1,0)$. The rotation degree is the winding number of $\mathbf{T}|_{\mathcal{L}}$, so

$$\gamma(\mathbf{F}, \mathcal{L}) = \frac{1}{2\pi} \oint_{\mathcal{L}} \frac{u\,dv - v\,du}{u^2 + v^2} = \deg(\mathbf{T}|_{\mathcal{L}}) = 0,$$

because a map admitting an extension to the disk is null-homotopic on the boundary.

証明終

Continuous Case.

The same extension argument works verbatim since $\mathbf{T}$ is continuous on D; no smoothness is needed once we appeal to homotopy of maps $D \to S^1$.

証明終

If D is multiply connected, we decompose it into simply connected regions.

Definition 1.2. ***Homotopy.***
Let $\mathbf{F}_0, \mathbf{F}_1$ be continuous non-degenerate vector fields on ∂D. A **continuous deformation** is a map $\mathbf{G} : \partial D \times [0,1] \to \mathbb{R}^2$ such that $\mathbf{G}(\mathbf{x}, 0) = \mathbf{F}_0(\mathbf{x})$ and $\mathbf{G}(\mathbf{x}, 1) = \mathbf{F}_1(\mathbf{x})$. If $\mathbf{G}(\mathbf{x}, \lambda) \neq \mathbf{0}$ for all $\lambda \in [0,1]$ and $\mathbf{x} \in \partial D$, it is a **non-degenerate deformation**, and $\mathbf{F}_0, \mathbf{F}_1$ are said to be **homotopic**.

定義

Proposition 1.3. ***Homotopy Invariance.***
Homotopic vector fields on ∂D have the same rotation degree.

命題

Proof

Let $I(\lambda) = \gamma(\mathbf{G}(\cdot, \lambda), \partial D)$. The integrand in the degree formula depends continuously on λ because $\mathbf{G}$ does and never vanishes; hence I is continuous on $[0,1]$. But $I(\lambda)$ is integer-valued for every λ, so continuity forces it to be constant. Therefore $I(0) = I(1)$, i.e. $\gamma(\mathbf{F}_0, \partial D) = \gamma(\mathbf{F}_1, \partial D)$.

■

Example 1.7. Brouwer Fixed-Point Theorem. Let $D \subset \mathbb{R}^2$ be a bounded closed convex region with a smooth boundary ∂D. Let $\mathbf{F} : D \to D$ be a continuous mapping. Prove that $\mathbf{F}$ has a fixed point in D.

範例

Proof

Assume $\mathbf{F}$ has no fixed point. Then $\mathbf{F}(\mathbf{x}) - \mathbf{x} \neq \mathbf{0}$ for all $\mathbf{x} \in D$. Consider the vector field $\mathbf{V}(\mathbf{x}) = \mathbf{F}(\mathbf{x}) - \mathbf{x}$ on ∂D. Since $\mathbf{V}$ is non-degenerate on the entire region D, by *Proposition 1.2*, $\gamma(\mathbf{V}, \partial D) = 0$.
However, let $\mathbf{n}(\mathbf{x})$ be the unit inward normal vector field on ∂D. Since ∂D is a simple closed curve, the rotation degree of the normal vector is $\gamma(\mathbf{n}, \partial D) = 1$. Since $\mathbf{F}$ maps D into itself, for any $\mathbf{x} \in \partial D$, the vector $\mathbf{F}(\mathbf{x}) - \mathbf{x}$ points into the region (or is tangent). Specifically, the angle between $\mathbf{F}(\mathbf{x}) - \mathbf{x}$ and $\mathbf{n}(\mathbf{x})$ is at most $\pi/2$, so:

$$(\mathbf{F}(\mathbf{x}) - \mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) \geq 0.$$

Construct the homotopy $\mathbf{G}(\mathbf{x}, \lambda) = \lambda[\mathbf{F}(\mathbf{x}) - \mathbf{x}] + (1-\lambda)\mathbf{n}(\mathbf{x})$. For $\lambda \in (0,1)$:

$$\mathbf{G} \cdot \mathbf{n} = \lambda(\mathbf{F} - \mathbf{x}) \cdot \mathbf{n} + (1-\lambda)\|\mathbf{n}\|^2 \geq 1 - \lambda > 0.$$

Thus $\mathbf{G}$ is non-degenerate. This implies $\mathbf{F} - \mathbf{x}$ is homotopic to $\mathbf{n}$, so $\gamma(\mathbf{V}, \partial D) = 1$. This contradicts the earlier deduction that the degree is 0. Thus, a fixed point must exist.

■

Example 1.8. Fundamental Theorem of Algebra. Prove that every polynomial $P_n(z)$ of degree $n \geq 1$ has at least one root in $\mathbb{C}$.

範例

Proof

Identify $\mathbb{C}$ with $\mathbb{R}^2$ via $z = x + iy$. Let $\mathbf{F}(x,y) = (\Re(P_n), \Im(P_n))$. Finding a root is equivalent to finding a point where $\mathbf{F}$ is degenerate. Assume $\mathbf{F}$ is non-degenerate everywhere. Then for any circle S_r of radius r, $\gamma(\mathbf{F}, S_r) = 0$.
Consider the polynomial $P_n(z) = z^n + \dots$ (monic WLOG). Let $\mathbf{F}_0$ correspond to z^n. Direct calculation shows $\gamma(\mathbf{F}_0, S_r) = n$. Construct the homotopy $G(z, \lambda) = \lambda z^n + (1-\lambda)P_n(z)$. For sufficiently large

$|z| = r$, the term z^n dominates the lower order terms. Specifically:

$$\bar{z}^n G(z, \lambda) = |z|^{2n} + o(|z|^{2n}) \quad \text{as } |z| \to \infty.$$

Thus G does not vanish on S_r for large r. Hence $\mathbf{F}$ is homotopic to $\mathbf{F}_0$, implying $\gamma(\mathbf{F}, S_r) = n$. Since $n \geq 1$, this contradicts $\gamma = 0$. Thus $\mathbf{F}$ must be degenerate somewhere.

■

Example 1.9. Miklós Schweitzer Competition 1995. Let f, g be integrable on $[0,1]$ with $\int_0^1 f = \int_0^1 g = 1$. Prove there exists $[a,b] \subset [0,1]$ such that $\int_a^b f = \int_a^b g = 1/2$.

範例

Proof

Define the region $D = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq y \leq 1\}$ and the vector field

$$\mathbf{G}(x,y) = \left(\int_x^y f(s)\, ds - \frac{1}{2}, \int_y^1 g(s)\, ds - \frac{1}{2} \right).$$

This field is continuous on D. The problem is to find a zero of $\mathbf{G}$. Assume none exist. The boundary ∂D consists of three segments:

1. The diagonal $x = y$.
2. The vertical segment $x = 0, 0 \leq y \leq 1$.
3. The horizontal segment $y = 1, 0 \leq x \leq 1$.

Observe the values on the axes:

$$\mathbf{G}(0,x) = \left(\int_0^x f - \frac{1}{2}, \int_x^1 g - \frac{1}{2} \right),$$

$$\mathbf{G}(x,1) = \left(\int_x^1 f - \frac{1}{2}, \int_1^1 g - \frac{1}{2} \right) = \left(1 - \int_0^x f - \frac{1}{2}, -\frac{1}{2} \right) = \left(\frac{1}{2} - \int_0^x f, -\frac{1}{2} \right).$$

Note that $\int_x^1 g - 1/2 = 1 - \int_0^x g - 1/2 = 1/2 - \int_0^x g$. The vectors at the three vertices are

$$A = \mathbf{G}(0,0) = \left(-\tfrac{1}{2}, \tfrac{1}{2}\right), \quad B = \mathbf{G}(0,1) = \left(\tfrac{1}{2}, -\tfrac{1}{2}\right), \quad C = \mathbf{G}(1,1) = \left(-\tfrac{1}{2}, -\tfrac{1}{2}\right).$$

Because $\mathbf{G}$ is nonzero on ∂D, its image is a closed curve in $\mathbb{R}^2 \setminus \{0\}$. By homotopy invariance of the degree, we may deform this curve (staying in $\mathbb{R}^2 \setminus \{0\}$) to the piecewise linear path $A \to B \to C \to A$, replacing the segment AB by a tiny detour that skirts the origin. That polygon clearly winds once around the origin, so $\gamma(\mathbf{G}, \partial D) = \pm 1$, in particular it is odd. By *Proposition 1.2*, $\mathbf{G}$ must have a zero in D.

■

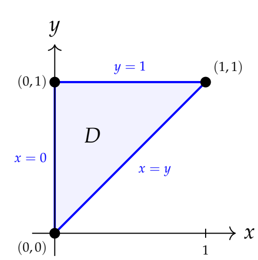


Figure 1.6: The domain $D = \{(x,y) : 0 \leq x \leq y \leq 1\}$ for the Schweitzer problem.

1.4 Exercises

1. **Evaluate via Green's Formula.** Apply Green's Formula to compute the following vector line integrals. The curves are oriented in the positive (counter-clockwise) direction.

   (a) $\oint_C (x^2 + xy)\,dx + (x^2 + y^2)\,dy$, where C is the square with vertices $(\pm 1, \pm 1)$.

   (b) $\oint_C \ln \frac{2+y}{1+x^2}\,dx + \frac{x(y+1)}{2+y}\,dy$, where C is the same square as above.

   (c) $\oint_C (x^2 - y^2)\,dx - 2xy\,dy$, where C is the boundary of the region defined by $x^2 + y^2 \leq 1, x \geq 0, y \geq x$.

   (d) $\oint_C \frac{x\,dy - y\,dx}{x^2 + y^2}$, where C is:

      (i) The arch of the cycloid $x = a(t - \sin t) - a\pi, y = a(1 - \cos t)$ for $t \in [0, 2\pi]$, closed by the x-axis.

      (ii) The arc of $(x-1)^2 + (y-1)^2 = 1$ from $(2, 1)$ to $(0, 1)$ via the upper semicircle, closed by the segment on $y = 1$.

2. **Identity for Radial Functions.** Let L be a piecewise smooth closed curve. If f is continuously differentiable, prove that:

   (a) $\oint_L f(xy)(y\,dx + x\,dy) = 0$.

   (b) $\oint_L f(x^2 + y^2)(x\,dx + y\,dy) = 0$.

3. **Flux Integral.** Calculate

$$\oint_C \frac{\partial u}{\partial n}\,ds,$$

   where $u = x^2 + y^2$, C is the circle $x^2 + y^2 = 6x$, and $\mathbf{n}$ is the unit outward normal vector.

4. **Directional Cosine Integral.** Let C be a piecewise smooth simple closed curve and $\mathbf{l}$ be a fixed constant vector. Prove that $\oint_C \cos(\mathbf{l}, \mathbf{n})\,ds = 0$.

5. **Generalised Winding Number.** Let C be a simple closed curve enclosing the origin. Let a_{ij} be constants such that $\Delta = a_{11}a_{22} - a_{12}a_{21} \neq 0$. Let $X = a_{11}x + a_{12}y$ and $Y = a_{21}x + a_{22}y$. Prove that:

$$\oint_C \frac{X\,dY - Y\,dX}{X^2 + Y^2} = 2\pi\,\mathrm{sgn}(\Delta).$$

6. **Line Integral on the Unit Circle.** Evaluate

$$\oint_L \frac{(x-y)\,dx + (x+4y)\,dy}{x^2+4y^2}$$

where L is the unit circle $x^2 + y^2 = 1$ traversed counter-clockwise.

7. **Area Calculations.** Use Green's area formula to find the area enclosed by:

   (a) The astroid generalisation: $x = a\cos^3 t, y = b\sin^3 t, 0 \le t \le 2\pi$.

   (b) The folium of Descartes loop: $x^3 + y^3 = 3axy$.

   (c) The Lamé curve: $(x/a)^{2n+1} + (y/b)^{2n+1} = C(x/a)^n(y/b)^n$, with $a, b, C > 0$.

8. **Path Independence.** Verify that the following integrals are path-independent and compute their values:

   (a) $\displaystyle\int_{(1,2)}^{(3,4)} \varphi(x)\,dx + \psi(y)\,dy$, for continuous φ, ψ.

   (b) $\displaystyle\int_{(1,0)}^{(6,8)} \frac{x\,dx + y\,dy}{x^2+y^2}$ along a path not passing through the origin.

9. **Integrating Factors.** Find a non-zero integrating factor $M(x, y)$ to make the following forms exact, and find the primitive potential:

   (a)

$$\omega = [-y\sqrt{x^2+y^2+1} - x(x^2+y^2)]\,dx + [x\sqrt{x^2+y^2+1} - y(x^2+y^2)]\,dy.$$

   (b)

$$\omega = x[(ay+bx)^3 + ay^3]\,dx + y[(ay+bx)^3 + bx^3]\,dy.$$

Advanced Line Integrals

1. **Bound Estimation.** Prove the inequality

$$\left|\int_C P\,dx + Q\,dy\right| \le ML,$$

where L is the arc length and $M = \max_C \sqrt{P^2+Q^2}$. Use this to show that $\lim_{R\to\infty} I_R = 0$, where

$$I_R = \oint_{x^2+y^2=R^2} \frac{y\,dx - x\,dy}{(x^2+xy+y^2)^2}.$$

2. **Logarithmic Potentials on Circles.** Calculate:

(a) $\int_{x^2+y^2=R^2} \ln\sqrt{(x-a)^2+y^2}\,ds$ for $|a| \neq R$.

(b) $\int_{x^2+y^2=R^2} \ln\sqrt{(x-a)^2+(y-b)^2}\,ds$ for $a^2+b^2 \neq R^2$.

3. **Potential Decay.** Let L be a simple closed curve. Let

$$u(x,y) = \oint_L f(\xi,\eta)\ln\sqrt{(x-\xi)^2+(y-\eta)^2}\,ds.$$

Prove that $u(x,y) \to 0$ as $x^2+y^2 \to \infty$ if and only if $\oint_L f\,ds = 0$.

4. **Mean Value Property Equivalence.** Let u be continuous on $\mathbb{R}^2$. Prove that the area mean value property (average over disk equals value at centre) holds for all $r > 0$ if and only if the boundary mean value property (average over circle equals value at centre) holds for all $r > 0$.

5. **Gradient Bound for Integral.** Let $f \in C^1(G)$ with $f = 0$ on ∂G, where G is the disk of radius a. Prove:

$$\left|\iint_G f(x,y)\,dx\,dy\right| \leq \frac{\pi}{3}a^3 \max_G |\nabla f|.$$

6. **High-Dimensional Mean Value Theorem.** Let $f \in C^1(D)$ where $D \subset \mathbb{R}^n$ is a ball of radius r. Prove there exists $p_0 \in \text{int}D$ such that:

$$\max_D f - \min_D f = |\nabla f(p_0)| \cdot 2r.$$

7. **Retraction Construction.** Assume $f : B \to B$ is a smooth map with no fixed points.

(a) Construct a map $g(x)$ by projecting x onto ∂B along the ray from $f(x)$ through x. Show g is well-defined and smooth.

(b) Verify that $g(x) = x$ for $x \in \partial B$ and $g(B) \subset \partial B$.

(c) Use the previous exercise to derive a contradiction, proving f must have a fixed point.

2

Surface Integrals

Following the natural progression from integration along curves, we extend our calculus to integration over surfaces in $\mathbb{R}^3$. We distinguish between integrals of scalar fields (measuring quantities such as surface area or mass) and integrals of vector fields (measuring flux). In this chapter, we develop the theory of the former, known as surface integrals of the first type.

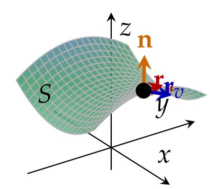


Figure 2.1: A parametric surface with tangent vectors $\mathbf{r}_u$, $\mathbf{r}_v$ and normal $\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v$.

2.1 Surfaces and Area

We begin by formalising the geometric object of study. Let S be a surface in $\mathbb{R}^3$. We generally describe S via a parametrisation.

> **Definition 2.1. *Parametric Surface.***
> A mapping $\mathbf{r} : D \to \mathbb{R}^3$ defined on a bounded region $D \subset \mathbb{R}^2$ is a **parametric surface** if $\mathbf{r}$ is a continuous vector-valued function
>
> $$\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v)).$$
>
> The surface S is **smooth** if $\mathbf{r}$ is continuously differentiable and the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ are linearly independent everywhere on D. That is, the normal vector is non-vanishing:
>
> $$\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}, \quad \text{where } \mathbf{r}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right).$$
>
> 定義

To define the area of a curved surface S, we employ an approximation method analogous to the rectification of curves, but with a subtlety required to avoid the "Schwarz lantern" paradox (where limits of inscribed polyhedra may not converge to the surface area).
Let S be a piecewise smooth surface (it may be closed or may have boundary made of piecewise smooth curves). We partition S into m sub-surfaces $S_1, \ldots, S_m$ using a mesh of piecewise smooth curves.
Let $d(S_i)$ denote the diameter of the i-th element. For each S_i, choose

an arbitrary point $M_i \in S_i$. Let T_i be the projection of the surface element S_i onto the tangent plane to S at M_i. The area of the planar region T_i is denoted by ΔT_i.

Definition 2.2. ***Surface Area.***
The area of the surface S, denoted $A(S)$, is defined as the limit of the sum of the areas of the tangential projections as the mesh size approaches zero:

$$A(S) = \lim_{\lambda \to 0} \sum_{i=1}^{m} \Delta T_i,$$

where $\lambda = \max_{1 \le i \le m}\{d(S_i)\}$. If this limit exists and is finite, S is said to be **rectifiable**.

定義

For a smooth parametric surface, the area element dS arises from the magnitude of the fundamental vector product. The infinitesimal area of the parallelogram spanned by $\mathbf{r}_u\, du$ and $\mathbf{r}_v\, dv$ is:

$$dS = \|\mathbf{r}_u \times \mathbf{r}_v\|\, du\, dv.$$

Using the identity $\|\mathbf{a} \times \mathbf{b}\|^2 = \|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a} \cdot \mathbf{b})^2$, we express this in terms of the coefficients of the first fundamental form.

Notation 2.1. First Fundamental Form We define the Gaussian coefficients:

$$E = \mathbf{r}_u \cdot \mathbf{r}_u = x_u^2 + y_u^2 + z_u^2,$$

$$F = \mathbf{r}_u \cdot \mathbf{r}_v = x_u x_v + y_u y_v + z_u z_v,$$

$$G = \mathbf{r}_v \cdot \mathbf{r}_v = x_v^2 + y_v^2 + z_v^2.$$

Then the surface area element is $dS = \sqrt{EG - F^2}\, du\, dv$.

記法

2.2 *The First Type of Surface Integral*

Let S be a rectifiable surface and $f : S \to \mathbb{R}$ a bounded function. Consider a partition $P = \{S_1, \ldots, S_m\}$ of S and sample points $M_i(\xi_i, \eta_i, \zeta_i) \in S_i$. Let ΔS_i be the area of the sub-surface S_i.

Definition 2.3. ***Scalar Surface Integral.***
The **surface integral of the first type** of f over S is defined as:

$$\iint_S f(x, y, z)\, dS = \lim_{\lambda(P) \to 0} \sum_{i=1}^{m} f(\xi_i, \eta_i, \zeta_i) \Delta S_i,$$

provided the limit exists independent of the partition and sample points.

定義

Evaluation Formulae

The evaluation of surface integrals reduces to double integrals over the parameter domain D.

Theorem 2.1. ***Evaluation on Parametric Surfaces.***
If S is defined by $\mathbf{r}(u,v)$ for $(u,v) \in D$, and f is continuous on S, then:

$$\iint_S f(x,y,z)\,dS = \iint_D f(x(u,v),y(u,v),z(u,v))\sqrt{EG-F^2}\,du\,dv.$$

定理

Corollary 2.1. *Existence for Continuous f.* If S is a smooth parametric surface and f is continuous on S, then the surface integral $\iint_S f\,dS$ exists.

推論

Proof

The composition $f \circ \mathbf{r}$ is continuous on the compact domain D, hence Riemann integrable. The evaluation formula expresses $\iint_S f\,dS$ as that double integral, so the limit in the Riemann-sum definition exists.

■

In the common case where the surface is the graph of a function $z = z(x,y)$ over a region D_{xy}, we may choose x and y as parameters. Then $\mathbf{r}(x,y) = (x,y,z(x,y))$. Calculating the partial derivatives:

$$\mathbf{r}_x = (1,0,z_x), \quad \mathbf{r}_y = (0,1,z_y).$$

It follows that $E = 1+z_x^2$, $G = 1+z_y^2$, and $F = z_x z_y$.

$$EG-F^2 = (1+z_x^2)(1+z_y^2)-(z_xz_y)^2 = 1+z_x^2+z_y^2+z_x^2z_y^2-z_x^2z_y^2 = 1+z_x^2+z_y^2.$$

Corollary 2.2. *Evaluation on Explicit Surfaces.* If S is given by $z = z(x,y)$ for $(x,y) \in D$, then:

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,z(x,y))\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy.$$

推論

Proof

Take the parametrisation $\mathbf{r}(x,y) = (x,y,z(x,y))$ used above. Substituting the computed coefficients E,G,F into $\sqrt{EG-F^2}$ gives $\sqrt{1+z_x^2+z_y^2}$. Applying the general evaluation formula with parameters $(u,v) = (x,y)$ yields the stated expression.

■

Example 2.1. Integral over a Cone. Let S be the portion of the cone $z = \sqrt{x^2+y^2}$ lying inside the cylinder $x^2 + y^2 = 2ax$ ($a > 0$). Calculate:

$$I = \iint_S (x^2y^2 + y^2z^2 + z^2x^2)\, dS.$$

範例

Solution

We represent S as the graph $z = \sqrt{x^2+y^2}$. The projection domain D is the disk $x^2+y^2 \le 2ax$. First, we calculate the area element.

