Fourier Series

Gudfit

Contents

0

Geometry of Inner Product Spaces

The study of Fourier series relies fundamentally on decomposing functions into superpositions of simpler "basis" functions (e^{inx}). To formalise this, we must view functions not merely as maps from domains to codomains, but as vectors in a space equipped with geometric structure. Just as $\mathbb{R}^3$ has lengths and angles defined by the dot product, function spaces require a notion of inner product to define orthogonality and convergence.

We begin by formalising the geometry of finite-dimensional vector spaces over $\mathbb{C}$. The transition from $\mathbb{R}^n$ to $\mathbb{C}$ necessitates careful handling of linearity to preserve positivity.

We assume familiarity with the definition of a vector space V over a field $\mathbb{F}$ (where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$). To measure lengths and angles, we require an additional structure.

Definition 0.1. ***Euclidean Space.***
Let V be a finite-dimensional vector space over $\mathbb{R}$. An **inner product** on V is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ satisfying:

1. **Bilinearity:** For all $u, v, w \in V$ and $\alpha, \beta \in \mathbb{R}$:

$$\langle \alpha u + v, w \rangle = \alpha \langle u, w \rangle + \langle v, w \rangle,$$

$$\langle u, \beta v + w \rangle = \beta \langle u, v \rangle + \langle u, w \rangle.$$

2. **Symmetry:** For all $v, w \in V$, $\langle v, w \rangle = \langle w, v \rangle$.
3. **Positive Definiteness:** For all $v \in V$, $\langle v, v \rangle \geq 0$, with equality if and only if $v = 0_V$.

The pair $(V, \langle \cdot, \cdot \rangle)$ is called a **Euclidean space**.

定義

Example 0.1. $\mathbb{R}^n$. The canonical example is $\mathbb{R}^n$ with the dot product $\langle v, w \rangle = v^T w = \sum v_i w_i$.

範例

Let $u, v, w \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$. We verify the axioms directly using the

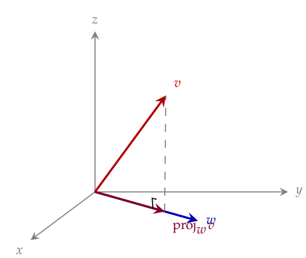


Figure 1: The inner product provides the geometric structure required to define projections and orthogonality in $\mathbb{R}^n$, derived from the algebraic axioms.

component-wise definition.

Symmetry.

Since multiplication in $\mathbb{R}$ is commutative ($v_i w_i = w_i v_i$):

$$\langle v, w \rangle = \sum_{i=1}^{n} v_i w_i = \sum_{i=1}^{n} w_i v_i = \langle w, v \rangle.$$

証明終

Bilinearity.

Linearity in the first argument follows from the distributivity of multiplication over addition in $\mathbb{R}$:

$$\langle \alpha u + v, w \rangle = \sum_{i=1}^{n} (\alpha u_i + v_i) w_i = \alpha \sum_{i=1}^{n} u_i w_i + \sum_{i=1}^{n} v_i w_i = \alpha \langle u, w \rangle + \langle v, w \rangle.$$

Linearity in the second argument is immediate by symmetry or by an identical expansion.

証明終

Positive Definiteness.

The square of a real number is non-negative, implying:

$$\langle v, v \rangle = \sum_{i=1}^{n} v_i^2 \geq 0.$$

If $\langle v, v \rangle = 0$, the sum of non-negative terms necessitates $v_i^2 = 0$ for all i. Hence $v_i = 0$ for all i, implying $v = 0_{\mathbb{R}^n}$. Conversely, $v = 0_{\mathbb{R}^n}$ implies $\langle v, v \rangle = 0$ trivially.

証明終

Fourier analysis inherently involves complex numbers via e^{inx}. Extending the definition of an inner product to a complex vector space V requires modification. If we strictly demanded bilinearity over $\mathbb{C}$, the positivity condition would collapse.