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{z}, \quad \frac{\partial z}{\partial y} = \frac{y}{z}.$$

$$dS = \sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}}\, dx\, dy = \sqrt{\frac{z^2+x^2+y^2}{z^2}}\, dx\, dy = \sqrt{\frac{2(x^2+y^2)}{x^2+y^2}}\, dx\, dy = \sqrt{2}\, dx\, dy.$$

Substituting $z^2 = x^2+y^2$ into the integrand:

$$f(x,y,z) = x^2y^2 + z^2(x^2+y^2) = x^2y^2 + (x^2+y^2)^2.$$

The integral becomes:

$$I = \sqrt{2}\iint_D [x^2y^2 + (x^2+y^2)^2]\, dx\, dy.$$

We employ polar coordinates. The boundary $x^2+y^2 = 2ax$ becomes $r = 2a\cos\theta$ for $\theta \in [-\pi/2, \pi/2]$.

$$I = \sqrt{2}\int_{-\pi/2}^{\pi/2} d\theta \int_0^{2a\cos\theta} [(r^2\cos^2\theta)(r^2\sin^2\theta) + r^4] r\, dr.$$

$$I = \sqrt{2}\int_{-\pi/2}^{\pi/2} (\cos^2\theta\sin^2\theta + 1)\, d\theta \int_0^{2a\cos\theta} r^5\, dr.$$

Evaluating the inner integral:

$$\int_0^{2a\cos\theta} r^5\, dr = \frac{1}{6}(2a\cos\theta)^6 = \frac{32}{3}a^6\cos^6\theta.$$

Thus:

$$I = \frac{32\sqrt{2}}{3}a^6 \int_{-\pi/2}^{\pi/2} (\cos^8\theta\sin^2\theta + \cos^6\theta)\, d\theta.$$

The integrand is even, so double the integral over $[0, \pi/2]$ and use the Beta–function identity $\int_0^{\pi/2} \sin^m\theta\cos^n\theta\, d\theta = \frac{1}{2}B\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$ (or, if preferred, apply the usual power-reduction formulas repeatedly to the same effect):

$$\int_{-\pi/2}^{\pi/2} \cos^8\theta\sin^2\theta\, d\theta = 2\cdot\tfrac{1}{2}B\left(\tfrac{3}{2}, \tfrac{9}{2}\right) = \frac{7\pi}{256},$$

$$\int_{-\pi/2}^{\pi/2} \cos^6\theta \, d\theta = 2 \cdot \tfrac{1}{2} B\left(\tfrac{1}{2}, \tfrac{7}{2}\right) = \frac{5\pi}{16}.$$

Combining, $\int_{-\pi/2}^{\pi/2} (\cos^8\theta \sin^2\theta + \cos^6\theta) \, d\theta = \frac{87\pi}{256}$, hence

$$I = \frac{32\sqrt{2}}{3} a^6 \cdot \frac{87\pi}{256} = \frac{29}{8}\sqrt{2}\pi a^6.$$

Alternative Method: Spherical Parametrisation We may also parametrise S using spherical coordinates. The cone equation $z = \sqrt{x^2+y^2}$ corresponds to the semi-vertical angle $\phi = \pi/4$. We parametrise S by:

$$x = \frac{r}{\sqrt{2}}\cos\theta, \quad y = \frac{r}{\sqrt{2}}\sin\theta, \quad z = \frac{r}{\sqrt{2}},$$

defined on a domain D in the (r, θ) plane. The cylinder $x^2 + y^2 = 2ax$ transforms to:

$$\frac{r^2}{2} = 2a\frac{r}{\sqrt{2}}\cos\theta \implies r = 2\sqrt{2}a\cos\theta.$$

Thus the parameter domain is $D = \{(r, \theta) : -\pi/2 \le \theta \le \pi/2, 0 \le r \le 2\sqrt{2}a\cos\theta\}$. We compute the coefficients of the first fundamental form with respect to the parameters (r, θ):

$$\mathbf{r}_r = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, 1), \quad \mathbf{r}_\theta = \frac{r}{\sqrt{2}}(-\sin\theta, \cos\theta, 0).$$

$$E = \mathbf{r}_r \cdot \mathbf{r}_r = 1, \quad G = \mathbf{r}_\theta \cdot \mathbf{r}_\theta = \frac{r^2}{2}, \quad F = \mathbf{r}_r \cdot \mathbf{r}_\theta = 0.$$

The area element is $dS = \sqrt{EG - F^2}\, dr\, d\theta = \frac{r}{\sqrt{2}}\, dr\, d\theta$. Substituting into the integrand $f = x^2y^2 + z^2(x^2+y^2)$:

$$x^2y^2 = \frac{r^4}{4}\cos^2\theta\sin^2\theta, \quad z^2(x^2+y^2) = \frac{r^2}{2}\cdot\frac{r^2}{2} = \frac{r^4}{4}.$$

The integral becomes:

$$I = \int_{-\pi/2}^{\pi/2} d\theta \int_0^{2\sqrt{2}a\cos\theta} \left(\frac{r^4}{4}\cos^2\theta\sin^2\theta + \frac{r^4}{4}\right)\frac{r}{\sqrt{2}}\, dr.$$

Evaluating the inner integral with respect to r yields $\frac{1}{6\sqrt{2}}(2\sqrt{2}a\cos\theta)^6(\cos^2\theta\sin^2\theta + 1)$, which simplifies to the same result:

$$I = \frac{29}{8}\sqrt{2}\pi a^6.$$

■

Geometric and Physical Applications

As with line integrals, surface integrals allow us to compute geometric and physical properties of surfaces.

Area. If $f(x, y, z) \equiv 1$, the integral yields the surface area $A(S) = \iint_S dS$.

Mass. If $\rho(x, y, z)$ is the surface mass density, the total mass is $m = \iint_S \rho\, dS$.

Centroid. The coordinates of the centre of mass (x_0, y_0, z_0) are given by:

$$x_0 = \frac{1}{m} \iint_S x\rho\, dS, \quad y_0 = \frac{1}{m} \iint_S y\rho\, dS, \quad z_0 = \frac{1}{m} \iint_S z\rho\, dS.$$

Example 2.2. Viviani's Surface. Find the area and the centroid of the surface portion of the upper hemisphere $z = \sqrt{a^2 - x^2 - y^2}$ cut out by the cylinder $x^2 + y^2 = ax$ $(a > 0)$.

範例

Solution

This surface is part of the boundary of the Viviani body (see *figure* 2.2). The domain D is the disk $x^2 + y^2 \le ax$. First, calculate dS for the sphere $x^2 + y^2 + z^2 = a^2$:

$$z_x = \frac{-x}{\sqrt{a^2 - x^2 - y^2}} = -\frac{x}{z}, \quad z_y = -\frac{y}{z}.$$

$$dS = \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}}\, dx\, dy = \sqrt{\frac{a^2}{z^2}}\, dx\, dy = \frac{a}{z}\, dx\, dy = \frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dx\, dy.$$

Area Calculation.

$$A = \iint_D \frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dx\, dy.$$

Using polar coordinates, the region D corresponds to $-\pi/2 \le \theta \le \pi/2$ and $0 \le r \le a\cos\theta$.

$$A = \int_{-\pi/2}^{\pi/2} d\theta \int_0^{a\cos\theta} \frac{a}{\sqrt{a^2 - r^2}} r\, dr.$$

Let $u = a^2 - r^2$, then $du = -2r\, dr$.

$$\int_0^{a\cos\theta} \frac{r}{\sqrt{a^2 - r^2}}\, dr = \left[-\sqrt{a^2 - r^2}\right]_0^{a\cos\theta} = a - a\sqrt{1 - \cos^2\theta} = a(1 - |\sin\theta|).$$

Thus,

$$A = a^2 \int_{-\pi/2}^{\pi/2} (1 - |\sin\theta|)\, d\theta = 2a^2 \int_0^{\pi/2} (1 - \sin\theta)\, d\theta.$$

$$A = 2a^2[\theta + \cos\theta]_0^{\pi/2} = 2a^2\left(\frac{\pi}{2} - 1\right) = (\pi - 2)a^2.$$

Centroid Calculation. By symmetry across the xz-plane, $y_0 = 0$. The density is uniform ($\rho \equiv 1$), so $m = A$. For x_0:

$$x_0 = \frac{1}{A} \iint_S x \, dS = \frac{1}{A} \iint_D \frac{ax}{\sqrt{a^2 - x^2 - y^2}} \, dx \, dy.$$

Switching to polar coordinates ($x = r \cos\theta$):

$$\iint_D \cdots = \int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta \int_0^{a\cos\theta} \frac{ar^2}{\sqrt{a^2 - r^2}} \, dr.$$

We employ the substitution $r = a \cos t$, so $dr = -a \sin t \, dt$. The limits transform from $0 \to a\cos\theta$ to $\pi/2 \to \theta$.

$$\int_0^{a\cos\theta} \frac{ar^2}{\sqrt{a^2 - r^2}} \, dr = \int_{\pi/2}^{\theta} \frac{a(a\cos t)^2}{a \sin t} (-a \sin t) \, dt = a^3 \int_{\theta}^{\pi/2} \cos^2 t \, dt.$$

Substituting this back into the expression for the moment (noting the factor of 2 from symmetry):

$$\iint_S x \, dS = 2 \int_0^{\pi/2} \cos\theta \left(a^3 \int_{\theta}^{\pi/2} \cos^2 t \, dt \right) d\theta.$$

Changing the order of integration over the triangular domain $0 \le \theta \le t \le \pi/2$:

$$\iint_S x \, dS = 2a^3 \int_0^{\pi/2} \cos^2 t \left(\int_0^t \cos\theta \, d\theta \right) dt = 2a^3 \int_0^{\pi/2} \cos^2 t \sin t \, dt.$$

Elementary evaluation yields:

$$\iint_S x \, dS = 2a^3 \left[-\frac{1}{3} \cos^3 t \right]_0^{\pi/2} = \frac{2}{3} a^3.$$

Therefore,

$$x_0 = \frac{2a^3/3}{(\pi - 2)a^2} = \frac{2a}{3(\pi - 2)}.$$

For z_0:

$$z_0 = \frac{1}{A} \iint_S z \, dS.$$

Since $dS = \frac{a}{z} dxdy$, the integrand simplifies remarkably: $z \, dS = z \frac{a}{z} dxdy = a \, dxdy$.

$$\iint_S z \, dS = \iint_D a \, dx \, dy = a \cdot \text{Area}(D).$$

The domain D is a circle of radius $a/2$, so $\text{Area}(D) = \pi(a/2)^2 = \pi a^2/4$.

$$\iint_S z \, dS = \frac{\pi a^3}{4}.$$

$$z_0 = \frac{\pi a^3/4}{(\pi - 2)a^2} = \frac{\pi a}{4(\pi - 2)}.$$

The centroid is $\left(\frac{2a}{3(\pi-2)}, 0, \frac{\pi a}{4(\pi-2)} \right)$.

∎

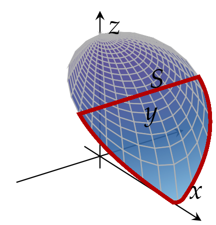


Figure 2.2: Viviani's surface S: the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$ cut by the cylinder $x^2 + y^2 = ax$.

2.3 *The Second Type of Surface Integral*

In vector calculus, we often integrate a vector field over a surface to compute flux. This leads to the second type of surface integral, which depends on the orientation of the surface.

Orientation of Surfaces

A smooth surface S is said to be **orientable** if it is possible to define a continuous unit normal vector field $\mathbf{n}(x, y, z)$ on S. An orientable surface has two sides; choosing a specific normal field $\mathbf{n}$ specifies the **orientation** (or "side") of the surface.

· For a closed surface (like a sphere), the convention is usually to choose the **outward** normal as positive.

· For a surface given by $z = z(x, y)$, the **upper** side is the one where $\mathbf{n} \cdot \mathbf{k} > 0$.

Non-orientable surfaces, such as the Möbius strip, do not admit a global consistent normal field and are excluded from this discussion.

Definition and Evaluation

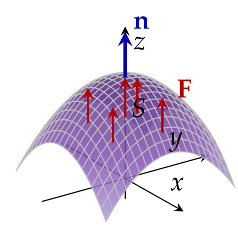


Figure 2.3: Flux of a vector field $\mathbf{F}$ through an oriented surface S: the integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$ measures net flow.

Let S be a piecewise smooth oriented surface. Let $\mathbf{F} = (P, Q, R)$ be a vector field defined on S. The flux of $\mathbf{F}$ across S is defined as the surface integral of the normal component of $\mathbf{F}$.

> **Definition 2.4.** *Vector Surface Integral.*
> The **surface integral of the second type** of $\mathbf{F}$ over S is denoted and defined by:
>
> $$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S (P\, dy\, dz + Q\, dz\, dx + R\, dx\, dy) = \iint_S (P, Q, R) \cdot \mathbf{n}\, dS.$$

定義

Equivalently, for a partition of S into small patches with representative points M_i, unit normals $\mathbf{n}_i$ consistent with the chosen orientation, and areas ΔS_i,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \lim_{\max \Delta S_i \to 0} \sum_i \mathbf{F}(M_i) \cdot \mathbf{n}_i\, \Delta S_i,$$

mirroring the Riemann-sum definition used for Type I integrals. The notation $dy\, dz$, $dz\, dx$, and $dx\, dy$ represents the projections of the area element $d\mathbf{S}$ onto the coordinate planes.

Evaluation via Projection

Just as with the first type, we evaluate these integrals by projecting onto coordinate planes. Signs are set by the components of the chosen unit normal $\mathbf{n}$:

- $\iint_S P\,dy\,dz$: project to D_{yz} and multiply by $\operatorname{sgn}(\mathbf{n}\cdot\mathbf{i})$.
- $\iint_S Q\,dz\,dx$: project to D_{zx} and multiply by $\operatorname{sgn}(\mathbf{n}\cdot\mathbf{j})$.
- $\iint_S R\,dx\,dy$: project to D_{xy} and multiply by $\operatorname{sgn}(\mathbf{n}\cdot\mathbf{k})$.

Equivalently, if a parametrisation gives the normal $\mathbf{r}_u \times \mathbf{r}_v$, use a plus sign when $(\mathbf{r}_u \times \mathbf{r}_v)\cdot\mathbf{n} > 0$ and a minus sign when it is negative; this keeps track of orientation when parameters run opposite to the chosen normal. For a graph $z = z(x,y)$ the upward normal is

$$\mathbf{n} = \frac{(-z_x, -z_y, 1)}{\sqrt{1+z_x^2+z_y^2}},$$

so $\operatorname{sgn}(\mathbf{n}\cdot\mathbf{k})$ decides the sign for $dx\,dy$, and analogous expressions hold if the surface is written as $x = x(y,z)$ or $y = y(x,z)$.

General Parametric Evaluation

If S is given by $\mathbf{r}(u,v) = (x,y,z)$, the vector area element is

$$d\mathbf{S} = (\mathbf{r}_u \times \mathbf{r}_v)\,du\,dv = (A,B,C)\,du\,dv,$$

where $A = \frac{\partial(y,z)}{\partial(u,v)}$, $B = \frac{\partial(z,x)}{\partial(u,v)}$, and $C = \frac{\partial(x,y)}{\partial(u,v)}$ are the Jacobians. Then:

$$\iint_S P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy = \pm\iint_D (PA+QB+RC)\,du\,dv.$$

The sign is chosen to match the orientation of S: if the parametric normal (A,B,C) agrees with the chosen orientation $\mathbf{n}$, use $+$; otherwise use $-$.

Example 2.3. Flux through a Hemisphere. Let Σ be the upper unit hemisphere $z = \sqrt{1-x^2-y^2}$ with the **inner** orientation (normal pointing towards the origin). Calculate:

$$I = \iint_\Sigma dy\,dz + dz\,dx + dx\,dy.$$

範例

Solution

Method 1: Coordinate Projection. $I = I_1 + I_2 + I_3$. Consider $I_1 = \iint_\Sigma dy\,dz$. The surface splits into two parts relative to the x-projection: the front ($x > 0$) and back ($x < 0$). However, for the sphere $x^2+y^2+z^2 = 1$, the inner normal points towards the origin. For the inward orientation $\mathbf{n} = -\mathbf{r}$ we have $n_x = -x$, so its sign flips across the plane $x = 0$: on the "front" side ($x > 0$) the normal points in the $-x$ direction, while on the "back" side ($x < 0$) it points in the $+x$ direction. Let $\Sigma = \Sigma_{\text{front}} \cup \Sigma_{\text{back}}$. On Σ_{back} (where $x < 0$), the normal has positive x-component. Thus we take $+\iint_{D_{yz}} dy\,dz$. On Σ_{front} (where $x > 0$), the normal has negative x-component. Thus we take $-\iint_{D_{yz}} dy\,dz$. By symmetry, the domain D_{yz} is the semi-disk $y^2 + z^2 \leq 1, z \geq 0$. The contributions cancel: $I_1 = 0$. Similarly, $I_2 = 0$. For $I_3 = \iint_\Sigma dx\,dy$, the normal $\mathbf{n}$ on the upper hemisphere points inwards (downwards), so $n_z < 0$. Thus we take the negative sign.

$$I_3 = -\iint_{D_{xy}} dx\,dy = -\text{Area}(D_{xy}) = -\pi(1)^2 = -\pi.$$

Total integral $I = -\pi$.

Method 2: Parametrisation. Using spherical coordinates: $x = \sin\phi\cos\theta, y = \sin\phi\sin\theta, z = \cos\phi$ for $\phi \in [0, \pi/2], \theta \in [0, 2\pi]$. The parametric normal vector is

$$\mathbf{N} = \frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta} = \sin\phi(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) = \sin\phi\,\mathbf{r}.$$

This vector points **outwards**. Since we require the inner orientation, we must take the negative sign.

$$I = -\iint_D [\underbrace{1 \cdot A}_{dydz} + \underbrace{1 \cdot B}_{dzdx} + \underbrace{1 \cdot C}_{dxdy}] d\phi d\theta$$

Substituting the Jacobians ($A = \sin^2\phi\cos\theta, B = \sin^2\phi\sin\theta, C = \sin\phi\cos\phi$):

$$I = -\int_0^{2\pi} d\theta \int_0^{\pi/2} (\sin^2\phi\cos\theta + \sin^2\phi\sin\theta + \sin\phi\cos\phi) d\phi.$$

The terms with $\cos\theta$ and $\sin\theta$ vanish upon integration over $[0, 2\pi]$.

$$I = -2\pi \int_0^{\pi/2} \sin\phi\cos\phi\,d\phi = -2\pi \left[\frac{1}{2}\sin^2\phi\right]_0^{\pi/2} = -\pi.$$

■

Example 2.4. Flux through a Boundary. Calculate

$$I = \iint_\Sigma (z + x)\,dy\,dz + (x + y)\,dz\,dx + (y + z)\,dx\,dy,$$

where Σ is the boundary of the solid $\Omega = \{x^2 + y^2 \leq 1, 0 \leq z \leq 1, x \geq 0, y \geq 0\}$ (first octant quarter-cylinder), oriented outwards.

範例

Solution

Let $\mathbf{F} = (z + x, x + y, y + z)$. We decompose Σ into 5 faces:

- Σ_1 (Curved surface $x^2 + y^2 = 1$): Projects to D_{yz} for the first term. Normal points out ($x, y > 0$).
- Σ_2 (Bottom $z = 0$): Normal $-\mathbf{k}$.
- Σ_3 (Top $z = 1$): Normal $+\mathbf{k}$.
- Σ_4 (Left $y = 0$): Normal $-\mathbf{j}$.
- Σ_5 (Back $x = 0$): Normal $-\mathbf{i}$.

The radial condition $x^2 + y^2 \leq 1$ does not restrict z, so the projections needed below are rectangles: $0 \leq y \leq 1, 0 \leq z \leq 1$ onto D_{yz} and $0 \leq x \leq 1, 0 \leq z \leq 1$ onto D_{zx}. We calculate term by term.

1. $\iint (z + x)\, dy\, dz$: Only Σ_1 and Σ_5 contribute (others have $dx = 0$ or normal $\perp$ $\mathbf{i}$). On Σ_5 ($x = 0$), the normal is $-\mathbf{i}$ (backwards). Projection is D_{yz}. Integral: $-\iint_{D_{yz}} (z + 0)\, dy\, dz$. On Σ_1 ($x = \sqrt{1 - y^2}$), normal has $x > 0$. Integral: $+\iint_{D_{yz}} (z + \sqrt{1 - y^2})\, dy\, dz$. Sum: $\iint_{D_{yz}} (\sqrt{1 - y^2})\, dy\, dz$. D_{yz} is the square $[0, 1] \times [0, 1]$. Integral $= \int_0^1 \int_0^1 \sqrt{1 - y^2}\, dy\, dz = 1 \cdot \frac{\pi}{4} = \frac{\pi}{4}$.
2. $\iint (x + y)\, dz\, dx$: By symmetry with the first term (swapping x, y and the relevant surfaces), this yields $\frac{\pi}{4}$.
3. $\iint (y + z)\, dx\, dy$: Only Σ_2 ($z = 0$) and Σ_3 ($z = 1$) contribute. On Σ_2 (normal down), integrand is $y + 0$. Integral: $-\iint_{D_{xy}} y\, dx\, dy$. On Σ_3 (normal up), integrand is $y + 1$. Integral: $+\iint_{D_{xy}} (y + 1)\, dx\, dy$. Sum: $\iint_{D_{xy}} 1\, dx\, dy = \text{Area}(D_{xy}) = \frac{\pi}{4}$.

Total $I = \frac{\pi}{4} + \frac{\pi}{4} + \frac{\pi}{4} = \frac{3\pi}{4}$.

■

Relation between Type I and Type II Integrals

The vector surface integral relates to the scalar surface integral via the normal vector.

Proposition 2.1. *Conversion Formula.*

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S (\mathbf{F} \cdot \mathbf{n})\, dS.$$

In coordinates, if $\mathbf{n} = (\cos\alpha, \cos\beta, \cos\gamma)$:

$$\iint_S P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy = \iint_S (P\cos\alpha + Q\cos\beta + R\cos\gamma)\,dS.$$

命題

Proof

For a parametrisation $\mathbf{r}(u,v)$ we have $d\mathbf{S} = (\mathbf{r}_u \times \mathbf{r}_v)\,du\,dv = \mathbf{n}\,dS$, so $\mathbf{F}\cdot d\mathbf{S} = (\mathbf{F}\cdot\mathbf{n})\,dS$. Expanding $\mathbf{r}_u \times \mathbf{r}_v = (A, B, C)$ shows $A = \partial(y,z)/\partial(u,v)$ etc., yielding the coordinate expression above.

■

This is particularly useful when the scalar integral simplifies due to symmetries in dS or $\mathbf{n}$.

Example 2.5. Simplification via Normal. Calculate

$$I = \iint_\Sigma xyz(y^2z^2 + z^2x^2 + x^2y^2)\,dS,$$

where Σ is the sphere portion $x^2 + y^2 + z^2 = a^2$ in the first octant.

範例

Solution

This is a Type I integral, but the integrand is complex. We observe the structure resembles a dot product. The integrand is $xyz(y^2z^2 + z^2x^2 + x^2y^2)$. Consider the vector field $\mathbf{F}$ and the normal $\mathbf{n} = (x/a, y/a, z/a)$. Note that

$$(y^3z^3, z^3x^3, x^3y^3)\cdot\mathbf{n} = y^3z^3\frac{x}{a} + z^3x^3\frac{y}{a} + x^3y^3\frac{z}{a} = \frac{1}{a}xyz(y^2z^2 + z^2x^2 + x^2y^2).$$

Thus the original integrand is $a(\mathbf{G}\cdot\mathbf{n})$ where $\mathbf{G} = (y^3z^3, z^3x^3, x^3y^3)$. We convert to a Type II integral:

$$I = a\iint_\Sigma \mathbf{G}\cdot d\mathbf{S} = a\iint_\Sigma y^3z^3\,dy\,dz + z^3x^3\,dz\,dx + x^3y^3\,dx\,dy.$$

By symmetry of the sphere and the function in the first octant, the three terms are equal.

$$I = 3a\iint_\Sigma x^3y^3\,dx\,dy.$$

This is now a simple integral over the quarter disk D_{xy}.

$$I = 3a\int_0^{\pi/2} d\theta \int_0^a (r^6\cos^3\theta\sin^3\theta) r\,dr = 3a\cdot\frac{a^8}{8}\int_0^{\pi/2}(\sin\theta\cos\theta)^3\,d\theta.$$

Using $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$:

$$I = \frac{3a^9}{8}\int_0^{\pi/2}\frac{1}{8}\sin^3 2\theta\,d\theta = \frac{3a^9}{64}\int_0^{\pi}\sin^3 u\,\frac{du}{2} = \frac{3a^9}{128}\cdot\frac{4}{3} = \frac{a^9}{32}.$$

■

Example 2.6. Viviani Flux. Calculate

$$I = \iint_\Sigma (y-z)\,dy\,dz + (z-x)\,dz\,dx + (x-y)\,dx\,dy$$

over the outer side of the sphere part $x^2+y^2+z^2 = 2Rx$ cut by $x^2+y^2 = 2rx$ ($z \geq 0$).

範例

Solution

The sphere equation is $(x-R)^2+y^2+z^2 = R^2$. The outward normal is $\mathbf{n} = \frac{1}{R}(x-R, y, z)$. We convert to Type I:

$$I = \iint_\Sigma \mathbf{F}\cdot\mathbf{n}\,dS = \frac{1}{R}\iint_\Sigma [(y-z)(x-R)+(z-x)y+(x-y)z]\,dS.$$

Expanding the term in brackets:

$$(xy - yR - zx + zR) + (zy - xy) + (xz - yz) = -yR + zR = R(z-y).$$

So $I = \iint_\Sigma \frac{1}{R}\cdot R(z-y)\,dS = \iint_\Sigma (z-y)\,dS$. The surface Σ is symmetric with respect to the plane $y = 0$ (since the defining equations are even in y). The function y is odd. Thus $\iint_\Sigma y\,dS = 0$.

$$I = \iint_\Sigma z\,dS.$$

For the sphere $x^2+y^2+z^2 = 2Rx$, we differentiate implicitly to find dS:

$$2x - 2R + 2zz_x = 0 \implies z_x = \frac{R-x}{z}, \quad z_y = -\frac{y}{z}.$$

$$dS = \sqrt{1+\frac{(R-x)^2+y^2}{z^2}}dxdy = \sqrt{\frac{z^2+(R-x)^2+y^2}{z^2}}dxdy = \sqrt{\frac{R^2}{z^2}}dxdy = \frac{R}{z}dxdy.$$

The integral becomes incredibly simple:

$$I = \iint_D z\left(\frac{R}{z}\right)dxdy = R\iint_D dxdy = R\cdot\text{Area}(D).$$

The domain D is the disk $x^2+y^2 \leq 2rx$, which has radius r. Area $= \pi r^2$.

$$I = \pi R r^2.$$

■

2.4 Exercises

1. **Basic Calculations.** Compute $\iint_S z^2\,dS$ where:
   (a) S is the upper part ($z \geq 0$) of the cone $z^2 = x^2+y^2$ cut by the

sphere $x^2 + y^2 + z^2 = R^2$.

(b) S is the conical surface parametrised by $x = r \sin\alpha \cos\theta, y = r \sin\alpha \sin\theta, z = r\cos\alpha$ for $0 \le r \le a, 0 \le \theta \le 2\pi$.

2. **Symmetry Exploitation.** Calculate

$$\iint_S (x + y + z)\, dS$$

over the upper unit hemisphere $x^2 + y^2 + z^2 = 1, z \ge 0$.

3. **Integration over the Sphere.** Evaluate

$$\iint_S (x + y + z)^2\, dS$$

where S is the unit sphere.

4. **Cone and Cylinder Intersection.** Find

$$\iint_S (x^4 - y^4 + y^2 z^2 - z^2 x^2 + 1)\, dS,$$

where S is the portion of the cone $z^2 = x^2 + y^2$ inside the cylinder $x^2 + y^2 = 2x$.

5. **Polyhedral and Parabolic Surfaces.** Compute $\iint_S |xyz|\, dS$ where:

(a) S is the octahedron $|x| + |y| + |z| = 1$.

(b) S is the part of the paraboloid $z = x^2 + y^2$ cut by the plane $z = 1$.

6. **Moment of Inertia Term.** Calculate

$$\iint_S (x^2 + y^2 + z^2)\, dS$$

where S is the boundary of the regular octahedron $|x| + |y| + |z| = a$.

7. **Parameter-Dependent Integral.** Let $f(x, y, z) = x^2 + y^2$ if $z \ge \sqrt{x^2 + y^2}$ and 0 otherwise. Calculate the function

$$F(t) = \iint_{x^2+y^2+z^2=t^2} f(x, y, z)\, dS.$$

8. **Potential of a Sphere.** Let S_t be the sphere of radius t centred at (x, y, z) (fixed, outside radius a). Let $f(\xi, \eta, \zeta) = 1$ inside the sphere $\xi^2 + \eta^2 + \zeta^2 < a^2$ and 0 outside. Calculate

$$F(t) = \iint_{S_t} f\, dS.$$

9. **Tetrahedral Surface.** Evaluate

$$\iint_S \frac{dS}{(1 + x + y)^2},$$

where S is the boundary of the tetrahedron $x + y + z \leq 1, x, y, z \geq 0$.

10. Viviani-Type Surface. Compute

$$\iint_S \frac{|x|}{z} \, dS,$$

where S is the part of the cylinder $x^2 + y^2 = 2ay$ cut out by $z = \sqrt{x^2 + y^2}$ and $z = 2a$.

11. Explicit Formula. Let Σ be given by $z = z(x, y)$ on domain D. Prove that:

$$\iint_\Sigma P \, dy \, dz + Q \, dz \, dx + R \, dx \, dy = \pm \iint_D (-Pz_x - Qz_y + R) \, dx \, dy.$$

Specify the sign for upper/lower orientations.

12. Symmetry Principles. Let Σ be symmetric about the xy-plane. Let Σ_1 be the upper part ($z > 0$).

(a) If $f(x, y, z) = -f(x, y, -z)$, prove $\iint_\Sigma f \, dS = 0$.

(b) If $R(x, y, z) = -R(x, y, -z)$, determine whether $\iint_\Sigma R \, dx \, dy$ is 0 or $2 \iint_{\Sigma_1} R \, dx \, dy$.

13. Projection Area. Let Σ be a planar region with area S and normal $\mathbf{n}$. If $\cos(\mathbf{n}, \mathbf{k}) = \mu$, prove the projected area is μS.

14. Flux through a Sphere. Compute

$$I_1 = \iint_\Sigma z \, dx \, dy$$

and

$$I_2 = \iint_\Sigma z^2 \, dx \, dy$$

for the sphere $x^2 + y^2 + z^2 = a^2$ with outward orientation. Explain the result of I_1 geometrically.

15. Gauss-Ostrogradsky Verification. Calculate the flux of $\mathbf{F} = (x, y, z)$ through the boundary of the cube $[0, 1]^3$ directly and compare with the volume integral of the sum of partial derivatives.

3
Gauss's Theorem

We now proceed to the three-dimensional analogue of Green's Formula. Gauss's Theorem establishes a fundamental link between a triple integral over a bounded region in $\mathbb{R}^3$ and a surface integral over its boundary.

3.1 *The Gauss Formula*

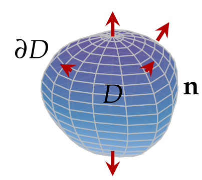


Figure 3.1: Gauss's Theorem: the outward flux of **F** through ∂D equals the integral of the sum of partial derivatives over D.

Let $D \subset \mathbb{R}^3$ be a bounded region whose boundary ∂D consists of a finite number of piecewise smooth closed orientable surfaces. We orient ∂D with the **outward** unit normal vector $\mathbf{n}$. Let P, Q, R be functions with continuous partial derivatives on $\bar{D}$.

> **Theorem 3.1.** ***Gauss's Formula.***
> The flux of the vector field $\mathbf{F} = (P, Q, R)$ across the boundary ∂D is equal to the triple integral of the sum of partial derivatives of P, Q, R over D:
>
> $$\iiint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx\,dy\,dz = \oiint_{\partial D} P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy.$$
>
> In terms of the surface integral of the first type, using the outward normal $\mathbf{n} = (\cos\alpha, \cos\beta, \cos\gamma)$:
>
> $$\iiint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dV = \oiint_{\partial D} (P\cos\alpha + Q\cos\beta + R\cos\gamma)\,dS.$$
>
> 定理

This result generalises *Green's Formula* to three dimensions. Just as Green's Formula relates domain integrals to boundary line integrals, Gauss's Formula relates volume integrals to boundary surface integrals. All surface integrals below are taken with the outward orientation unless explicitly stated otherwise.

> **Example 3.1.** Flux through a Surface of Revolution. Calculate the

surface integral

$$I = \iint_{\Sigma} 4xz\,dy\,dz - 2yz\,dz\,dx + (1 - z^2)\,dx\,dy,$$

where Σ is the surface of revolution generated by the curve $z = e^y$ $(0 \leq y \leq a)$ rotating around the z-axis. The surface is oriented via the "lower" side (the normal has a negative z-component).

範例

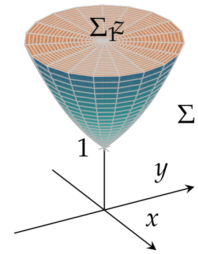


Figure 3.2: The region D enclosed by the surface of revolution Σ and the top disk Σ_1.

Solution

The equation of the surface is $z = e^{\sqrt{x^2+y^2}}$ for $x^2 + y^2 \leq a^2$. Let $P = 4xz$, $Q = -2yz$, and $R = 1 - z^2$. We observe that the sum of partial derivatives is zero:

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 4z - 2z - 2z = 0.$$

Direct calculation is cumbersome due to the parametrisation. Instead, we apply Gauss's Formula. The surface Σ is not closed. We close the region by adding the top disk Σ_1 at $z = e^a$ defined by $x^2 + y^2 \leq a^2$. Let D be the solid region bounded by Σ and Σ_1. The boundary $\partial D = \Sigma \cup \Sigma_1$. We must determine the orientation. The problem specifies the "lower" side of Σ. Since Σ forms the bottom/sides of the cup-shaped region D (described by $1 \leq z \leq e^a$ and $0 \leq r = \sqrt{x^2+y^2} \leq \ln z$ so that $z = e^r$), the outward normal to D points downwards on Σ. This matches the specified orientation. On Σ_1, the outward normal is $\mathbf{k}$ (upward). By *theorem 3.1*:

$$\iiint_D 0\,dV = \iint_{\Sigma} \mathbf{F} \cdot d\mathbf{S} + \iint_{\Sigma_1} \mathbf{F} \cdot d\mathbf{S}.$$

Thus $I = -\iint_{\Sigma_1} \mathbf{F} \cdot d\mathbf{S}$. On Σ_1, $z = e^a$ is constant, so $dz = 0$. The normal is $(0,0,1)$, so we project onto the xy-plane:

$$\iint_{\Sigma_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{x^2+y^2 \leq a^2} R(x, y, e^a)\,dx\,dy = \iint_{x^2+y^2 \leq a^2} (1 - e^{2a})\,dx\,dy.$$

$$= (1 - e^{2a}) \cdot \pi a^2.$$

Therefore:

$$I = -(1 - e^{2a})\pi a^2 = (e^{2a} - 1)\pi a^2.$$

■

3.2 *Singularities and Domain Excavation*

A powerful application of Gauss's Formula arises in calculating integrals of fields with singularities. If the field is undefined at a point

P_0 inside the closed surface, we cannot apply the theorem directly to the interior. Instead, we "excavate" the singularity by surrounding P_0 with a small sphere S_ε, applying the theorem to the region between the outer surface and S_ε.

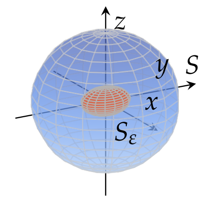


Figure 3.3: Excavation around a singularity: the region D_ε lies between the outer sphere S and the inner ellipsoid S_ε.

Example 3.2. Flux of an Anisotropic Field. Calculate the surface integral

$$I = \oiint_S \frac{x\,dy\,dz + y\,dz\,dx + z\,dx\,dy}{(ax^2 + by^2 + cz^2)^{3/2}},$$

where S is the unit sphere $x^2 + y^2 + z^2 = 1$ with the outward orientation, and $a, b, c > 0$.

範例

Solution

Let P, Q, R be the components of the integrand. The denominator vanishes at the origin, which lies inside S.

Method 1: Gauss's Formula (Excavation). We compute the partial derivative sum for $\mathbf{r} \neq \mathbf{0}$. Let $\rho = (ax^2 + by^2 + cz^2)^{1/2}$. Then $P = x\rho^{-3}$.

$$\frac{\partial P}{\partial x} = \rho^{-3} + x(-3\rho^{-4})\frac{\partial \rho}{\partial x} = \rho^{-3} - 3x\rho^{-4} \cdot \frac{1}{2}\rho^{-1}(2ax) = \rho^{-3} - 3ax^2\rho^{-5}.$$

Summing the partial derivatives:

$$3\rho^{-3} - 3\rho^{-5}(ax^2 + by^2 + cz^2) = 3\rho^{-3} - 3\rho^{-5}(\rho^2) = 0.$$

Since this sum is zero everywhere except the origin, the flux through S is equal to the flux through any small closed surface surrounding the origin. We choose a surface S_ε tailored to the symmetry of the denominator: the ellipsoid $ax^2 + by^2 + cz^2 = \varepsilon^2$. Let D_ε be the region between S and S_ε. By Gauss's Formula:

$$\oiint_S \mathbf{F} \cdot d\mathbf{S} - \oiint_{S_\varepsilon} \mathbf{F} \cdot d\mathbf{S} = \iiint_{D_\varepsilon} 0\,dV = 0.$$

(Note the sign is minus because the standard outward normal of D_ε on the inner boundary S_ε points towards the origin, while we define the integral over S_ε with the outward normal relative to the small ellipsoid itself). Thus $I = \oiint_{S_\varepsilon} \mathbf{F} \cdot d\mathbf{S}$. On S_ε, the denominator is $(\varepsilon^2)^{3/2} = \varepsilon^3$.

$$I = \frac{1}{\varepsilon^3} \oiint_{S_\varepsilon} x\,dy\,dz + y\,dz\,dx + z\,dx\,dy.$$

We apply Gauss's Formula *again* to the integral on the RHS, regarding it as an integral over the solid ellipsoid E_ε defined by

$ax^2 + by^2 + cz^2 \leq \varepsilon^2$. The integrand is $\mathbf{r} \cdot d\mathbf{S}$, so the sum of partial derivatives is 3.

$$\oiint_{S_\varepsilon} \mathbf{r} \cdot d\mathbf{S} = \iiint_{E_\varepsilon} 3\, dx\, dy\, dz = 3\,\mathrm{Vol}(E_\varepsilon).$$

The volume of the ellipsoid is $\frac{4\pi}{3} \frac{\varepsilon^3}{\sqrt{abc}}$.

$$I = \frac{1}{\varepsilon^3} \cdot 3 \cdot \frac{4\pi\varepsilon^3}{3\sqrt{abc}} = \frac{4\pi}{\sqrt{abc}}.$$

Method 2: Direct Parametrisation. Parametrise the unit sphere S by spherical coordinates (φ, θ).

$$x = \sin\varphi\cos\theta, \quad y = \sin\varphi\sin\theta, \quad z = \cos\varphi.$$

The vector area element matches the outward normal: $A = \sin^2\varphi\cos\theta$, $B = \sin^2\varphi\sin\theta$, $C = \sin\varphi\cos\varphi$. Substituting into the integral:

$$I = \int_0^{2\pi} \int_0^{\pi} \frac{xA + yB + zC}{(ax^2 + by^2 + cz^2)^{3/2}}\, d\varphi\, d\theta.$$

The numerator simplifies to $\sin\varphi$ (since $x^2 + y^2 + z^2 = 1$). The denominator term is $D(\varphi, \theta) = a\sin^2\varphi\cos^2\theta + b\sin^2\varphi\sin^2\theta + c\cos^2\varphi$.

$$I = 8\int_0^{\pi/2} d\theta \int_0^{\pi/2} \frac{\sin\varphi\, d\varphi}{(D(\varphi, \theta))^{3/2}}.$$

Let $u = \cos\varphi$, then $du = -\sin\varphi d\varphi$. The limits become $1 \to 0$. The denominator becomes $K\sin^2\varphi + c\cos^2\varphi = K(1-u^2) + cu^2 = K - (K-c)u^2$, where $K = a\cos^2\theta + b\sin^2\theta$.

$$\int_0^1 \frac{du}{[K - (K-c)u^2]^{3/2}} = \frac{1}{K\sqrt{c}}.$$

(Using the standard integral $\int_0^1 (A - Bt^2)^{-3/2} dt = \frac{1}{A\sqrt{A-B}}$). Here $A = K$ and $A - B = K - (K-c) = c$. Thus:

$$I = \frac{8}{\sqrt{c}} \int_0^{\pi/2} \frac{d\theta}{a\cos^2\theta + b\sin^2\theta} = \frac{8}{\sqrt{c}} \int_0^{\pi/2} \frac{\sec^2\theta\, d\theta}{a + b\tan^2\theta}.$$

Let $t = \tan\theta$.

$$I = \frac{8}{\sqrt{c}} \int_0^{\infty} \frac{dt}{a + bt^2} = \frac{8}{\sqrt{c}} \left[\frac{1}{\sqrt{ab}} \arctan\left(\sqrt{\frac{b}{a}} t \right) \right]_0^{\infty}.$$

$$I = \frac{8}{\sqrt{abc}} \cdot \frac{\pi}{2} = \frac{4\pi}{\sqrt{abc}}.$$

■

Note

Using Gauss's Formula for regions with singularities requires precise identification of the "hole" to be excised. The choice of the auxiliary surface (sphere vs. ellipsoid) can significantly simplify the subsequent calculation.

3.3 *Volume by Surface Integrals*

Just as Green's Formula yields a method for computing the area of a planar region via a line integral along its boundary, Gauss's Formula allows us to calculate the volume of a solid region using surface integrals.
Let $\Omega \subset \mathbb{R}^3$ be a bounded closed region with a piecewise smooth boundary $\partial\Omega$. Let $V(\Omega)$ denote its volume.

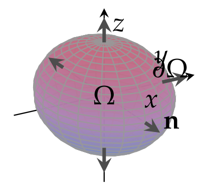


Figure 3.4: Volume via surface integral: $V = \frac{1}{3}\oiint_{\partial\Omega} \mathbf{r}\cdot\mathbf{n}\,dS$.