Note

Suppose $\langle \cdot, \cdot \rangle$ were bilinear over $\mathbb{C}$. Then for any $v \neq 0$:

$$\langle iv, iv \rangle = i \langle v, iv \rangle = i^2 \langle v, v \rangle = -\langle v, v \rangle.$$

If $\langle v, v \rangle > 0$, then $\langle iv, iv \rangle < 0$, violating positive definiteness.

To maintain $\langle v, v \rangle \geq 0$, the map must be *conjugate* linear in one argument. We adopt the mathematical convention (anti-linear in the second argument).

Definition 0.2. ***Hermitian Inner Product*.**
Let V be a finite-dimensional vector space over $\mathbb{C}$. A **Hermitian inner**

product is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ satisfying:

1. **Sesquilinearity:** It is linear in the first argument and anti-linear in the second. For $v_1, v_2, w \in V$ and $\alpha \in \mathbb{C}$:

$$\langle \alpha v_1 + v_2, w \rangle = \alpha \langle v_1, w \rangle + \langle v_2, w \rangle,$$
$$\langle v, \alpha w_1 + w_2 \rangle = \bar{\alpha} \langle v, w_1 \rangle + \langle v, w_2 \rangle.$$

2. **Hermitian Symmetry:** For all $v, w \in V$,

$$\langle v, w \rangle = \overline{\langle w, v \rangle}.$$

3. **Positive Definiteness:** For all $v \in V$, $\langle v, v \rangle \geq 0$, with equality if and only if $v = 0_V$.

The pair $(V, \langle \cdot, \cdot \rangle)$ is called a **Hermitian space**.

定義

Remark.

Hermitian symmetry ensures $\langle v, v \rangle = \overline{\langle v, v \rangle}$, so $\langle v, v \rangle$ is always real, making the positivity condition well-defined.

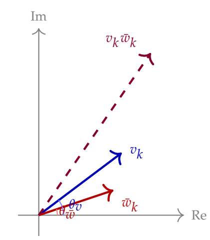


Figure 2: In $\mathbb{C}^n$, the inner product encodes both geometric projection and relative phase.

Example 0.2. The Standard Hermitian Product. On $\mathbb{C}^n$, for column vectors $v = (v_1, \ldots, v_n)^T$ and $w = (w_1, \ldots, w_n)^T$, we define:

$$\langle v, w \rangle = \sum_{k=1}^{n} v_k \overline{w_k} = w^\dagger v,$$

where $w^\dagger = (w^T)^*$ is the conjugate transpose.

範例

The inner product induces a natural notion of length (norm) and proximity (metric).

Definition 0.3. ***Induced Norm.***
Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. The **norm** of a vector v is

$$\|v\| = \sqrt{\langle v, v \rangle}.$$

The **distance** between vectors v and w is defined as $d(v, w) = \|v - w\|$.

定義

A vector v is a **unit vector** if $\|v\| = 1$.
The Cauchy-Schwarz inequality governs the geometry of inner product spaces, ensuring that the inner product cannot exceed the product of the lengths of the vectors.

Theorem 0.1. ***Cauchy-Schwarz Inequality.***
For all $v, w \in V$:
$$|\langle v, w\rangle| \leq \|v\|\|w\|.$$
Equality holds if and only if v and w are linearly dependent.

定理

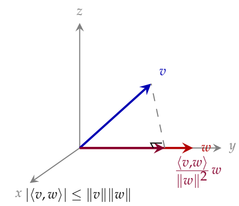


Figure 3: Cauchy-Schwarz geometrically: the projection $|\langle v, w\rangle|/\|w\|$ cannot exceed $\|v\|$.

The case $\langle v, w\rangle = 0$ is trivial. We assume $\langle v, w\rangle \neq 0$.