Corollary 3.1. *Volume Formulas*. The volume of Ω is given by the surface integrals over the boundary $\partial\Omega$, oriented with the outward normal $\mathbf{n} = (\cos\alpha, \cos\beta, \cos\gamma)$:

$$V(\Omega) = \oiint_{\partial\Omega} x\,dy\,dz = \oiint_{\partial\Omega} y\,dz\,dx = \oiint_{\partial\Omega} z\,dx\,dy.$$

Symmetrising these expressions yields the vector form (using the chosen outward orientation; take absolute value if a different orientation is used):

$$V(\Omega) = \frac{1}{3}\oiint_{\partial\Omega} x\,dy\,dz + y\,dz\,dx + z\,dx\,dy = \frac{1}{3}\oiint_{\partial\Omega} (\mathbf{r}\cdot\mathbf{n})\,dS.$$

推論

Proof

Apply Gauss's Formula (*theorem* 3.1) to the vector fields $\mathbf{F}_1 = (x, 0, 0)$, $\mathbf{F}_2 = (0, y, 0)$, and $\mathbf{F}_3 = (0, 0, z)$. For $\mathbf{F}_1$, the partial derivative sum is 1. Thus:

$$\iiint_{\Omega} 1\,dV = \oiint_{\partial\Omega} x\,dy\,dz.$$

The other identities follow similarly. Averaging the three results gives the symmetric form involving the sum of partial derivatives equal to 3.

■

Parametric Evaluation

When the boundary $\partial\Omega$ is given by a parametric representation $\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$ for $(u, v) \in D$, the symmetric

volume formula transforms into a determinant integral over the parameter domain.
Recall the Jacobians of the surface components:

$$A = \frac{\partial(y,z)}{\partial(u,v)}, \quad B = \frac{\partial(z,x)}{\partial(u,v)}, \quad C = \frac{\partial(x,y)}{\partial(u,v)}.$$

Substituting these into the relation $P\,dy\,dz + \cdots = (PA + QB + RC)\,du\,dv$:

$$V(\Omega) = \frac{1}{3}\left|\iint_D (xA + yB + zC)\,du\,dv\right|.$$

Expressing A, B, C explicitly as determinants yields a compact form involving the scalar triple product of the position vector and its tangents.

Proposition 3.1. ***Parametric Volume Formula.***

$$V(\Omega) = \frac{1}{3}\left|\iint_D \det(\mathbf{r}, \mathbf{r}_u, \mathbf{r}_v)\,du\,dv\right| = \frac{1}{3}\left|\iint_D \begin{vmatrix} x & y & z \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}\,du\,dv\right|.$$

命題

Volumes in Spherical Coordinates

A particularly useful application arises when the surface is defined by a radial function $r = r(\varphi, \theta)$ in spherical coordinates, where φ is the polar angle (colatitude) and θ is the azimuthal angle. The surface parametrisation is:

$$\mathbf{r}(\varphi, \theta) = r(\varphi, \theta)(\sin\varphi\cos\theta,\, \sin\varphi\sin\theta,\, \cos\varphi).$$

Corollary 3.2. *Spherical Volume.* If Ω is the star-shaped region $0 \le \rho \le r(\varphi, \theta)$ for $(\varphi, \theta) \in D$ with $r(\varphi, \theta) \ge 0$, then:

$$V(\Omega) = \frac{1}{3}\iint_D r^3(\varphi, \theta)\sin\varphi\,d\varphi\,d\theta.$$

推論

Proof

We can verify this directly via the surface integral or by triple integration.

Method 1: Surface Integral. We compute the determinant $\det(\mathbf{r}, \mathbf{r}_\varphi, \mathbf{r}_\theta)$. Let $\mathbf{e}_\rho$ be the radial unit vector. Then $\mathbf{r} = r\mathbf{e}_\rho$. Differentiation yields $\mathbf{r}_\varphi = r_\varphi\mathbf{e}_\rho + r\mathbf{e}_\varphi$ and $\mathbf{r}_\theta = r_\theta\mathbf{e}_\rho + r\sin\varphi\mathbf{e}_\theta$. The cross product is:

$$\mathbf{r}_\varphi \times \mathbf{r}_\theta = (r_\varphi\mathbf{e}_\rho + r\mathbf{e}_\varphi) \times (r_\theta\mathbf{e}_\rho + r\sin\varphi\mathbf{e}_\theta).$$

Ignoring terms with $\mathbf{e}_\rho \times \mathbf{e}_\rho = 0$, the only term with a radial component comes from $(r\mathbf{e}_\varphi) \times (r \sin\varphi \mathbf{e}_\theta) = r^2 \sin\varphi(\mathbf{e}_\varphi \times \mathbf{e}_\theta) = r^2 \sin\varphi \mathbf{e}_\rho$. Thus, the dot product with $\mathbf{r}$ is:

$$\mathbf{r} \cdot (\mathbf{r}_\varphi \times \mathbf{r}_\theta) = (r\mathbf{e}_\rho) \cdot (\cdots + r^2 \sin\varphi \mathbf{e}_\rho) = r^3 \sin\varphi.$$

Applying the parametric formula yields the result.

Method 2: Triple Integral. Integrating the volume element $\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$:

$$V = \iint_D \sin\varphi \, d\varphi \, d\theta \int_0^{r(\varphi,\theta)} \rho^2 \, d\rho = \iint_D \frac{r^3(\varphi,\theta)}{3} \sin\varphi \, d\varphi \, d\theta.$$

■

Example 3.3. Volume of a Cardioid of Revolution. Calculate the volume enclosed by the surface given in spherical coordinates by $r(\varphi,\theta) = a(1+\cos\varphi)$ (where $a > 0, 0 \le \varphi \le \pi$).

範例

Solution

Using the spherical volume formula with $D = [0,\pi] \times [0,2\pi]$:

$$V = \frac{1}{3}\int_0^{2\pi} d\theta \int_0^{\pi} a^3(1+\cos\varphi)^3 \sin\varphi \, d\varphi.$$

The θ integral gives 2π. For the φ integral, let $u = 1 + \cos\varphi$. Then $du = -\sin\varphi \, d\varphi$. Limits: $\varphi = 0 \implies u = 2$; $\varphi = \pi \implies u = 0$.

$$V = \frac{2\pi a^3}{3}\int_0^2 u^3 \, du = \frac{2\pi a^3}{3}\left[\frac{u^4}{4}\right]_0^2 = \frac{2\pi a^3}{3} \cdot \frac{16}{4} = \frac{8\pi a^3}{3}.$$

■

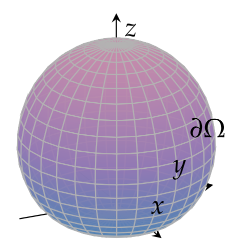


Figure 3.5: The cardioid of revolution $r = a(1 + \cos\varphi)$ in spherical coordinates. Its volume is $\frac{8\pi a^3}{3}$.

3.4 Exercises

1. **Flux through Simple Closed Surfaces.** Use Gauss's Formula to compute the following vector surface integrals.
   (a) $\oiint_S y(x-z)\,dy\,dz + z^2\,dz\,dx + (y^2+xz)\,dx\,dy$, where S is the surface of the cube $[0,a]^3$ with inner orientation.
   (b) $\oiint_\Sigma (x^3+x)\,dy\,dz + (y^2-xz)\,dz\,dx + (z^3+z)\,dx\,dy$, where Σ is the sphere $x^2+y^2+z^2 = 2z$ with outer orientation.
   (c) Let Σ be the surface of revolution obtained by rotating the region bounded by $z = 1 - y^2$ and $z = 0$ in the yz-plane

around the z-axis (outer orientation). Calculate the surface integral:

$$\oiint_{\Sigma} \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) dy\,dz + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) dz\,dx + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) dx\,dy,$$

where $A_1 = x^3 - x^2y + z^3, A_2 = xy^2 + y^3, A_3 = xz + z^2$.

2. **Open Surfaces and Auxiliary Caps.** Calculate the following integrals by closing the surface and applying Gauss's Formula.

   (a) $\iint_{\Sigma} (x^2 \cos\alpha + y^2 \cos\beta + z^2 \cos\gamma)\, dS$, where Σ is the conical surface $z^2 = x^2 + y^2$ for $0 \le z \le h$, with downward normal $(\cos\alpha, \cos\beta, \cos\gamma)$.

   (b) $\iint_{\Sigma} x^3\, dy\,dz + y^3\, dz\,dx + z^3\, dx\,dy$, where Σ is the upper hemisphere $x^2 + y^2 + z^2 = a^2$ $(z \ge 0)$ with upper orientation.

   (c) $\iint_{\Sigma} \left(\frac{x^3}{a^3} + y^3z^3 \right) dy\,dz + \left(\frac{y^3}{b^3} + z^3x^3 \right) dz\,dx + \left(\frac{z^3}{c^3} + x^3y^3 \right) dx\,dy,$ where Σ is the part of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ with $x \ge 0$, oriented towards negative x.

3. **Volume of a Cone.** Let Σ be a conical surface $F(x, y, z) = 0$ with vertex at the origin. Let Π be the plane $Ax + By + Cz = D$. Prove that the volume of the cone formed by Σ and Π is $V = \frac{1}{3}SH$, where S is the base area on Π and H is the perpendicular height from the origin to Π.

   > *Remark.*
   >
   > Use the vector volume formula $V = \frac{1}{3} \iint \mathbf{r} \cdot \mathbf{n}\, dS$ and consider the contribution from the lateral surface.

4. **Volume of a Lemniscate Surface.** Find the volume of the solid enclosed by the surface $(x^2 + y^2 + z^2)^2 = a^2xy$.

   > *Remark.*
   >
   > Use spherical coordinates.

5. **Flux on a Hyperboloid.** Calculate $\iint_{\Sigma} (x^3 + y^3)\, dy\,dz + (x^3 + 2x^2y)\, dz\,dx - x^2z\, dx\,dy$, where Σ is the portion of the hyperboloid $x^2 + y^2 - z^2 = 1$ between $z = 0$ and $z = \sqrt{3}$, oriented outwards.

6. **Paraboloid Flux.** Let $V = \{(x, y, z) \mid x^2 + y^2 < z < 1\}$ and $S = \partial V$. Calculate the outward flux:

$$\oiint_{S} yz\, dz\,dx + (x^2 + y^2)z\, dx\,dy.$$

7. **Mixed Flux Integral.** Evaluate the surface integral

$$\oiint_S z\, dy\, dz + \cos y\, dz\, dx + dx\, dy$$

over the outer side of the unit sphere $x^2 + y^2 + z^2 = 1$.

4
Stokes' Theorem

In the previous chapters, we established Green's Formula, which relates a line integral along a simple closed curve in the plane to a double integral over the enclosed region. We also developed Gauss's Formula, linking surface flux to volume integrals. We now complete this triad of fundamental theorems with Stokes' Formula (often called Stokes' Theorem). This result generalises Green's Formula to oriented surfaces in $\mathbb{R}^3$, providing a profound connection between the circulation of a vector field along a boundary curve and a surface integral involving its partial derivatives.

4.1 The Stokes Formula

Let Σ be a piecewise smooth oriented surface in $\mathbb{R}^3$, bounded by a piecewise smooth, simple closed curve $\partial\Sigma$. We adopt the **right-hand rule** convention for orientation: if one's right hand curls in the direction of the traversal of $\partial\Sigma$, the thumb points in the direction of the unit normal vector $\mathbf{n}$ of Σ.
Let P, Q, R be functions with continuous partial derivatives on a region containing Σ.

Theorem 4.1. ***Stokes' Formula.***
The line integral of the vector field $\mathbf{F} = (P, Q, R)$ along the boundary $\partial\Sigma$ is equal to the surface integral of the determinant of partial derivatives over Σ. In coordinate form:

$$\oint_{\partial\Sigma} P\,dx + Q\,dy + R\,dz = \iint_{\Sigma} \begin{vmatrix} dy\,dz & dz\,dx & dx\,dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}.$$

Using the relation between surface integrals of the first and second types (*proposition* 2.1), this may be written in terms of the directional cosines

$(\cos\alpha, \cos\beta, \cos\gamma)$ of the normal $\mathbf{n}$:

$$\oint_{\partial\Sigma} \mathbf{F}\cdot d\mathbf{r} = \iint_{\Sigma} \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} dS.$$

The integrand is composed of the terms:

$$\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\cos\alpha + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\cos\beta + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\cos\gamma.$$

定理

Note

If the surface Σ lies entirely in the xy-plane, the normal is $\mathbf{k} = (0,0,1)$. The determinant simplifies to $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$, and Stokes' Formula reduces directly to Green's Formula (*theorem* 1.1).

Example 4.1. Cube Section Circulation. Calculate the circulation

$$I = \oint_C (y^2 - z^2)\,dx + (z^2 - x^2)\,dy + (x^2 - y^2)\,dz,$$

where C is the intersection of the boundary of the cube $\Omega = \{0 \le x, y, z \le a\}$ and the plane $x + y + z = \frac{3}{2}a$. The orientation of C is counter-clockwise when viewed from the positive z-axis.

範例

Solution

Calculating the integral directly would require parametrising the six segments of the hexagonal intersection shown in *figure* 4.1. Instead, we apply Stokes' Formula. Let Σ be the planar region enclosed by C on the plane $x + y + z = \frac{3}{2}a$. The normal vector to the plane is $(1,1,1)$. Normalising gives $\mathbf{n} = \frac{1}{\sqrt{3}}(1,1,1)$. Because $\mathbf{n}$ has positive z-component, the stated "counter-clockwise when viewed from $+z$" boundary orientation agrees with the right-hand rule for Stokes.

We compute the terms for the surface integral from $\mathbf{F} = (y^2 - z^2, z^2 - x^2, x^2 - y^2)$:

$$\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) = \frac{\partial}{\partial y}(x^2 - y^2) - \frac{\partial}{\partial z}(z^2 - x^2) = -2y - 2z,$$
$$\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) = \frac{\partial}{\partial z}(y^2 - z^2) - \frac{\partial}{\partial x}(x^2 - y^2) = -2z - 2x,$$
$$\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \frac{\partial}{\partial x}(z^2 - x^2) - \frac{\partial}{\partial y}(y^2 - z^2) = -2x - 2y.$$

The normal vector is $\mathbf{n} = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. The integrand is the dot product of these terms with $\mathbf{n}$:

$$\frac{1}{\sqrt{3}}[(-2y-2z)+(-2z-2x)+(-2x-2y)] = -\frac{2}{\sqrt{3}}(2x+2y+2z) = -\frac{4}{\sqrt{3}}(x+y+z).$$

On the surface Σ, we have $x+y+z = \frac{3}{2}a$. Therefore, the integrand is constant:

$$-\frac{4}{\sqrt{3}}\left(\frac{3}{2}a\right) = -2\sqrt{3}a.$$

The integral becomes:

$$I = \iint_{\Sigma} -2\sqrt{3}a\, dS = -2\sqrt{3}a \cdot \text{Area}(\Sigma).$$

The intersection of the cube with this plane is a regular hexagon. The distance from the origin to the plane is $h = \frac{3a/2}{\sqrt{3}} = \frac{\sqrt{3}}{2}a$, which passes through the centre of the cube. The hexagon vertices are the midpoints of the cube edges. The side length is $s = \frac{a}{\sqrt{2}}$. The area of a regular hexagon is $\frac{3\sqrt{3}}{2}s^2 = \frac{3\sqrt{3}}{2}\left(\frac{a^2}{2}\right) = \frac{3\sqrt{3}}{4}a^2$. Substituting this area:

$$I = -2\sqrt{3}a \cdot \left(\frac{3\sqrt{3}}{4}a^2\right) = -\frac{9}{2}a^3.$$

■

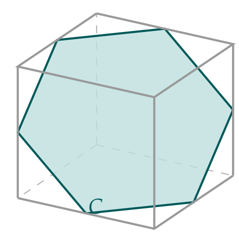


Figure 4.1: The hexagonal intersection C of the plane $x+y+z = \frac{3}{2}a$ and the cube.

Example 4.2. Stokes' Theorem on an Intersection Curve. Use Stokes' formula to calculate

$$I = \oint_C (y^2+z^2)\,dx + (z^2+x^2)\,dy + (x^2+y^2)\,dz,$$

where C is the intersection of the sphere $x^2+y^2+z^2 = 2Rx$ and the cylinder $x^2+y^2 = 2rx$ $(0 < r < R, z > 0)$. The boundary orientation is the one induced by the outward normal of the spherical cap via the right-hand rule (equivalently, traversing C keeps the smaller cap on the left).

範例

Solution

Let Σ be the portion of the sphere $x^2+y^2+z^2 = 2Rx$ lying inside the cylinder, oriented with the outward normal. The boundary $\partial\Sigma$ corresponds to the curve C with the specified orientation.

The vector field is $\mathbf{F} = (y^2+z^2, z^2+x^2, x^2+y^2)$. The terms for the surface integral are:

$$\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right).$$

Substituting the components:

$$(2y - 2z, 2z - 2x, 2x - 2y).$$

The surface Σ is part of the sphere $(x-R)^2 + y^2 + z^2 = R^2$. The unit outward normal vector $\mathbf{n}$ is:

$$\mathbf{n} = \frac{1}{R}(x - R, y, z).$$

We evaluate the integrand for Stokes' formula:

$$\begin{aligned} &\frac{1}{R}\left[(2y-2z)(x-R) + (2z-2x)y + (2x-2y)z\right] \\ =& \frac{2}{R}\left[(xy - yR - zx + zR) + (yz - xy) + (xz - yz)\right]. \end{aligned}$$

Upon expansion, the terms xy, zx, and yz cancel:

$$\frac{2}{R}(zR - yR) = 2(z - y).$$

Thus, the integral becomes:

$$I = \iint_\Sigma 2(z-y)\,dS = 2\iint_\Sigma z\,dS - 2\iint_\Sigma y\,dS.$$

The surface Σ and the domain D ($x^2 + y^2 \leq 2rx$) are symmetric with respect to the plane $y = 0$. Since the function $f(x,y,z) = y$ is odd with respect to y, the integral $\iint_\Sigma y\,dS$ vanishes. We are left with:

$$I = 2\iint_\Sigma z\,dS.$$

For the sphere $(x-R)^2 + y^2 + z^2 = R^2$, projecting to the xy-plane gives $dS = \frac{R}{z}\,dx\,dy$ (standard Jacobian for $z = \sqrt{2Rx - x^2 - y^2}$); multiplying by z yields the handy identity $z\,dS = R\,dx\,dy$. Therefore, we reduce the surface integral to a double integral over the projection domain D:

$$I = 2\iint_D z\left(\frac{R}{z}\right)\,dx\,dy = 2R\iint_D dx\,dy = 2R \cdot \text{Area}(D).$$

The domain D is the disk $x^2 + y^2 \leq 2rx$, which has radius r and area πr^2.

$$I = 2R(\pi r^2) = 2\pi r^2 R.$$

■

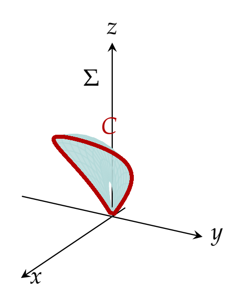


Figure 4.2: The surface Σ on the sphere cut by the cylinder, bounded by the curve C.

4.2 Theoretical Consequences

Stokes' Formula allows us to prove general properties of vector fields on closed surfaces.

Proposition 4.1. ***Closed Surface Integral.***
Let Σ be a piecewise smooth closed surface enclosing a volume Ω. If $\mathbf{F} = (P, Q, R)$ is a vector field with continuous partial derivatives, then:

$$\oiint_{\Sigma} \begin{vmatrix} \cos(\mathbf{n}, x) & \cos(\mathbf{n}, y) & \cos(\mathbf{n}, z) \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} dS = 0.$$

命題

We present two proofs to illustrate the consistency of vector calculus.

Gauss's Formula

Assume $\mathbf{F}$ is twice continuously differentiable. Apply Gauss's Formula (*theorem* 3.1). The integrand is:

$$\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\cos\alpha + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\cos\beta + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\cos\gamma.$$

Applying Gauss's Formula converts this surface integral into a triple integral over Ω:

$$\iiint_{\Omega} \left[\frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right] dx\,dy\,dz.$$

By symmetry of mixed partial derivatives (Schwarz's Theorem), terms cancel (e.g., $\frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 R}{\partial y \partial x} = 0$). The volume integral is identically zero.

証明終

Stokes' Formula (Splitting Argument)

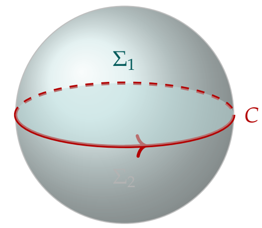
This method requires only first-order derivatives on Σ. Divide the closed surface Σ into two patches Σ_1 and Σ_2 by introducing a simple closed curve C on Σ. Orient Σ with the outward normal $\mathbf{n}$. Apply Stokes' Formula to each patch:

$$\iint_{\Sigma_1} (\dots)\, dS = \oint_{\partial\Sigma_1} P\,dx + Q\,dy + R\,dz,$$

$$\iint_{\Sigma_2} (\dots)\, dS = \oint_{\partial\Sigma_2} P\,dx + Q\,dy + R\,dz.$$

Note that the boundaries $\partial\Sigma_1$ and $\partial\Sigma_2$ are the same curve C, but their induced orientations are opposite (see *figure* 4.3). Let C be oriented consistently with Σ_1. Then $\partial\Sigma_2$ is traversed in the reverse direction, so the second integral is the negative of the first. Summing the two integrals gives zero.

証明終



$\partial\Sigma_1 = C, \partial\Sigma_2 = -C$

Figure 4.3: Splitting a closed surface Σ into Σ_1 and Σ_2 along curve C. The induced orientations on C cancel.

4.3 *Conditions for Path Independence in Space*

We now extend the conditions for path independence of line integrals, previously established for planar regions, to three-dimensional space. The result relies fundamentally on Stokes' Formula, which links the circulation of a field to the derivatives of its components. However, the validity of this extension depends on the topological nature of the domain.

Simply Connected Regions in $\mathbb{R}^3$

In the plane, a region is simply connected if it contains no "holes". In $\mathbb{R}^3$, the concept is slightly more subtle. For Stokes' Formula to imply that vanishing partial derivative terms lead to a vanishing circulation, we require that every closed curve C in the domain Ω bounds a surface Σ that lies entirely within Ω.

Definition 4.1. ***Surface Simply Connected Region.***
A region $\Omega \subset \mathbb{R}^3$ is said to be **surface simply connected** (or simply connected) if for every piecewise smooth simple closed curve $C \subset \Omega$, there exists a piecewise smooth orientable surface $\Sigma \subset \Omega$ such that $\partial\Sigma = C$.

定義

Example 4.3. Topological Examples.

1. **Concentric Spheres:** The region between two concentric spheres (a spherical shell) is surface simply connected. Any closed loop within the shell can be "shrunk" or spanned by a surface without hitting the central void.
2. **Cylindrical Hole:** The region $\Omega = \{(x, y, z) \in \mathbb{R}^3 \mid 1 < x^2 + y^2\}$ (space with a cylinder removed) is *not* simply connected. A loop encircling the z-axis cannot be spanned by a surface within Ω.
3. **Torus:** The interior of a torus is not simply connected.

範例

Equivalence of Conditions

Theorem 4.2. ***Conservative Fields in Space.***
Let Ω be a surface simply connected region in $\mathbb{R}^3$. Let $\mathbf{F} = (P, Q, R)$

be a vector field with continuous partial derivatives on Ω. The following conditions are equivalent:

1. **Irrotational Field:** The cross-partial derivatives are equal everywhere in Ω:
$$\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.$$
2. **Zero Circulation:** For any piecewise smooth closed curve $C \subset \Omega$, $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$.
3. **Path Independence:** For any path $L \subset \Omega$ from A to B, the integral $\int_L \mathbf{F} \cdot d\mathbf{r}$ depends only on the endpoints A, B.
4. **Exact Differential:** There exists a scalar potential φ on Ω such that $\mathbf{F} = \nabla\varphi$ (i.e., $d\varphi = P\,dx + Q\,dy + R\,dz$).

定理

Proof

We outline the cyclic implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)$.

$(1) \Rightarrow (2)$: Let C be a closed curve. Since Ω is surface simply connected, there exists a surface $\Sigma \subset \Omega$ with $\partial\Sigma = C$. By Stokes' Formula, the integrand involving the differences of partial derivatives vanishes (since they are equal), so:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_\Sigma 0\,dS = 0.$$

$(2) \iff (3)$: Standard argument identical to the planar case (see *theorem* 1.2).

$(3) \Rightarrow (4)$: Fix a base point $M_0(x_0, y_0, z_0)$. Define $\varphi(M) = \int_{M_0}^{M} \mathbf{F} \cdot d\mathbf{r}$. Since the integral is path-independent, φ is well-defined. Differentiating φ with respect to x, y, z recovers P, Q, R.

$(4) \Rightarrow (1)$: If $\mathbf{F} = \nabla\varphi$, then the mixed partials are equal (e.g., $\frac{\partial P}{\partial y} = \frac{\partial^2\varphi}{\partial y\partial x} = \frac{\partial^2\varphi}{\partial x\partial y} = \frac{\partial Q}{\partial x}$).

■

When these conditions are met, the potential function $\varphi(x, y, z)$ can be recovered by integrating along a piecewise linear path parallel to the axes from (x_0, y_0, z_0) to (x, y, z):

$$\varphi(x, y, z) = \int_{x_0}^{x} P(t, y_0, z_0)\,dt + \int_{y_0}^{y} Q(x, t, z_0)\,dt + \int_{z_0}^{z} R(x, y, t)\,dt + C.$$

Example 4.4. Recovering the Potential. Consider the differential form

$$\omega = z\left(\frac{1}{x^2y} - \frac{1}{x^2+z^2}\right)dx + \frac{z}{xy^2}dy + \left(\frac{x}{x^2+z^2} - \frac{1}{xy}\right)dz.$$

Determine if a potential exists, and if so, find it.

範例

Solution

Let P, Q, R be the coefficients of dx, dy, dz. We verify the equality of partial derivatives on the domain $x, y > 0$.

1. $\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(zx^{-1}y^{-2}) = -zx^{-2}y^{-2}$. $\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(zx^{-2}y^{-1}) = -zx^{-2}y^{-2}$. (Match)
2. $\frac{\partial R}{\partial y} = \frac{\partial}{\partial y}(-x^{-1}y^{-1}) = x^{-1}y^{-2}$. $\frac{\partial Q}{\partial z} = \frac{\partial}{\partial z}(z(xy^2)^{-1}) = (xy^2)^{-1}$. (Match)
3. $\frac{\partial P}{\partial z} = \frac{1}{x^2y} - \frac{1(x^2+z^2)-z(2z)}{(x^2+z^2)^2} = \frac{1}{x^2y} - \frac{x^2-z^2}{(x^2+z^2)^2}$. $\frac{\partial R}{\partial x} = \frac{1(x^2+z^2)-x(2x)}{(x^2+z^2)^2} - (-x^{-2}y^{-1}) = \frac{z^2-x^2}{(x^2+z^2)^2} + \frac{1}{x^2y}$. (Match)

Since the cross-partials match, a potential φ exists. We compute it using two methods.

Path Integration. We integrate from $(x_0, y_0, 0)$ to (x, y, z).

We choose $z_0 = 0$ to simplify terms involving z. Path: $L_1 : (x_0, y_0, 0) \to (x, y_0, 0)$; $L_2 :\to (x, y, 0)$; $L_3 :\to (x, y, z)$.

- Along L_1 ($y = y_0, z = 0, dz = 0, dy = 0$): $P(x, y_0, 0) = 0(\dots) = 0$. Integral is 0.
- Along L_2 ($x = x, z = 0, dz = 0, dx = 0$): $Q(x, y, 0) = 0/(xy^2) = 0$. Integral is 0.
- Along L_3 ($x = x, y = y$ fixed, z varies): Integrand is $R(x, y, z) = \frac{x}{x^2+z^2} - \frac{1}{xy}$.

$$\varphi = \int_0^z \left(\frac{x}{x^2+t^2} - \frac{1}{xy}\right)dt = \left[\arctan\frac{t}{x} - \frac{t}{xy}\right]_0^z = \arctan\frac{z}{x} - \frac{z}{xy}.$$

Thus $\varphi(x, y, z) = \arctan\frac{z}{x} - \frac{z}{xy} + C$.

Indefinite Integration. Since $\frac{\partial\varphi}{\partial z} = R$, we integrate R with respect to z:

$$\varphi = \int\left(\frac{x}{x^2+z^2} - \frac{1}{xy}\right)dz = \arctan\frac{z}{x} - \frac{z}{xy} + \psi(x, y).$$

Now differentiate with respect to y:

$$\frac{\partial\varphi}{\partial y} = -\frac{z}{x}(-y^{-2}) + \frac{\partial\psi}{\partial y} = \frac{z}{xy^2} + \frac{\partial\psi}{\partial y}.$$

Equating to $Q = \frac{z}{xy^2}$ implies $\frac{\partial \psi}{\partial y} = 0$, so $\psi = \psi(x)$. Differentiate with respect to x:

$$\frac{\partial \varphi}{\partial x} = \frac{1}{1+(z/x)^2}\left(-\frac{z}{x^2}\right) - z(-x^{-2}y^{-1}) + \psi'(x) = \frac{-z}{x^2+z^2} + \frac{z}{x^2 y} + \psi'(x).$$

Equating to P shows $\psi'(x) = 0$. Thus ψ is a constant.

$$\varphi(x,y,z) = \arctan\frac{z}{x} - \frac{z}{xy} + C.$$

■

4.4 Exercises

1. **Exact Differentials.** Prove that the following forms are exact and find their primitives:
   (a) $(x^2 - 2yz)\,dx + (y^2 - 2xz)\,dy + (z^2 - 2xy)\,dz$.
   (b) $\left[\frac{x}{(x^2-y^2)^2} - \frac{1}{x} + 2x^2\right] dx + \left[\frac{1}{y} - \frac{y}{(x^2-y^2)^2} + 3y^3\right] dy + 5z^3\,dz$.

2. **Path Independence Calculation.** Evaluate

$$\int_{(1,2,3)}^{(6,1,1)} yz\,dx + xz\,dy + xy\,dz.$$

3. **Radial Field Work.** Let C be any piecewise smooth path from a point on the sphere $r = a$ to a point on the sphere $r = b$ ($b > a$). Prove:

$$\int_C r^3(x\,dx + y\,dy + z\,dz) = \frac{1}{5}(b^5 - a^5).$$

4. **Plane Area via Determinant.** Let C be a simple closed curve on the plane $x\cos\alpha + y\cos\beta + z\cos\gamma = p$, enclosing an area S. If the orientation of C and the normal vector $(\cos\alpha, \cos\beta, \cos\gamma)$ form a right-handed system, show that:

$$\oint_C \begin{vmatrix} dx & dy & dz \\ \cos\alpha & \cos\beta & \cos\gamma \\ x & y & z \end{vmatrix} = 2S.$$

5. **Intersection Circulation.** Calculate

$$\oint_C (y-z)\,dx + (z-x)\,dy + (x-y)\,dz,$$

where C is the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 1$, oriented counter-clockwise when viewed from the positive x-axis.

6. **Path Integral on a Cone.** Find

$$\int_C (z^3 + 3x^2y)\, dx + (x^3 + 3y^2z)\, dy + (y^3 + 3z^2x)\, dz,$$

where C is the intersection of the cone $z = \sqrt{a^2 - x^2 - y^2}$ and the plane $x = y$, from $A(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}, 0)$ to $B(-\frac{a}{\sqrt{2}}, -\frac{a}{\sqrt{2}}, 0)$. Note that the path goes over the apex of the cone.

7. **Stokes on an Open Surface.** Use Stokes' formula to compute:

$$\int_C e^{x+z}\{[(x+1)y^2 + 1]\, dx + 2xy\, dy + xy^2\, dz\},$$

where C is the intersection arc of the half-cylinder $|x| + |y| = a$ ($y > 0$) and the plane $y = z$ from $(-a, 0, 0)$ to $(a, 0, 0)$.

8. **Vanishing Circulation.** Let C be any piecewise smooth simple closed curve. Let f, g, h be continuous functions. Prove that:

$$\oint_C [f(x) - yz]\, dx + [g(y) - xz]\, dy + [h(z) - xy]\, dz = 0.$$

9. **Surface Integral.** Calculate

$$\iint_\Sigma \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x - z & x^3 - yz & -3xy^2 \end{vmatrix} dS,$$

where Σ is the upper hemisphere $x^2 + y^2 + z^2 = R^2$ ($z \geq 0$) with the *lower* (inner) normal orientation.

10. **Great Circle Circulation.** Find

$$\oint_C y\, dx + z\, dy + x\, dz,$$

where C is the great circle formed by $x^2 + y^2 + z^2 = a^2$ and $x + y + z = 0$, oriented counter-clockwise viewed from the positive z-axis.

5
Outer Product of Vectors, Exterior Differentiation, and the General Stokes Formula

Although differential forms and exterior differentiation may not appear in every standard curriculum, they have become fundamental tools in modern analysis. Their concise expression and structure provide significant convenience when addressing fundamental problems in calculus.

5.1 Outer Product of Vectors

We begin by considering two linearly independent vectors $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ in the plane $\mathbb{R}^2$. Let Π be the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.
We stipulate an orientation for this area: when $\mathbf{a}$ rotates counterclockwise to $\mathbf{b}$, the area of the parallelogram is positive; otherwise, it is negative. From analytic geometry, the signed area of Π under this definition is given by the second-order determinant:

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

Definition 5.1. ***Outer Product in*** $\mathbb{R}^2$**.**
The **outer product** of vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as:

$$\mathbf{a} \wedge \mathbf{b} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

定義

Note
Here $\wedge$ denotes the determinant (oriented area) of two vectors. Later, the same symbol is used for the wedge product of differential forms; context distinguishes the two, but both encode orientation.

Proposition 5.1. ***Properties of the Outer Product.***
The outer product operation satisfies the following properties for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$:

1. **Antisymmetry**: $\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$. It follows immediately that $\mathbf{a} \wedge \mathbf{a} = 0$.

2. **Linear Distribution Law**:

$$\begin{aligned} \mathbf{a} \wedge (\mathbf{b} + \mathbf{c}) &= \mathbf{a} \wedge \mathbf{b} + \mathbf{a} \wedge \mathbf{c}, \\ (\mathbf{a} + \mathbf{b}) \wedge \mathbf{c} &= \mathbf{a} \wedge \mathbf{c} + \mathbf{b} \wedge \mathbf{c}, \\ (\lambda \mathbf{a}) \wedge \mathbf{b} &= \mathbf{a} \wedge (\lambda \mathbf{b}) = \lambda(\mathbf{a} \wedge \mathbf{b}). \end{aligned}$$

命題

Proof

These properties follow directly from the algebraic properties of the determinant.

1. **Antisymmetry:** Swapping the columns of a determinant changes its sign:

$$\mathbf{b} \wedge \mathbf{a} = \begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix} = - \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = -(\mathbf{a} \wedge \mathbf{b}).$$

2. **Linearity:** The determinant is linear in each of its columns. For the first distribution law:

$$\mathbf{a} \wedge (\mathbf{b} + \mathbf{c}) = \begin{vmatrix} a_1 & a_2 \\ b_1 + c_1 & b_2 + c_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} = \mathbf{a} \wedge \mathbf{b} + \mathbf{a} \wedge \mathbf{c}.$$

   Scalar multiplication follows similarly from factoring constants out of rows or columns.

■

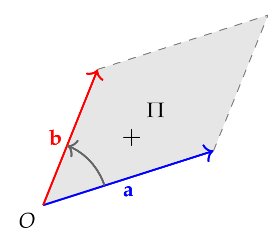


Figure 5.1: The parallelogram Π spanned by **a** and **b**. The outer product is positive when the rotation from **a** to **b** is counter-clockwise.

We extend this definition to higher dimensions.

Definition 5.2. ***Outer Product in*** $\mathbb{R}^n$.
Let $\mathbf{a}_i = (a_{i1}, a_{i2}, \ldots, a_{in})$ for $i = 1, 2, \ldots, n$ be vectors in $\mathbb{R}^n$. The **outer product** is defined as the determinant:

$$\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \cdots \wedge \mathbf{a}_n = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix}.$$

定義

Proof

The determinant is an alternating multilinear map in its rows (or columns): it is linear in each row separately and changes sign when two rows are interchanged. Both facts are standard consequences of Laplace expansion. Since the n-vector outer product is exactly this determinant, the two-dimensional properties in *proposition* 5.1 carry over verbatim: swapping any two $\mathbf{a}_i$ reverses the sign (antisymmetry), and each row is linear in the corresponding vector (multilinearity).

■

In the specific case where $n = 3$, the outer product of three linearly independent vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ represents the oriented volume of the parallelepiped having these vectors as edges.

- When $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ form a **right-handed system**, the volume is positive.
- Otherwise, the volume is negative.

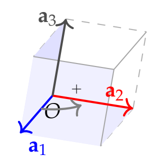


Figure 5.2: The parallelepiped spanned by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ in $\mathbb{R}^3$. The outer product gives positive volume when the vectors form a right-handed system.

5.2 *Differential Forms*

The geometric intuition of the outer product allows us to construct a rigorous algebraic framework for integration and differentiation in higher dimensions. We begin by revisiting the total differential of a continuously differentiable function $f : U \to \mathbb{R}$ on a region $U \subset \mathbb{R}^n$. Recall that the total differential is given by:

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$

Here, the differentials $dx_1, dx_2, \ldots, dx_n$ are traditionally viewed as independent increments of the variables. In the language of differential forms, we reinterpret these dx_i as basis vectors of a linear space, independent of the specific values of x. The differential df is thus a vector in the space spanned by this basis.

First-Order Differential Forms

We formalise the space of these objects. Let $U \subseteq \mathbb{R}^n$ be a region and let $C^k(U)$ denote the set of k-times continuously differentiable functions on U.

Definition 5.3. ***1-Form.***
A **first-order differential form** (or simply a **1-form**) on U is an expression of the type:

$$\omega = a_1(x)\, dx_1 + a_2(x)\, dx_2 + \cdots + a_n(x)\, dx_n,$$

where the coefficients $a_i(x)$ are continuous functions on U (i.e., $a_i \in$

$C^0(U)$).

定義

The set of all such 1-forms on U is denoted by $\Lambda^1(U)$.

Note

If the coefficient functions $a_i(x)$ belong to $C^k(U)$ for some integer $k \geq 1$, we say the form is of class C^k. Under standard pointwise addition and scalar multiplication by functions in $C^0(U)$, $\Lambda^1(U)$ forms a linear space (specifically, a module) over the ring of continuous functions.

Higher-Order Differential Forms

To define forms of higher degree, we construct a new basis by taking the outer product of the differentials dx_i. We denote the outer product of dx_i and dx_j by the symbol $\wedge$ (read as "wedge"). Consistent with the properties of the outer product of vectors derived in the previous section, we impose the following algebraic rules:

1. **Antisymmetry**: $dx_i \wedge dx_j = -dx_j \wedge dx_i$ for all i, j.
2. **Vanishing Property**: $dx_i \wedge dx_i = 0$ for all i.

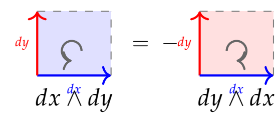


Figure 5.3: Antisymmetry of the wedge product: swapping the order reverses the orientation.

2-Forms

From the set of differentials $\{dx_1, \ldots, dx_n\}$, we can form ordered pairs $dx_i \wedge dx_j$. Due to antisymmetry, we only require basis elements where the indices are strictly increasing.

Definition 5.4. ***2-Form.***
A **second-order differential form** (or **2-form**) is an element of the linear space $\Lambda^2(U)$ spanned by the basis elements $\{dx_i \wedge dx_j \mid 1 \leq i < j \leq n\}$. The **standard form** of a 2-form is:

$$\omega = \sum_{1 \leq i < j \leq n} g_{ij}(x)\, dx_i \wedge dx_j,$$

where $g_{ij}(x)$ are function coefficients.

定義

The dimension of this basis over the function space is the binomial coefficient $\binom{n}{2}$.

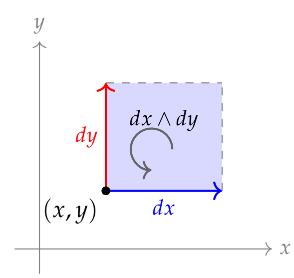


Figure 5.4: Visualisation of the basis 2-form $dx \wedge dy$ in $\mathbb{R}^2$ as an oriented area element.

General p-Forms

We generalise this construction to arbitrary order p. A basis element is formed by the wedge product of p differentials:

$$dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p}.$$

The antisymmetry rule extends to these products: swapping any two adjacent terms changes the sign of the product.

$$\cdots \wedge dx_u \wedge dx_v \wedge \cdots = -(\cdots \wedge dx_v \wedge dx_u \wedge \dots).$$

Consequently, if any index is repeated (i.e., $i_r = i_s$ for $r \neq s$), the entire product vanishes.

Definition 5.5. ***p*-*Form*.**
A **p-th order differential form** (or **p-form**) is an element of the linear space $\Lambda^p(U)$ with basis elements:

$$\{dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p} \mid 1 \leq i_1 < i_2 < \cdots < i_p \leq n\}.$$

The **standard form** is given by:

$$\omega = \sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} g_{i_1 i_2 \dots i_p}(x)\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p}.$$

定義

There are $\binom{n}{p}$ such basis elements. We note two important boundary cases:

1. **Top-dimensional forms (Λ^n):** Since there is only one way to choose n distinct indices from n possibilities (up to permutation), the space Λ^n is 1-dimensional over the functions. Any n-form can be written as:
$$\omega = g(x)\, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n.$$
2. **Zero forms (Λ^0):** By convention, a 0-form is simply a scalar function on U. Thus $\Lambda^0(U) = C^0(U)$, and the function $g(x) \equiv 1$ serves as a basis.
3. **Vanishing forms ($p > n$):** If $p > n$, any product of p differentials from a set of n must contain a repetition (by the Pigeonhole Principle). Thus, $dx_{i_1} \wedge \cdots \wedge dx_{i_p} = 0$, and $\Lambda^p = \{0\}$.

5.3 *Outer Product of Differential Forms*

Having defined the spaces Λ^p, we now construct the algebra of differential forms by introducing the outer product operation on the direct sum space $\Lambda = \Lambda^0 + \Lambda^1 + \cdots + \Lambda^n$. This space has dimension $\sum_{k=0}^{n} C_n^k = 2^n$. Any element $\omega \in \Lambda$ can be written as $\omega = \omega_0 + \cdots + \omega_n$, where $\omega_i \in \Lambda^i$.

Definition 5.6. ***Outer Product of Forms*.**
Let $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ and $dx_J = dx_{j_1} \wedge \cdots \wedge dx_{j_q}$ be basis ele-

ments. Their outer product is defined as:

$$dx_I \wedge dx_J = dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_q}.$$

This yields a $(p+q)$-form. If the index sets $\{i_1, \ldots, i_p\}$ and $\{j_1, \ldots, j_q\}$ share any common elements, then $dx_I \wedge dx_J = 0$. For general forms $\omega = \sum_I g_I(x)dx_I \in \Lambda^p$ and $\eta = \sum_J h_J(x)dx_J \in \Lambda^q$, the product is:

$$\omega \wedge \eta = \sum_{I,J} g_I(x)h_J(x)\, dx_I \wedge dx_J.$$

For a 0-form $f \in \Lambda^0$, we define $f \wedge \omega = f\omega = \sum_I f(x)g_I(x)dx_I$.

定義

Proposition 5.2. ***Properties of the Outer Product of Forms.***
The operation $\wedge$ satisfies the following properties:

1. **Dimensional Vanishing:** If $\omega \in \Lambda^p, \eta \in \Lambda^q$ and $p+q > n$, then $\omega \wedge \eta = 0$.