Case 1: Real Inner Product

Assume $\langle v, w\rangle \in \mathbb{R}$. Consider the function $P(t) = \|tv + w\|^2$ for $t \in \mathbb{R}$. By positive definiteness, $P(t) \geq 0$ for all t. Expanding using bilinearity (or sesquilinearity with real coefficients):

$$\begin{aligned} P(t) &= \langle tv + w, tv + w\rangle \\ &= t^2\langle v, v\rangle + t\langle v, w\rangle + t\langle w, v\rangle + \langle w, w\rangle \\ &= \|v\|^2 t^2 + 2\langle v, w\rangle t + \|w\|^2. \end{aligned}$$

Since $P(t)$ is a non-negative quadratic polynomial, its discriminant Δ must be non-positive:

$$\Delta = (2\langle v, w\rangle)^2 - 4\|v\|^2\|w\|^2 \leq 0.$$

This implies $4\langle v, w\rangle^2 \leq 4\|v\|^2\|w\|^2$, yielding $|\langle v, w\rangle| \leq \|v\|\|w\|$.

証明終

Case 2: Complex Inner Product

In the general case, $\langle v, w\rangle \in \mathbb{C}$. We can reduce this to the real case by rotating v. Let $\alpha = \frac{\langle w, v\rangle}{|\langle w, v\rangle|}$. This is a complex number of modulus 1. Define $\tilde{v} = \alpha v$. Then:

$$\langle \tilde{v}, w\rangle = \alpha\langle v, w\rangle = \frac{\overline{\langle v, w\rangle}}{|\langle v, w\rangle|}\langle v, w\rangle = \frac{|\langle v, w\rangle|^2}{|\langle v, w\rangle|} = |\langle v, w\rangle|.$$

Since $\langle \tilde{v}, w\rangle$ is real (indeed $\langle \tilde{v}, w\rangle = |\langle v, w\rangle|$), we apply Case 1 to $\tilde{v}$ and w:
$$|\langle \tilde{v}, w\rangle| \leq \|\tilde{v}\|\|w\|.$$
Substituting back, $|\langle v, w\rangle| \leq |\alpha|\|v\|\|w\| = \|v\|\|w\|$.

証明終

The properties of the induced norm follow from *theorem* 0.1.

Proposition 0.1. ***Properties of the Norm.***
For all $v, w \in V$ and λ in the base field ($\mathbb{R}$ or $\mathbb{C}$):

1. $\|v\| \geq 0$, and $\|v\| = 0 \iff v = 0$.
2. $\|\lambda v\| = |\lambda|\|v\|$ (Homogeneity).
3. $\|v + w\| \leq \|v\| + \|w\|$ (Triangle Inequality).

命題

Proof

The first two follow from the definition of the inner product. We prove the Triangle Inequality.

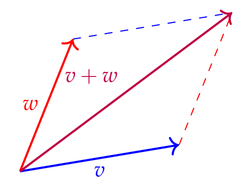
$$\begin{aligned}\|v+w\|^2 &= \langle v+w, v+w\rangle \\ &= \|v\|^2 + \langle v,w\rangle + \langle w,v\rangle + \|w\|^2 \\ &= \|v\|^2 + 2\mathrm{Re}(\langle v,w\rangle) + \|w\|^2.\end{aligned}$$

Since $\mathrm{Re}(z) \leq |z|$, we have $2\mathrm{Re}(\langle v,w\rangle) \leq 2|\langle v,w\rangle|$. By Cauchy-Schwarz:

$$\begin{aligned}\|v+w\|^2 &\leq \|v\|^2 + 2\|v\|\|w\| + \|w\|^2 \\ &= (\|v\| + \|w\|)^2.\end{aligned}$$

Taking the square root yields the result.

■



$\|v+w\| \leq \|v\| + \|w\|$

Figure 4: The Triangle Inequality: The length of the sum vector is at most the sum of the lengths of the constituent vectors.

Convergence in Normed Spaces

The definition of distance allows us to discuss limits and convergence, a prerequisite for defining infinite sums such as Fourier series.