2. **Graded Commutativity:** For $\omega \in \Lambda^p, \eta \in \Lambda^q$:

$$\omega \wedge \eta = (-1)^{pq} \eta \wedge \omega.$$

3. **Distributivity and Associativity:** For any $\omega, \eta, \sigma \in \Lambda$:

$$(\omega + \eta) \wedge \sigma = \omega \wedge \sigma + \eta \wedge \sigma,$$

$$\sigma \wedge (\omega + \eta) = \sigma \wedge \omega + \sigma \wedge \eta,$$

$$(\omega \wedge \eta) \wedge \sigma = \omega \wedge (\eta \wedge \sigma).$$

命題

Proof

The properties stem from the definition of the outer product on basis elements and its linear extension.

1. **Dimensional Vanishing:** A basis p-form dx_I uses p distinct differentials; a basis q-form dx_J uses q distinct differentials. If $p + q > n$, the combined list of indices must repeat some dx_k, and antisymmetry forces $dx_k \wedge dx_k = 0$, so every summand in $\omega \wedge \eta$ vanishes.

2. **Graded Commutativity:** Swapping dx_I (length p) past dx_J (length q) involves pq swaps of 1-forms; each swap contributes a factor -1, yielding $(-1)^{pq}$ overall.

3. **Distributivity and Associativity:** Let $\omega = \sum g_I dx_I$, $\eta = \sum h_J dx_J$,

and $\sigma = \sum k_K dx_K$. Then

$$(\omega + \eta) \wedge \sigma = \sum_{I,K} (g_I + h_I) k_K \, dx_I \wedge dx_K = \omega \wedge \sigma + \eta \wedge \sigma,$$

and similarly on the right. For associativity, note that $(dx_I \wedge dx_J) \wedge dx_K = dx_I \wedge (dx_J \wedge dx_K)$ because concatenating the ordered list of differentials does not depend on parenthesisation; linearity extends this to general forms.

■

Corollary 5.1. *Self-Product of Forms.* If $\omega \in \Lambda^p$ and $\omega \neq 0$:

- If p is **odd**, then $\omega \wedge \omega = 0$.
- If p is **even**, $\omega \wedge \omega$ is not necessarily zero.

推論

Proof

By graded commutativity, $\omega \wedge \omega = (-1)^{p^2} \omega \wedge \omega$. Bringing terms to one side gives $(1 - (-1)^{p^2}) \, \omega \wedge \omega = 0$. If p is odd, $(-1)^{p^2} = -1$, so the factor is 2 and the only solution is $\omega \wedge \omega = 0$. If p is even, $(-1)^{p^2} = 1$, the prefactor vanishes, and no cancellation forces the square to be zero—indeed, example 5.1 shows it can be non-zero.

■

Note

Unlike the outer product of vectors in $\mathbb{R}^n$ (where $\mathbf{a} \wedge \mathbf{a} = 0$), the outer product of differential forms allows for non-zero squares when the degree is even.

Example 5.1. Non-zero Square in $\mathbb{R}^4$. Consider the 2-form $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$ in $\mathbb{R}^4$.

$$\begin{aligned}
\omega \wedge \omega &= (dx_1 \wedge dx_2 + dx_3 \wedge dx_4) \wedge (dx_1 \wedge dx_2 + dx_3 \wedge dx_4) \\
&= (dx_1 \wedge dx_2) \wedge (dx_1 \wedge dx_2) + (dx_1 \wedge dx_2) \wedge (dx_3 \wedge dx_4) \\
&\quad + (dx_3 \wedge dx_4) \wedge (dx_1 \wedge dx_2) + (dx_3 \wedge dx_4) \wedge (dx_3 \wedge dx_4).
\end{aligned}$$

Terms with repeated indices vanish. Using the property that $\alpha \wedge \beta = \beta \wedge \alpha$ for 2-forms (since $(-1)^{2 \times 2} = 1$):

$$\omega \wedge \omega = 2 \, dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4.$$

範例

Exterior Differentiation

We introduce the exterior differentiation operator $d : \Lambda \to \Lambda$.

Definition 5.7. ***Exterior Derivative.***
Let $U \subset \mathbb{R}^n$.

1. For a differentiable function f (0-form), df is the total differential:

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$

2. For a p-form $\omega = g(x) dx_{i_1} \wedge \cdots \wedge dx_{i_p}$, defined on U:

$$d\omega = \sum_{j=1}^{n} \frac{\partial g}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

3. The operator is extended to all of Λ by linearity: $d(\alpha\omega + \beta\eta) = \alpha d\omega + \beta d\eta$.

定義

Example 5.2. Derivative of Basis Elements. Let $\omega = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. Then $d\omega = 0$.

範例

Proof

Regard ω as $1 \cdot dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. Then $d\omega = d(1) \wedge \omega = 0 \wedge \omega = 0$.

■

Example 5.3. Calculation in $\mathbb{R}^3$. Let $\omega = P\,dx + Q\,dy + R\,dz$ be a C^1 1-form in $\mathbb{R}^3$. Calculate $d\omega$.

範例

Solution

$$d\omega = dP \wedge dx + dQ \wedge dy + dR \wedge dz.$$

Expanding $dP = \frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy + \frac{\partial P}{\partial z} dz$:

$$dP \wedge dx = \left(\frac{\partial P}{\partial y} dy + \frac{\partial P}{\partial z} dz \right) \wedge dx = \frac{\partial P}{\partial y} dy \wedge dx + \frac{\partial P}{\partial z} dz \wedge dx.$$

Summing all terms and using antisymmetry ($dy \wedge dx = -dx \wedge dy$, etc.):

$$d\omega = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz \wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy.$$

■

Example 5.4. Second Derivative of a Function. Let $f \in \Lambda^0$ be of class C^2. Then $d^2 f = 0$.

範例

Proof

$$d^2 f = d\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i\right) = \sum_{i=1}^{n} d\left(\frac{\partial f}{\partial x_i}\right) \wedge dx_i = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \wedge dx_i.$$

We split the sum into $i < j$ and $i > j$. Using $\frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ and $dx_j \wedge dx_i = -dx_i \wedge dx_j$:

$$d^2 f = \sum_{i<j} \left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_j \partial x_i}\right) dx_i \wedge dx_j = 0.$$

■

Proposition 5.3. ***Properties of Exterior Differentiation.***

1. **Leibniz Rule:** If $\omega \in \Lambda^p$ and $\eta \in \Lambda^q$ are C^1, then:

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta.$$

2. **Nilpotency:** If ω is C^2, then $d(d\omega) = 0$.

命题

Proof

It suffices to prove these for basis elements. Let $\omega = f dx_I$ where I is a multi-index of length p, and $\eta = g dx_J$ where J is a multi-index of length q.

1. **Leibniz Rule:** Note that $\omega \wedge \eta = (fg) dx_I \wedge dx_J$.

$$\begin{aligned} d(\omega \wedge \eta) &= d(fg) \wedge dx_I \wedge dx_J \\ &= (g\,df + f\,dg) \wedge dx_I \wedge dx_J \\ &= (g\,df \wedge dx_I \wedge dx_J) + (f\,dg \wedge dx_I \wedge dx_J). \end{aligned}$$

   For the first term, we can commute g to the front (scalar): $(df \wedge dx_I) \wedge (g dx_J) = d\omega \wedge \eta$. For the second term, we must move the 1-form dg past the p-form dx_I. This requires p transpositions, introducing a factor of $(-1)^p$:

$$f(dg \wedge dx_I) \wedge dx_J = f((-1)^p dx_I \wedge dg) \wedge dx_J = (-1)^p (f dx_I) \wedge (dg \wedge dx_J) = (-1)^p \omega \wedge d\eta.$$

   Summing these gives the result.

2. **Nilpotency:** Using the Leibniz rule on $\omega = f dx_I$ (regarding dx_I as a form with constant coefficient 1, so $d(dx_I) = 0$):

$$d(d\omega) = d(df \wedge dx_I) = d(df) \wedge dx_I - df \wedge d(dx_I).$$

> We know $d(df) = d^2 f = 0$ (from the previous Example) and $d(dx_I) = 0$ (derivative of basis). Thus $d^2\omega = 0$.
>
> ■

Transformation and Jacobi Determinant

Let A be a linear transformation on $\mathbb{R}^3$ with matrix (a_{ij}). Let Ω be a cube with side lengths α, β, γ. Its volume corresponds to the vector outer product:

$$V = \alpha\mathbf{i} \wedge \beta\mathbf{j} \wedge \gamma\mathbf{k} = \alpha\beta\gamma(\mathbf{i} \wedge \mathbf{j} \wedge \mathbf{k}).$$

The image set $A(\Omega)$ is a parallelepiped spanned by the transformed vectors ¸, ȷ, ı. Its oriented volume is:

$$¸ \wedge ȷ \wedge ı = \det A \cdot (\alpha\beta\gamma\, \mathbf{i} \wedge \mathbf{j} \wedge \mathbf{k}) = (\det A)V.$$

If $\det A > 0$, the vectors form a right-handed system; otherwise, they form a left-handed system. The factor $\det A$ is the *oriented* volume scaling; the unsigned volume scales by $|\det A|$.
For a general bounded closed set Ω, using the definition of multiple integrals:

$$\mathrm{Vol}(A(\Omega)) = \iiint_{A(\Omega)} dx\, dy\, dz = \det A \iiint_{\Omega} du\, dv\, dw = (\det A)\mathrm{Vol}(\Omega).$$

Thus, $\det A$ represents the volume expansion coefficient.

- If $\det A > 0$, the transformation is **direction-preserving**.
- If $|\det A| = 1$, the transformation is **volume-preserving** (area-preserving).

This extends to differentiable mappings $f : \mathbb{R}^3 \to \mathbb{R}^3$. Near a point P_0, f is approximated by its linearisation with determinant equal to the Jacobian:

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)}.$$

If $J > 0$ everywhere, f preserves direction; if $|J| = 1$, f preserves volume.

5.4 *Change of Variables in Multiple Integrals*

The exterior product of differential forms provides a natural and rigorous mechanism for handling the change of variables in integration. Let the volume element in a right-handed Cartesian coordinate system $O - xyz$ be $dx\, dy\, dz$. Geometrically, this represents the volume of an infinitesimal cube. In the language of forms, we denote this as the wedge product $dx \wedge dy \wedge dz$.

Consider a coordinate transformation $T : D \subset \mathbb{R}^3 \to \mathbb{R}^3$ given by:

$$x = x(u,v,w), \quad y = y(u,v,w), \quad z = z(u,v,w).$$

Let the volume element in the parameter space be $du \wedge dv \wedge dw$. Using the properties of the wedge product and the total differential, we compute the transformed volume form directly:

$$dx \wedge dy \wedge dz = \left(\frac{\partial x}{\partial u} du + \dots\right) \wedge \left(\frac{\partial y}{\partial u} du + \dots\right) \wedge \left(\frac{\partial z}{\partial u} du + \dots\right).$$

Terms with repeated differentials (like $du \wedge du$) vanish. The remaining terms assemble into the Jacobian determinant:

$$dx \wedge dy \wedge dz = \frac{\partial(x,y,z)}{\partial(u,v,w)} du \wedge dv \wedge dw.$$

This yields the change of variables formula:

$$\int_{T(D)} f(x,y,z)\, dx \wedge dy \wedge dz = \int_D f(x(u,v,w), y(u,v,w), z(u,v,w)) \frac{\partial(x,y,z)}{\partial(u,v,w)}\, du \wedge dv \wedge dw.$$

Proof

Substitute the total differentials $dx = \sum_u x_u\, du + x_v\, dv + x_w\, dw$ (and similarly for dy, dz) into $dx \wedge dy \wedge dz$. By antisymmetry every term containing a repeated factor such as $du \wedge du$ vanishes. Exactly one term survives for each permutation of (du, dv, dw); its coefficient is the corresponding signed minor of the Jacobian matrix. Collecting these gives the determinant $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ times $du \wedge dv \wedge dw$. Integrating both sides over D yields the stated change-of-variables formula and automatically tracks orientation via the sign of the determinant.

■

Note

Unlike the standard scalar change of variables formula involving

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|,$$

the form-based version **does not** require absolute values. The orientation is handled automatically: if the Jacobian is negative, the orientation of the integration domain is reversed, and the wedge product $du \wedge dv \wedge dw$ carries that sign. Also, note the order sensitivity: $dx \wedge dy \wedge dz$ is the positive volume element, while $dy \wedge dx \wedge dz = -dx \wedge dy \wedge dz$ corresponds to a left-handed system.

Example 5.5. Polar Coordinates. For $x = r\cos\theta, y = r\sin\theta$:

$$dx \wedge dy = \frac{\partial(x,y)}{\partial(r,\theta)} dr \wedge d\theta = (\cos\theta \cdot r\cos\theta - (-r\sin\theta)\sin\theta)\, dr \wedge d\theta = r\, dr \wedge d\theta.$$

範例

The General Stokes Formula

We now arrive at the unification of the fundamental theorems of vector calculus. By interpreting integrands as differential forms, we see that the Fundamental Theorem of Calculus, Green's Theorem, Gauss's Theorem, and Stokes' Theorem are all special cases of a single result.
Let M be an oriented region (manifold) of dimension k, and let ∂M be its boundary with the induced orientation.

1. **Newton-Leibniz:** $M = [a, b]$. $\partial M = \{b\} - \{a\}$. For a 0-form f:

$$\int_a^b df(x) = f(b) - f(a) \implies \int_M df = \int_{\partial M} f.$$

2. **Green's Theorem:** $D \subset \mathbb{R}^2$. $\omega = Pdx + Qdy$. Then $d\omega = (\partial_x Q - \partial_y P) dx \wedge dy$.

$$\oint_{\partial D} Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy \implies \oint_{\partial D} \omega = \int_D d\omega.$$

3. **Classical Stokes' Theorem:** $\Sigma \subset \mathbb{R}^3$. $\omega = Pdx + Qdy + Rdz$. Then $d\omega$ corresponds to $(\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$.

$$\oint_{\partial \Sigma} \mathbf{F} \cdot d\mathbf{r} = \iint_\Sigma (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \implies \oint_{\partial \Sigma} \omega = \int_\Sigma d\omega.$$

4. **Gauss's Theorem:** $\Omega \subset \mathbb{R}^3$. $\omega = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy$. Then $d\omega = (\nabla \cdot \mathbf{F}) dx \wedge dy \wedge dz$.

$$\oiint_{\partial \Omega} \mathbf{F} \cdot d\mathbf{S} = \iiint_\Omega \nabla \cdot \mathbf{F} \, dV \implies \oint_{\partial \Omega} \omega = \int_\Omega d\omega.$$

These observations lead to one of the most celebrated results in analysis:

Theorem 5.1. ***General Stokes Formula.***
Let M be an oriented smooth k-dimensional region (manifold with boundary), and let ω be a smooth $(k-1)$-form with compact support on M. Then:

$$\int_M d\omega = \oint_{\partial M} \omega.$$

定理

This formula states that the integral of the derivative of a form over the interior is equal to the integral of the form itself over the boundary. It encapsulates the essence of calculus: local variations (derivative) sum up to a global boundary value. This elegant unification paves the way for advanced studies in differential geometry and topology.

5.5 Exercises

1. **Geometric Interpretation.** Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ and consider the associated 1-forms $\alpha = a_1 dx + a_2 dy + a_3 dz$ and $\beta = b_1 dx + b_2 dy + b_3 dz$. Show that the coefficients of the 2-form $\alpha \wedge \beta$ are the components of the cross product $\mathbf{a} \times \mathbf{b}$, and that the norm of this 2-form (use the Euclidean norm of its coefficient vector) is equal to the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.
2. **Triple Product.** Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$. Show that the scalar triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ corresponds to the coefficient of the standard volume form $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$ in the expansion of $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$.
3. **Basis Expansion.** Let $\omega \in \Lambda^2(\mathbb{R}^4)$. Write ω in the standard basis $\{dx_i \wedge dx_j \mid 1 \leq i < j \leq 4\}$.
4. **Coordinate Transformation.** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the polar coordinate map $(r, \theta) \mapsto (r \cos\theta, r \sin\theta)$. Express the form $\omega = x\,dy - y\,dx$ in terms of $r, \theta, dr, d\theta$.
5. **Algebraic Properties.** Verify the graded commutativity property directly for 1-forms $\alpha, \beta, \gamma \in \Lambda^1(\mathbb{R}^n)$:

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma).$$

   Does this property hold if β is a 2-form?
6. **Computing Derivatives.** Let $\omega = xyz\,dx + x^2\,dy + z^2\,dz$. Compute $d\omega$. Check explicitly that $d(d\omega) = 0$.
7. **Volume Expansion.** Let A be a 3×3 matrix with $\det A = -2$. If Ω is the unit cube, describe the geometric effect of the transformation $T(\mathbf{x}) = A\mathbf{x}$ on Ω in terms of volume and orientation.
8. **Spherical Coordinates.** Compute the exterior product $dx \wedge dy \wedge dz$ in spherical coordinates $(x, y, z) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)$ directly using the properties of the wedge product. Use this to set up the integral for the volume of a ball of radius R.
9. **Generalized Polar Coordinates.** Let $x = ar\cos\theta, y = br\sin\theta$. Compute the area element $dx \wedge dy$ in terms of r, θ and use it to find the area of the ellipse $x^2/a^2 + y^2/b^2 \leq 1$.
10. **Integration by Parts.** (Advanced/optional—assumes integration of forms on a k-manifold.) Let M be a compact oriented k-dimensional manifold with boundary. Let $\alpha \in \Lambda^p(M)$ and

$\beta \in \Lambda^{k-p-1}(M)$. Prove the integration by parts formula:

$$\int_M d\alpha \wedge \beta = \int_{\partial M} \alpha \wedge \beta + (-1)^{p+1} \int_M \alpha \wedge d\beta.$$

> *Remark.*
> Hint: Apply Stokes' Theorem to $\omega = \alpha \wedge \beta$ and use the Leibniz rule for $d(\alpha \wedge \beta)$.

6
Field Theory (Introduction)

Historically, the concept of a field arose in physics to describe continuous quantities distributed over space, such as temperature, gravitational, or electromagnetic fields. Mathematically, we classify these into scalar fields (assigning a magnitude to each point) and vector fields (assigning a magnitude and direction). Having developed the machinery of surface and line integrals, we now formalise the differential operators — divergence and curl — that characterise the local behaviour of these fields, and unify the integral theorems of the previous chapters.

6.1 Divergence

We begin by quantifying the "outward flow" of a vector field from a point.

> **Definition 6.1.** ***Divergence.***
> Let D be a region in $\mathbb{R}^3$ and let $\mathbf{F} : D \to \mathbb{R}^3$ be a vector field defined by $\mathbf{F}(x, y, z) = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, where P, Q, R have continuous partial derivatives. The **divergence** of $\mathbf{F}$ is the scalar function defined by:
>
> $$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$
>
> In $\mathbb{R}^2$, for a field $\mathbf{F}(x, y) = P\mathbf{i} + Q\mathbf{j}$, the divergence is similarly defined as:
>
> $$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$
>
> 定義

This operator measures the rate at which "fluid" (represented by the vector field) expands or compresses at a point. Positive divergence indicates a source; negative divergence indicates a sink.

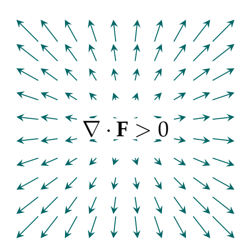


Figure 6.1: A vector field with positive divergence acts as a source.

6.2 The Generalised Divergence Theorem

The notion of divergence allows us to express the integral theorems of vector calculus in a unified, coordinate-free manner.
Recall *Gauss's Formula* (*theorem 3.1*). In terms of divergence, the relation between the volume integral over a region $D \subset \mathbb{R}^3$ and the flux through its boundary ∂D becomes:

$$\iiint_D \operatorname{div} \mathbf{F} \, dV = \oiint_{\partial D} \mathbf{F} \cdot \mathbf{n} \, dS.$$

Here, $\mathbf{n}$ is the unit outward normal to ∂D.
Similarly, Green's Formula in the plane (specifically the normal form, derived from *Green's Formula*) relates the double integral over a planar region D to the flux across its boundary curve ∂D. If $\mathbf{F} = (P, Q)$, then:

$$\iint_D \operatorname{div} \mathbf{F} \, dA = \oint_{\partial D} \mathbf{F} \cdot \mathbf{n} \, ds = \oint_{\partial D} -Q \, dx + P \, dy.$$

> **Theorem 6.1.** ***The Divergence Theorem.***
> Let D be a bounded region in $\mathbb{R}^n$ ($n = 1, 2, 3$) with a piecewise smooth boundary ∂D oriented by the unit outward normal $\mathbf{n}$. If $\mathbf{F}$ is a continuously differentiable vector field on $\bar{D}$, then:
>
> $$\int_D \operatorname{div} \mathbf{F} \, dV = \oint_{\partial D} \mathbf{F} \cdot \mathbf{n} \, dS.$$
>
> 定理

This statement unifies the fundamental theorems of calculus across dimensions:

Dimension $n = 1$***:*** The region D is an interval $[a, b]$. The boundary ∂D consists of the endpoints $\{a, b\}$. The "normals" are $\mathbf{n}(b) = 1$ and $\mathbf{n}(a) = -1$. The divergence is simply the derivative $f'(x)$. The theorem yields the Newton-Leibniz formula:

$$\int_a^b f'(x) \, dx = f(b) \cdot (1) + f(a) \cdot (-1) = f(b) - f(a).$$

Dimension $n = 2$***:*** This yields the vector form of Green's Theorem.

Dimension $n = 3$***:*** This yields Gauss's Theorem.

Proof for a Rectangular Solid

Suppose $\mathbf{F}$ is differentiable near the rectangular solid $E = [x_1, x_2] \times [y_1, y_2] \times [z_1, z_2]$. We denote the six faces of the solid as follows:

- S_1: $x = x_1$, $(y, z) \in [y_1, y_2] \times [z_1, z_2]$ with inward normal $-\mathbf{i}$. Area element $d\mathbf{S} = -\mathbf{i} \, dy \, dz$.

- S_2: $x = x_2$, $(y, z) \in [y_1, y_2] \times [z_1, z_2]$ with outward normal $\mathbf{i}$. Area element $d\mathbf{S} = \mathbf{i}\, dy\, dz$.
- S_3: $y = y_1$, $(x, z) \in [x_1, x_2] \times [z_1, z_2]$ with inward normal $-\mathbf{j}$. Area element $d\mathbf{S} = -\mathbf{j}\, dx\, dz$.
- S_4: $y = y_2$, $(x, z) \in [x_1, x_2] \times [z_1, z_2]$ with outward normal $\mathbf{j}$. Area element $d\mathbf{S} = \mathbf{j}\, dx\, dz$.
- S_5: $z = z_1$, $(x, y) \in [x_1, x_2] \times [y_1, y_2]$ with inward normal $-\mathbf{k}$. Area element $d\mathbf{S} = -\mathbf{k}\, dx\, dy$.
- S_6: $z = z_2$, $(x, y) \in [x_1, x_2] \times [y_1, y_2]$ with outward normal $\mathbf{k}$. Area element $d\mathbf{S} = \mathbf{k}\, dx\, dy$.

The rectangular geometry ensures that only one component of $\mathbf{F} = (P, Q, R)$ contributes to the flux through any given face. We compute the net flux through the pair of faces perpendicular to the x-axis, S_1 and S_2:

$$\begin{aligned}
\Phi_{12} &= \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} \\
&= \int_{z_1}^{z_2} \int_{y_1}^{y_2} \mathbf{F}(x_1, y, z) \cdot (-\mathbf{i})\, dy\, dz + \int_{z_1}^{z_2} \int_{y_1}^{y_2} \mathbf{F}(x_2, y, z) \cdot (\mathbf{i})\, dy\, dz \\
&= \int_{z_1}^{z_2} \int_{y_1}^{y_2} [P(x_2, y, z) - P(x_1, y, z)]\, dy\, dz.
\end{aligned}$$

By the Fundamental Theorem of Calculus, the difference in P can be expressed as an integral of its derivative:

$$P(x_2, y, z) - P(x_1, y, z) = \int_{x_1}^{x_2} \frac{\partial P}{\partial x}\, dx.$$

Substituting this back:

$$\Phi_{12} = \int_{z_1}^{z_2} \int_{y_1}^{y_2} \int_{x_1}^{x_2} \frac{\partial P}{\partial x}\, dx\, dy\, dz.$$

Similarly, for the faces S_3 and S_4 perpendicular to the y-axis:

$$\begin{aligned}
\Phi_{34} &= \iint_{S_3} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_4} \mathbf{F} \cdot d\mathbf{S} \\
&= \int_{z_1}^{z_2} \int_{x_1}^{x_2} [Q(x, y_2, z) - Q(x, y_1, z)]\, dx\, dz \\
&= \int_{z_1}^{z_2} \int_{x_1}^{x_2} \int_{y_1}^{y_2} \frac{\partial Q}{\partial y}\, dy\, dx\, dz.
\end{aligned}$$

And for the faces S_5 and S_6 perpendicular to the z-axis:

$$\begin{aligned}
\Phi_{56} &= \iint_{S_5} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_6} \mathbf{F} \cdot d\mathbf{S} \\
&= \int_{y_1}^{y_2} \int_{x_1}^{x_2} [R(x, y, z_2) - R(x, y, z_1)]\, dx\, dy \\
&= \int_{y_1}^{y_2} \int_{x_1}^{x_2} \int_{z_1}^{z_2} \frac{\partial R}{\partial z}\, dz\, dx\, dy.
\end{aligned}$$

The total flux over the boundary ∂E is the sum of the fluxes through all faces. By the linearity of the triple integral, we sum the results:

$$\begin{aligned}
\oiint_{\partial E} \mathbf{F} \cdot d\mathbf{S} &= \Phi_{12} + \Phi_{34} + \Phi_{56} \\
&= \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dz\, dy\, dx \\
&= \iiint_E \operatorname{div} \mathbf{F}\, dV.
\end{aligned}$$

This completes the proof for the rectangular solid. ■

Proof for a Cylindrical Region

Consider a cylindrical region $E = \{(x, y, z) \mid x^2 + y^2 \leq R^2,\ 0 \leq z \leq h\}$. The boundary ∂E consists of three parts: the bottom disk S_{bot} ($z = 0$), the top disk S_{top} ($z = h$), and the lateral surface S_{lat} ($x^2 + y^2 = R^2$). We aim to show:

$$\oiint_{\partial E} \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_E \nabla \cdot \mathbf{F}\, dV.$$

Vertical Flux contribution: The outward normals on the top and bottom are $\mathbf{k}$ and $-\mathbf{k}$ respectively. The flux of the vertical component $R\mathbf{k}$ is:

$$\begin{aligned}
\Phi_{vertical} &= \iint_{S_{top}} \mathbf{F} \cdot \mathbf{k}\, dS + \iint_{S_{bot}} \mathbf{F} \cdot (-\mathbf{k})\, dS \\
&= \iint_{x^2+y^2 \leq R^2} [R(x, y, h) - R(x, y, 0)]\, dx\, dy.
\end{aligned}$$

By the Fundamental Theorem of Calculus with respect to z:

$$R(x, y, h) - R(x, y, 0) = \int_0^h \frac{\partial R}{\partial z}\, dz.$$

Thus:

$$\Phi_{vertical} = \iint_D \left(\int_0^h \frac{\partial R}{\partial z}\, dz \right) dx\, dy = \iiint_E \frac{\partial R}{\partial z}\, dV.$$

Horizontal Flux contribution: On the lateral surface S_{lat}, the outward normal is radial: $\mathbf{n} = (\cos\theta, \sin\theta, 0)$ in cylindrical coordinates. The area element is $dS = R\,d\theta\,dz$. Only the horizontal components P and Q contribute to the flux through S_{lat}.

$$\Phi_{lat} = \int_0^h \int_0^{2\pi} (P\cos\theta + Q\sin\theta) R\,d\theta\,dz.$$

Consider the horizontal divergence $\nabla_H \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$. We integrate this over the volume E:

$$\iiint_E \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dV = \int_0^h \left[\iint_{x^2+y^2 \le R^2} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dx\,dy\right] dz.$$

For each fixed z, the inner double integral can be transformed via Green's Theorem (or the 2D Divergence Theorem) into a line integral over the boundary circle $x^2 + y^2 = R^2$:

$$\iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA = \oint_{\partial D} (P\mathbf{i} + Q\mathbf{j}) \cdot \mathbf{n}_{2D}\,ds.$$

Here $\mathbf{n}_{2D} = (\cos\theta, \sin\theta)$ and $ds = R\,d\theta$.

$$\oint_{\partial D} (P\cos\theta + Q\sin\theta) R\,d\theta.$$

Integrating this result from $z = 0$ to $z = h$ yields exactly the expression for Φ_{lat}.

Summing the vertical and horizontal contributions:

$$\oiint_{\partial E} \mathbf{F} \cdot \mathbf{n}\,dS = \Phi_{vertical} + \Phi_{lat} = \iiint_E \frac{\partial R}{\partial z}\,dV + \iiint_E \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dV = \iiint_E \operatorname{div} \mathbf{F}\,dV.$$

This confirms the theorem for the cylindrical region. ■

Applications to Integration

The Divergence Theorem provides a mechanism for transferring derivatives from the interior of a region to its boundary, generalizing integration by parts.

> **Proposition 6.1.** ***Integration by Parts in*** $\mathbb{R}^3$**.**
> Let $\Omega \subset \mathbb{R}^3$ be a bounded region with piecewise smooth boundary Σ, oriented outwards. If u, v are continuously differentiable scalar functions on $\bar{\Omega}$, then:
>
> $$\iiint_\Omega u\frac{\partial v}{\partial x}\,dV = \oiint_\Sigma uv\,dy\,dz - \iiint_\Omega v\frac{\partial u}{\partial x}\,dV.$$
>
> 命题

Proof

Consider the vector field $\mathbf{F} = (uv, 0, 0)$. The divergence of $\mathbf{F}$ is:

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(uv) + 0 + 0 = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}.$$

Applying the Divergence Theorem (*theorem* 6.1) to $\mathbf{F}$:

$$\iiint_\Omega \left(u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} \right) dV = \oiint_\Sigma \mathbf{F} \cdot \mathbf{n} \, dS.$$

The surface integral term is the flux of $(uv, 0, 0)$. In coordinate form, $\mathbf{F} \cdot \mathbf{n} \, dS = P \, dy \, dz = uv \, dy \, dz$. Thus:

$$\iiint_\Omega u\frac{\partial v}{\partial x} \, dV + \iiint_\Omega v\frac{\partial u}{\partial x} \, dV = \oiint_\Sigma uv \, dy \, dz.$$

Rearranging terms yields the result.

■

Corollary 6.1. *Integral of a Derivative.* Setting $v(x, y, z) \equiv 1$ in *proposition* 6.1 yields:

$$\iiint_\Omega \frac{\partial u}{\partial x} \, dV = \oiint_\Sigma u \, dy \, dz = \oiint_\Sigma u \cos\alpha \, dS,$$

where $\cos\alpha$ is the x-component of the outward normal. Analogous formulae hold for partial derivatives with respect to y and z.

推論

Proof

We apply *proposition* 6.1 with $v(x, y, z) \equiv 1$. Since v is constant, $\frac{\partial v}{\partial x} = 0$. The integration by parts formula simplifies to:

$$0 = \oiint_\Sigma u \, dy \, dz - \iiint_\Omega \frac{\partial u}{\partial x} \, dV.$$

Rearranging gives the result. Alternatively, apply Gauss's Theorem directly to the vector field $\mathbf{F} = (u, 0, 0)$.

■

6.3 *Curl*

While divergence measures the expansion of a field at a point, the **curl** measures its local rotation or circulation.

Definition 6.2. ***Curl.***
Let $\mathbf{F}(x, y, z) = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a vector field on a region $D \subset \mathbb{R}^3$, where P, Q, R are continuously differentiable. The **curl** of $\mathbf{F}$ is the vec-

tor field defined by:

$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

The determinant notation provides a convenient mnemonic:

$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}.$$

定義

With this notation, *Stokes' Formula* (*theorem* 4.1) takes a concise vector form. If Σ is an oriented surface with unit normal $\mathbf{n}$ and boundary $\partial\Sigma$ oriented by the right-hand rule with tangent $\boldsymbol{\tau}$:

$$\iint_{\Sigma} (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \oint_{\partial\Sigma} \mathbf{F} \cdot \boldsymbol{\tau}\, ds.$$

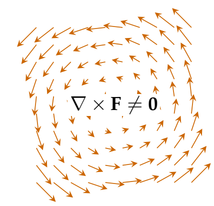


Figure 6.2: A vector field with non-zero curl exhibits local rotation.

6.4 *The Hamilton Operator*

The symbols $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$ introduced above are instances of the **Hamilton operator** (or del operator), denoted ∇.

Definition 6.3. ***The Del Operator.***
The operator ∇ is defined formally as a vector of partial differential operators:

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$

Its action depends on the operand:

- **Gradient:** If f is a scalar function, $\nabla f = \operatorname{grad} f$.
- **Divergence:** If $\mathbf{F}$ is a vector field, $\nabla \cdot \mathbf{F} = \operatorname{div}\mathbf{F}$ (formal dot product).
- **Curl:** If $\mathbf{F}$ is a vector field, $\nabla \times \mathbf{F} = \operatorname{curl}\mathbf{F}$ (formal cross product).

定義

The operator ∇ is linear. For constants α, β, scalar functions f, g, and vector fields $\mathbf{a}, \mathbf{b}$, the following identities hold:

$$\nabla(\alpha f + \beta g) = \alpha \nabla f + \beta \nabla g \tag{6.1}$$

$$\nabla \cdot (\alpha \mathbf{a} + \beta \mathbf{b}) = \alpha \nabla \cdot \mathbf{a} + \beta \nabla \cdot \mathbf{b} \tag{6.2}$$

$$\nabla \times (\alpha \mathbf{a} + \beta \mathbf{b}) = \alpha \nabla \times \mathbf{a} + \beta \nabla \times \mathbf{b} \tag{6.3}$$

Product Rules and Second-Order Identities

The product rules for ∇ combine the logic of the Leibniz rule for differentiation with vector algebra.

Proposition 6.2. ***Vector Calculus Identities.***
Let f, g be differentiable scalar fields and $\mathbf{a}, \mathbf{b}$ be differentiable vector fields.

1. $\nabla(fg) = g\nabla f + f\nabla g$.
2. $\nabla \cdot (f\mathbf{a}) = f(\nabla \cdot \mathbf{a}) + \mathbf{a} \cdot \nabla f$.
3. $\nabla \times (f\mathbf{a}) = f(\nabla \times \mathbf{a}) + (\nabla f) \times \mathbf{a}$.
4. $\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$.

Furthermore, if the fields are twice continuously differentiable:

5. $\nabla \cdot (\nabla \times \mathbf{a}) = 0$ (Divergence of curl is zero).
6. $\nabla \times (\nabla f) = \mathbf{0}$ (Curl of gradient is zero).

命题

Remark.

Identities (5) and (6) are crucial for classifying fields. Identity (6) implies that conservative fields are irrotational, while (5) implies that solenoidal fields (divergence-free) can often be expressed as the curl of a **vector potential**.

We prove the fourth, fifth and sixth identity to illustrate the manipulation of these operators.

Proof of Identity 4

Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$. The cross product is given by the determinant:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

The divergence is the sum of partial derivatives of the components.

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \frac{\partial}{\partial x}(a_2 b_3 - a_3 b_2) - \frac{\partial}{\partial y}(a_1 b_3 - a_3 b_1) + \frac{\partial}{\partial z}(a_1 b_2 - a_2 b_1).$$

We apply the product rule to each term. We group terms containing components of $\mathbf{b}$ without derivatives, and terms containing components of $\mathbf{a}$ without derivatives. Let I_1 be the terms where derivatives act on $\mathbf{a}$:

$$I_1 = b_3 \frac{\partial a_2}{\partial x} - b_2 \frac{\partial a_3}{\partial x} - b_3 \frac{\partial a_1}{\partial y} + b_1 \frac{\partial a_3}{\partial y} + b_2 \frac{\partial a_1}{\partial z} - b_1 \frac{\partial a_2}{\partial z}.$$

Rearranging by components of $\mathbf{b}$:

$$I_1 = b_1 \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} \right) + b_2 \left(\frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x} \right) + b_3 \left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y} \right) = \mathbf{b} \cdot (\nabla \times \mathbf{a}).$$

Similarly, let I_2 be the terms where derivatives act on $\mathbf{b}$:

$$I_2 = a_2 \frac{\partial b_3}{\partial x} - a_3 \frac{\partial b_2}{\partial x} - a_1 \frac{\partial b_3}{\partial y} + a_3 \frac{\partial b_1}{\partial y} + a_1 \frac{\partial b_2}{\partial z} - a_2 \frac{\partial b_1}{\partial z}.$$

Rearranging by components of **a** reveals a sign change relative to the curl formula:

$$I_2 = -a_1\left(\frac{\partial b_3}{\partial y} - \frac{\partial b_2}{\partial z}\right) - a_2\left(\frac{\partial b_1}{\partial z} - \frac{\partial b_3}{\partial x}\right) - a_3\left(\frac{\partial b_2}{\partial x} - \frac{\partial b_1}{\partial y}\right) = -\mathbf{a}\cdot(\nabla\times\mathbf{b}).$$

Thus $\nabla\cdot(\mathbf{a}\times\mathbf{b}) = I_1 + I_2 = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})$.

証明終

Proof of Identity 5

Let $\mathbf{a} = (P, Q, R)$ be twice continuously differentiable. The curl is:

$$\nabla\times\mathbf{a} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

The divergence of this vector field is:

$$\nabla\cdot(\nabla\times\mathbf{a}) = \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right).$$

Expanding the terms:

$$\frac{\partial^2 R}{\partial x\partial y} - \frac{\partial^2 Q}{\partial x\partial z} + \frac{\partial^2 P}{\partial y\partial z} - \frac{\partial^2 R}{\partial y\partial x} + \frac{\partial^2 Q}{\partial z\partial x} - \frac{\partial^2 P}{\partial z\partial y}.$$

Since **a** is twice continuously differentiable, Schwarz's Theorem guarantees the equality of mixed partial derivatives (e.g., $\frac{\partial^2 R}{\partial x\partial y} = \frac{\partial^2 R}{\partial y\partial x}$). Grouping matching terms:

$$\left(\frac{\partial^2 R}{\partial x\partial y} - \frac{\partial^2 R}{\partial y\partial x}\right) + \left(\frac{\partial^2 P}{\partial y\partial z} - \frac{\partial^2 P}{\partial z\partial y}\right) + \left(\frac{\partial^2 Q}{\partial z\partial x} - \frac{\partial^2 Q}{\partial x\partial z}\right) = 0 + 0 + 0 = 0.$$

証明終

Proof of Identity 6

Let f be a twice continuously differentiable scalar field. The gradient is $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$. The curl of the gradient is:

$$\nabla\times(\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}.$$

The **i**-component is:

$$\left(\frac{\partial}{\partial y}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y\partial z} - \frac{\partial^2 f}{\partial z\partial y}.$$

By Schwarz's Theorem, this difference is zero. Similarly, the **j**-component involves $\frac{\partial^2 f}{\partial z\partial x} - \frac{\partial^2 f}{\partial x\partial z} = 0$, and the **k**-component involves $\frac{\partial^2 f}{\partial x\partial y} - \frac{\partial^2 f}{\partial y\partial x} = 0$. Thus, $\nabla\times(\nabla f) = \mathbf{0}$.

証明終

Example 6.1. Mean Value Property for Harmonic Functions. Let $u(x, y, z)$ be a scalar field defined on a ball $B_R(\mathbf{M}_0)$ centred at $\mathbf{M}_0$ with radius R. Suppose the surface integral of the normal derivative vanishes on every concentric sphere $B_\rho(\mathbf{M}_0)$ for $0 < \rho \leq R$:

$$\oiint_{\partial B_\rho} \frac{\partial u}{\partial n}\, dS = 0.$$

Show that the value of u at the centre is the average of its values on the boundary surface ∂B_R:

$$u(\mathbf{M}_0) = \frac{1}{4\pi R^2} \oiint_{\partial B_R} u\, dS.$$

範例

Solution

We use spherical parametrisation about $\mathbf{M}_0$:

$$x = x_0 + \rho \sin\varphi \cos\theta, \quad y = y_0 + \rho \sin\varphi \sin\theta, \quad z = z_0 + \rho \cos\varphi.$$

On the sphere ∂B_ρ, the unit normal is $\mathbf{n} = \mathbf{e}_\rho$. The normal derivative is the directional derivative in the radial direction:

$$\frac{\partial u}{\partial n} = \nabla u \cdot \mathbf{n} = \frac{\partial u}{\partial \rho}.$$

Using the scaling relation of surface elements established in Chapter 25 ($dS_\rho = \rho^2 dS_1$, where dS_1 is the element on the unit sphere):

$$0 = \oiint_{\partial B_\rho} \frac{\partial u}{\partial \rho}\, dS_\rho = \oiint_{\partial B_1} \frac{\partial u}{\partial \rho}(\mathbf{M}_0 + \rho \mathbf{n})\, \rho^2 dS_1.$$

Since $\rho > 0$, we may divide by ρ^2 and move the derivative outside the integral (differentiation under the integral sign):

$$\frac{d}{d\rho}\left(\oiint_{\partial B_1} u(\mathbf{M}_0 + \rho \mathbf{n})\, dS_1 \right) = 0.$$

This implies that the integral $I(\rho) = \oiint_{\partial B_1} u(\mathbf{M}_0 + \rho \mathbf{n})\, dS_1$ is constant for $\rho \in (0, R]$. We equate the value at $\rho = R$ to the limit as $\rho \to 0^+$. At $\rho = R$:

$$I(R) = \frac{1}{R^2} \oiint_{\partial B_R} u\, dS_R.$$

As $\rho \to 0^+$, by continuity of u, $u(\mathbf{M}_0 + \rho \mathbf{n}) \to u(\mathbf{M}_0)$. Thus:

$$\lim_{\rho \to 0} I(\rho) = \oiint_{\partial B_1} u(\mathbf{M}_0)\, dS_1 = u(\mathbf{M}_0) \cdot \text{Area}(\partial B_1) = 4\pi u(\mathbf{M}_0).$$

Equating the two expressions:

$$4\pi u(\mathbf{M}_0) = \frac{1}{R^2} \oiint_{\partial B_R} u \, dS.$$

Rearranging yields the result. ■

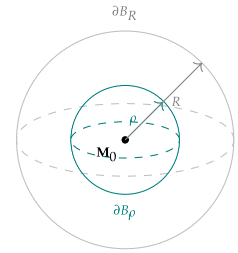


Figure 6.3: The mean value property relates the value at the centre $\mathbf{M}_0$ to the average over the sphere ∂B_R.

6.5 *Exercises*

1. **Basic Calculation.** Let $\mathbf{F}(x, y, z) = (x^2 y, y^2 z, z^2 x)$. Compute $\operatorname{curl} \mathbf{F}$. Is this field irrotational?

2. **Irrotational Field.** Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Prove that $\operatorname{curl} \mathbf{r} = \mathbf{0}$. If $\mathbf{c}$ is a constant vector, prove that $\operatorname{curl}(\mathbf{c} \times \mathbf{r}) = 2\mathbf{c}$.

3. **Operator Identities.** Verify the identity $\nabla(fg) = f\nabla g + g\nabla f$ for scalar fields f, g.

4. **Radial Fields.** Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Let $\mathbf{r} = (x, y, z)$ and $r = |\mathbf{r}|$. Compute:
   (a) $\operatorname{grad} f(r)$
   (b) $\operatorname{div}(f(r)\mathbf{r})$
   (c) $\operatorname{curl}(f(r)\mathbf{r})$

5. **Solenoidal Radial Field.** Find the function $f(r)$ such that the field $f(r)\mathbf{r}$ is solenoidal (divergence-free) for $r > 0$.

6. **Cross Product Divergence.** Verify the identity $\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$ for the specific fields $\mathbf{a} = (y, z, x)$ and $\mathbf{b} = (z, x, y)$.

7. **Integral Definitions.** Let $V \subset D \subset \mathbb{R}^3$ be a volume with boundary Σ, diameter $\delta(V)$, and volume $|V|$. Let $\mathbf{n}$ be the outward normal. Prove that for any $\mathbf{p}_0 \in V$:
   (a) $\operatorname{div} \mathbf{A}(\mathbf{p}_0) = \lim_{\delta(V) \to 0} \frac{1}{|V|} \oiint_{\Sigma} \mathbf{A} \cdot \mathbf{n} \, dS.$
   (b) $\operatorname{curl} \mathbf{A}(\mathbf{p}_0) = \lim_{\delta(V) \to 0} \frac{1}{|V|} \oiint_{\Sigma} \mathbf{n} \times \mathbf{A} \, dS.$
   (c) $\operatorname{grad} \varphi(\mathbf{p}_0) = \lim_{\delta(V) \to 0} \frac{1}{|V|} \oiint_{\Sigma} \varphi \mathbf{n} \, dS.$

7
The Laplace Operator and Harmonic Functions

We now turn our attention to the second-order differential operator that governs diffusion, electrostatics, and gravitation: the Laplace operator.

7.1 The Laplacian

Definition 7.1. ***Laplace Operator.***
The Laplace operator, denoted by Δ or ∇^2, is the divergence of the gradient:

$$\Delta u = \nabla \cdot (\nabla u).$$

In Cartesian coordinates for $\mathbb{R}^3$, if $u(x, y, z)$ is twice differentiable:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

Analogously in $\mathbb{R}^2$, $\Delta u = u_{xx} + u_{yy}$.

定義

Green's Identities

The integration by parts formula for the Laplacian yields Green's Identities, which are fundamental to the theory of partial differential equations.

Theorem 7.1. ***Green's First Identity.***
Let $\Omega \subset \mathbb{R}^3$ be a bounded region with piecewise smooth boundary Σ, and let u, v be twice continuously differentiable functions on $\bar{\Omega}$. Then:

$$\iiint_\Omega (v\Delta u + \nabla u \cdot \nabla v)\, dV = \oiint_\Sigma v \frac{\partial u}{\partial n}\, dS.$$

Here $\frac{\partial u}{\partial n} = \nabla u \cdot \mathbf{n}$ is the directional derivative along the outward unit normal $\mathbf{n}$.

定理

Proof

We apply the Divergence Theorem (*theorem* 6.1) to the vector field $\mathbf{F} = v\nabla u$. First, compute the divergence using the product rule (*proposition* 6.2):

$$\nabla \cdot (v\nabla u) = \nabla v \cdot \nabla u + v(\nabla \cdot \nabla u) = \nabla v \cdot \nabla u + v\Delta u.$$

By the Divergence Theorem:

$$\iiint_\Omega \nabla \cdot (v\nabla u)\, dV = \oiint_\Sigma (v\nabla u) \cdot \mathbf{n}\, dS.$$

Substituting the divergence expression and noting $(v\nabla u) \cdot \mathbf{n} = v(\nabla u \cdot \mathbf{n}) = v\frac{\partial u}{\partial n}$ completes the proof.

∎

Corollary 7.1. *Integral of the Laplacian.* Taking $v \equiv 1$ in Green's First Identity yields:

$$\iiint_\Omega \Delta u\, dV = \oiint_\Sigma \frac{\partial u}{\partial n}\, dS.$$

This states that the total "generation" of the field inside Ω (measured by Δu) equals the net flux of the gradient through the boundary.

推論

Proof

Set $v(x, y, z) \equiv 1$ in *theorem* 7.1. Then $\nabla v = \mathbf{0}$. The identity becomes:

$$\iiint_\Omega (1 \cdot \Delta u + \nabla u \cdot \mathbf{0})\, dV = \oiint_\Sigma 1 \cdot \frac{\partial u}{\partial n}\, dS,$$

which simplifies immediately to the result. Alternatively, apply the Divergence Theorem directly to $\mathbf{F} = \nabla u$.

∎

Theorem 7.2. ***Green's Second Identity.***
Under the same conditions as *theorem* 7.1:

$$\iiint_\Omega (v\Delta u - u\Delta v)\, dV = \oiint_\Sigma \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right) dS.$$

定理

Proof

Write Green's First Identity for the pair (u, v) and then for (v, u) (interchanging the roles). Subtracting the second equation from the first cancels the symmetric term $\nabla u \cdot \nabla v$, leaving the result.

∎

Spherical Means and the Radial Laplacian

The behaviour of the Laplacian is closely tied to the average value of functions over spheres. We formalise the relationship between the spherical mean and the radial derivatives.
Let $h(x, y, z)$ be a twice continuously differentiable function. For a fixed point $\mathbf{M} \in \mathbb{R}^3$, we define the **spherical mean** $M_h(\mathbf{M}, r)$ on the sphere $\partial B_r(\mathbf{M})$ of radius r:

$$M_h(\mathbf{M}, r) = \frac{1}{4\pi r^2} \oiint_{\partial B_r(\mathbf{M})} h(\xi, \eta, \zeta)\, dS_r.$$

Proposition 7.1. ***Differential Equation for Spherical Means.***
The spherical mean M_h satisfies the radial differential equation:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} \right) M_h(\mathbf{M}, r) = \Delta M_h(\mathbf{M}, r),$$

where the Laplacian Δ on the right acts on the spatial coordinates of the centre $\mathbf{M}$.

命題

Proof

Differentiability: By rescaling to the unit sphere using $\boldsymbol{\xi} = \mathbf{M} + r\boldsymbol{\alpha}$ (where $\boldsymbol{\alpha} \in \partial B_1(\mathbf{0})$), we write:

$$M_h(\mathbf{M}, r) = \frac{1}{4\pi} \oiint_{\partial B_1(\mathbf{0})} h(\mathbf{M} + r\boldsymbol{\alpha})\, dS_1.$$

Differentiation under the integral sign shows M_h is C^2 in r and $\mathbf{M}$.

First Derivative: Differentiating with respect to r:

$$\frac{\partial M_h}{\partial r} = \frac{1}{4\pi} \oiint_{\partial B_1} \nabla h(\mathbf{M} + r\boldsymbol{\alpha}) \cdot \boldsymbol{\alpha}\, dS_1.$$

Scaling back to the radius r sphere (where $\boldsymbol{\alpha} = \mathbf{n}$):

$$\frac{\partial M_h}{\partial r} = \frac{1}{4\pi r^2} \oiint_{\partial B_r(\mathbf{M})} \frac{\partial h}{\partial n}\, dS_r.$$

Applying *corollary* 7.1 (Gauss's Theorem for gradients), we convert the surface integral to a volume integral:

$$\frac{\partial M_h}{\partial r} = \frac{1}{4\pi r^2} \iiint_{B_r(\mathbf{M})} \Delta h\, dV.$$

Second Derivative: We differentiate (7.1) with respect to r. Using the rule for differentiating a volume integral with variable radius

($\frac{d}{dr}\int_0^r \cdots = \ldots$), or simply product rule on $r^{-2} \times$ Integral:

$$\frac{\partial^2 M_h}{\partial r^2} = -\frac{2}{4\pi r^3} \iiint_{B_r(\mathbf{M})} \Delta h \, dV + \frac{1}{4\pi r^2} \oiint_{\partial B_r(\mathbf{M})} \Delta h \, dS_r.$$

Combining Terms:

$$\frac{\partial^2 M_h}{\partial r^2} + \frac{2}{r}\frac{\partial M_h}{\partial r} = \frac{1}{4\pi r^2} \oiint_{\partial B_r(\mathbf{M})} \Delta h \, dS_r.$$

The RHS is exactly the spherical mean of the function Δh. However, since the Laplacian commutes with the translation involved in defining the mean (or by differentiating under the integral sign on the unit sphere form):

$$\Delta_{\mathbf{M}} M_h = \frac{1}{4\pi} \oiint_{\partial B_1} \Delta_{\mathbf{M}} h(\mathbf{M} + r\boldsymbol{\alpha}) \, dS_1 = \frac{1}{4\pi r^2} \oiint_{\partial B_r} \Delta h \, dS_r.$$

Thus, the radial operator equals the spatial Laplacian.

■

Corollary 7.2. *Behaviour at Origin.*

$$\lim_{r \to 0^+} \frac{\partial}{\partial r} M_h(\mathbf{M}, r) = 0.$$

推論

Proof

From (7.1), $\frac{\partial M_h}{\partial r}$ is the average value of Δh over the ball B_r times $\frac{1}{3}r$ (since volume is $\frac{4}{3}\pi r^3$). As $r \to 0$, this term vanishes (boundedness of Δh).

■

7.2 *Harmonic Functions*

The Laplace operator gives rise to one of the most important classes of functions in analysis.

Definition 7.2. ***Harmonic Function.***
A twice continuously differentiable function $u : \Omega \to \mathbb{R}$ is called **harmonic** on the region Ω if it satisfies Laplace's equation:

$$\Delta u = 0$$

everywhere in Ω.

定義

Harmonic functions exhibit remarkable properties related to averages and extrema, which we now establish.

The Mean Value Property

We previously encountered the mean value property in the context of example 6.1. Here, we state it formally as a fundamental property of harmonic functions.

Theorem 7.3. ***Mean Value Formula.***
Let u be harmonic on a region Ω. For any point $\mathbf{M}_0 \in \Omega$ and any radius $R > 0$ such that the closed ball $\bar{B}_R(\mathbf{M}_0)$ is contained in Ω:

$$u(\mathbf{M}_0) = \frac{1}{4\pi R^2} \oiint_{\partial B_R(\mathbf{M}_0)} u(x, y, z)\, dS.$$

In other words, the value of a harmonic function at the centre of a sphere is the average of its values on the surface of the sphere.

定理

Proof

Consider the spherical mean $M_u(\mathbf{M}_0, \rho)$ defined in the previous section. By the radial differential equation for spherical means:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) M_u(\mathbf{M}_0, r) = \Delta M_u(\mathbf{M}_0, r).$$

Since u is harmonic ($\Delta u = 0$), the spatial Laplacian of its mean is also zero. Thus M_u satisfies the ODE:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial M_u}{\partial r}\right) = 0.$$

This implies $r^2 \frac{\partial M_u}{\partial r} = C$. Since $\frac{\partial M_u}{\partial r} \to 0$ as $r \to 0$ (from the previous section), we must have $C = 0$. Consequently, $\frac{\partial M_u}{\partial r} = 0$, so $M_u(\mathbf{M}_0, r)$ is constant with respect to r. By continuity, $\lim_{r \to 0} M_u(\mathbf{M}_0, r) = u(\mathbf{M}_0)$. Thus $M_u(\mathbf{M}_0, R) = u(\mathbf{M}_0)$.

■

The Maximum Principle

The mean value property implies that a harmonic function cannot have local "peaks" or "valleys" inside its domain, as a peak would require the value at the centre to exceed the surrounding average.

Theorem 7.4. ***Maximum Principle.***
Let u be a harmonic function on a connected region Ω. If u attains its supremum or infimum at an interior point of Ω, then u is constant on Ω.

定理

Proof

Suppose u attains its maximum value K at an interior point $\mathbf{M}_0 \in \Omega$. Assume for contradiction that u is not constant. Then the set of points where $u < K$ is non-empty. Let $U = \{\mathbf{x} \in \Omega : u(\mathbf{x}) = K\}$. Since u is continuous, U is closed in Ω. Since Ω is connected, if U were also open, then U would be all of Ω. We show U is open. Let $\mathbf{x} \in U$. Choose a ball $B_R(\mathbf{x}) \subset \Omega$. By the Mean Value Formula:

$$K = u(\mathbf{x}) = \frac{1}{4\pi R^2} \oiint_{\partial B_R(\mathbf{x})} u\, dS.$$

Since $u \leq K$ everywhere, if there were any point on the sphere where $u < K$, continuity would imply $u < K$ on a small patch, strictly lowering the average below K. Thus $u = K$ on the entire sphere. Since this holds for all $0 < \rho < R$, $u = K$ on the whole ball $B_R(\mathbf{x})$. This proves U is open. Since U is non-empty, open, and closed in the connected set Ω, we have $U = \Omega$. Thus u is constant, proving the contrapositive.

■

Corollary 7.3. *Boundary Extrema.* If Ω is a bounded region and u is continuous on $\bar{\Omega}$ and harmonic on Ω, then the maximum and minimum values of u are attained on the boundary $\partial\Omega$.

推論

Proof

Since $\bar{\Omega}$ is compact and u is continuous, u attains a global maximum at some point $\mathbf{M} \in \bar{\Omega}$. If $\mathbf{M} \in \partial\Omega$, the result holds. If $\mathbf{M} \in \Omega$ (interior), then by the Maximum Principle, u is constant on the connected component of Ω containing $\mathbf{M}$. By continuity, u is constant on the closure of that component, so the value at the boundary is the same maximum value. Thus, the maximum is always attained on the boundary. The same logic applies to the minimum.

■

Note

This principle is powerful for proving uniqueness of solutions to boundary value problems. If two harmonic functions agree on the boundary, their difference is harmonic and zero on the boundary; by the maximum principle, the difference must be zero everywhere.

7.3 *The Poisson Integral Formula*

In the theory of differential equations, a central problem is the Dirichlet problem: given a continuous function f on the boundary of a region Ω, does there exist a harmonic function u on the interior of Ω such that $u|_{\partial\Omega} = f$? We resolve this affirmatively for the case where Ω is a disk in $\mathbb{R}^2$.
To do so, we introduce a tool from harmonic analysis.

Definition 7.3. ***Approximate Identity.***
An **approximate identity** is a family of 2π-periodic functions $\{K_r(t)\}_{0\leq r<1}$ satisfying:

1. **Positivity:** $K_r(t) \geq 0$.
2. **Normalisation:** $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_r(t)\,dt = 1$.
3. **Concentration:** For any $\delta > 0$, $\lim\limits_{r\to 1^-} \max_{\delta\leq|t|\leq\pi} K_r(t) = 0$.

Common examples include the **Heat kernel** and the **Fejér kernel**. These kernels "concentrate" mass near $t = 0$ as $r \to 1$, allowing us to recover a function from its weighted average.

See Fourier Series Notes.

定義

Proposition 7.2. ***Convergence of Convolution.***
Let $\{K_r\}$ be an approximate identity and f be a continuous 2π-periodic function. Then the convolution $u_r(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta - t)K_r(t)\,dt$ satisfies:

$$\lim_{r\to 1^-} u_r(\theta) = f(\theta)$$

uniformly in θ.

命題

Proof

Let $\varepsilon > 0$. Since f is continuous on a compact circle, it is uniformly continuous. Choose $\delta > 0$ such that $|t| < \delta \implies |f(\theta - t) - f(\theta)| < \varepsilon/2$. Using the normalisation property of K_r:

$$|u_r(\theta) - f(\theta)| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(\theta - t) - f(\theta)]K_r(t)\,dt\right|.$$

We split the integral into I_1 (where $|t| < \delta$) and I_2 (where $\delta \leq |t| \leq \pi$). For I_1:

$$\frac{1}{2\pi}\int_{|t|<\delta}|f(\theta - t) - f(\theta)|K_r(t)\,dt < \frac{\varepsilon}{2}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi} K_r(t)\,dt = \frac{\varepsilon}{2}.$$

For I_2: Let $M = \sup|f|$. Then $|f(\theta - t) - f(\theta)| \leq 2M$.

$$I_2 \leq 2M \cdot \max_{\delta\leq|t|\leq\pi} K_r(t).$$

By the concentration property, for r sufficiently close to 1, this maximum is less than $\frac{\varepsilon}{4M}$. Thus $I_2 < \varepsilon/2$. Hence $|u_r(\theta) - f(\theta)| < \varepsilon$.

■

Theorem 7.5. *Poisson Integral Formula.*
Let $f(\theta)$ be a continuous, 2π-periodic function representing boundary values on the unit circle. The function u defined on the unit disk $D = \{(r, \theta) : 0 \leq r < 1\}$ by:

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} f(\varphi) P_r(\theta - \varphi)\, d\varphi,$$

where $P_r(t) = \frac{1-r^2}{1-2r\cos t + r^2}$ is the **Poisson kernel**, satisfies the following properties:

1. u is harmonic in D (i.e., $\Delta u = 0$).
2. u approaches the boundary values continuously:

$$\lim_{(r,\theta)\to(1,\theta_0)} u(r, \theta) = f(\theta_0)$$

   for every $\theta_0 \in [0, 2\pi)$.

定理

Uniqueness.

Suppose two such functions u and v exist. Their difference $w = u - v$ is harmonic in D and continuous on $\bar{D}$ with $w|_{\partial D} = 0$. By the Maximum Principle (*theorem* 7.4), the maximum and minimum of w are both 0. Thus $w \equiv 0$, proving uniqueness.

証明終

Existence and Harmonicity.

Motivated by the polar form of the Laplacian $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$, we observe that $r^n \cos n\theta$ and $r^n \sin n\theta$ are harmonic for $n \geq 0$. We construct u as a power series suggested by the Fourier series of f:

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta),$$

where a_n, b_n are the Fourier coefficients of f. Substituting the integral definitions of a_n, b_n and summing the geometric series (using complex exponentials) yields the kernel:

$$\frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos n(\theta - \varphi) = \operatorname{Re}\left[\frac{1}{2} + \sum_{n=1}^{\infty} (re^{i(\theta-\varphi)})^n\right] = \operatorname{Re}\left[\frac{1}{2} + \frac{z}{1-z}\right] = \frac{1}{2}\frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2}.$$

Since the series converges uniformly for $r \leq r_0 < 1$, term-by-term differentiation is valid. Since each term $r^n \cos n\theta$ is harmonic, the sum u is harmonic.

証明終

Boundary Behaviour.

The Poisson kernel $P_r(t)$ acts as an "approximate identity" (similar to the heat kernel or Dirichlet kernel discussed in *definition* 7.3). Specifically:

- $P_r(t) > 0$.
- $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)\,dt = 1$.
- For any $\delta > 0$, $\lim_{r\to 1^-} \max_{\delta\le|t|\le\pi} P_r(t) = 0$.

Using these properties, the convolution integral converges to $f(\theta_0)$ as $r \to 1$ (see the proof in *proposition* 7.2).

証明終

Corollary 7.4. *Poisson Formula for Disk of Radius R.* If u is harmonic on the open disk $B_R(0)$ and continuous on its closure, then for any point (r, θ) with $0 \le r < R$:

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} u(R,\varphi)\frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta-\varphi) + r^2}\,d\varphi.$$

推論

This formula allows us to recover the values of a harmonic function anywhere inside a disk solely from its values on the boundary circle.

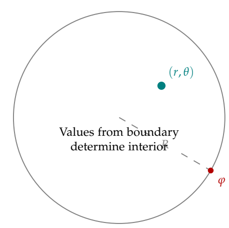


Figure 7.1: The Poisson integral reconstructs the harmonic function inside the disk from boundary data.

7.4 *Exercises*

1. **Vector Calculus Identity.** Prove the identity $\nabla(\nabla\cdot\mathbf{a}) - \nabla\times(\nabla\times\mathbf{a}) = \Delta\mathbf{a}$, where the Laplacian of a vector $\mathbf{a} = (a_1, a_2, a_3)$ is defined component-wise as $(\Delta a_1, \Delta a_2, \Delta a_3)$.
2. **Energy Method for Uniqueness.** Let u be harmonic on Ω and twice continuously differentiable on $\bar{\Omega}$.
   (a) Prove the identity:
   $$\iiint_\Omega |\nabla u|^2\,dV = \oiint_\Sigma u\frac{\partial u}{\partial n}\,dS.$$
   (b) Suppose $u = 0$ on the boundary Σ. Show that $\nabla u = \mathbf{0}$ everywhere in Ω, and thus u is constant (and hence zero).
   (c) Deduce that the solution to the Dirichlet problem ($\Delta u = f$ in Ω, $u = g$ on Σ) is unique.
3. **Solvability Condition.** Consider the Neumann problem:
   $$\Delta u = f \text{ in } D, \quad \frac{\partial u}{\partial n} = g \text{ on } \partial D.$$

Prove that a necessary condition for a solution to exist is:

$$\iiint_D f\,dV = \oiint_{\partial D} g\,dS.$$

4. **Volume Mean Value Property.** Using the surface mean value property, prove that if u is harmonic on a ball $B_R(\mathbf{M}_0)$, its value at the centre is the average over the volume:

$$u(\mathbf{M}_0) = \frac{1}{\frac{4}{3}\pi R^3} \iiint_{B_R(\mathbf{M}_0)} u(x,y,z)\,dV.$$

5. **Regularity.** Prove that harmonic functions are infinitely differentiable (C^∞).

> *Remark.*
>
> Hint: Use the Mean Value Property and differentiation under the integral sign.

6. **Composition Property.** Let f be a non-constant harmonic function on a connected open set. Let $g : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable. Prove that if the composition $g \circ f$ is also harmonic, then g must be a linear function.

7. **Direct Verification.** Verify by direct calculation that the Poisson integral

$$u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - \phi) + r^2} f(\phi)\,d\phi$$

satisfies Laplace's equation in polar coordinates.