Definition 0.4. ***Convergence of a Sequence.***
Let $(V, \|\cdot\|)$ be a normed vector space. A sequence of vectors $\{v_n\}_{n=1}^{\infty} \subset V$ is said to **converge** to a limit $v \in V$ if

$$\lim_{n\to\infty} \|v_n - v\| = 0.$$

We write $v_n \to v$ or $\lim_{n\to\infty} v_n = v$.

定義

In finite-dimensional spaces such as $\mathbb{C}^n$, convergence in norm is equivalent to coordinate-wise convergence. However, in infinite-dimensional function spaces (the setting of Fourier analysis), convergence in norm (e.g., mean square convergence) does not necessarily imply pointwise convergence. The geometry established here provides the robust framework required to navigate these subtleties.

0.1 Orthogonal and Orthonormal Families

The most distinct feature of inner product spaces is the ability to define perpendicularity, or orthogonality, which generalises the intuitive geometric concept to arbitrary dimensions.

Definition 0.5. ***Orthogonality.***
Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space.

1. Two vectors $v, w \in V$ are **orthogonal**, denoted $v \perp w$, if $\langle v, w \rangle = 0$.
2. A family of vectors $\mathcal{F} = \{v_1, \ldots, v_p\}$ is an **orthogonal family** if $v_i \perp v_j$ for all $i \neq j$.
3. A family $\mathcal{F}$ is an **orthonormal family** if it is orthogonal and every vector has unit norm: $\|v_i\| = 1$ for all i.

定義

Using the Kronecker delta symbol, defined as $\delta_{ij} = 1$ if $i = j$ and 0 otherwise, the condition for a family $\{v_i\}$ to be orthonormal can be written compactly as:

$$\langle v_i, v_j \rangle = \delta_{ij}.$$

Remark (Normalization).

Any non-zero vector v can be normalised to a unit vector $\hat{v} = \frac{v}{\|v\|}$.

Orthogonality simplifies the calculation of norms for sums of vectors, leading to a generalisation of the Pythagorean theorem.

Proposition 0.2. ***Pythagorean Identity.***
Let $u, v \in V$. If $u \perp v$, then:

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2.$$

命題

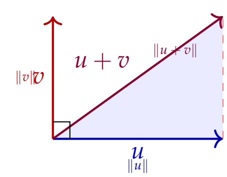


Figure 5: The Pythagorean identity: when $u \perp v$, the squared length of the hypotenuse equals the sum of the squared lengths of the legs.

Proof

Expanding the squared norm:

$$\begin{aligned}
\|u + v\|^2 &= \langle u + v, u + v \rangle \\
&= \|u\|^2 + \langle u, v \rangle + \langle v, u \rangle + \|v\|^2 \\
&= \|u\|^2 + 2\text{Re}(\langle u, v \rangle) + \|v\|^2.
\end{aligned}$$

Since $u \perp v$, $\langle u, v \rangle = 0$, and the result follows.

■

Note

In complex vector spaces, the converse is false. The condition $\|u + v\|^2 = \|u\|^2 + \|v\|^2$ implies only that $\text{Re}(\langle u, v \rangle) = 0$, not that the inner product itself vanishes. For example, in $\mathbb{C}^2$, let $u = (1, 0)^T$ and $v = (i, 0)^T$. Then $\|u + v\|^2 = |1 + i|^2 = 2$ and $\|u\|^2 + \|v\|^2 = 1 + 1 = 2$, yet $\langle u, v \rangle = 1 \cdot (-i) = -i \neq 0$.

A crucial consequence of orthogonality is that it enforces linear independence.

Theorem 0.2. ***Independence of Orthogonal Families.***
Let $\{v_1, \ldots, v_p\}$ be an orthogonal family of non-zero vectors in V. Then

the family is linearly independent.

定理

Proof

Suppose $\sum_{i=1}^{p} \alpha_i v_i = 0$ for some scalars α_i. We must show all α_i vanish. Take the inner product of the sum with a specific vector v_j:

$$\left\langle \sum_{i=1}^{p} \alpha_i v_i, v_j \right\rangle = \langle 0, v_j \rangle = 0.$$

Using linearity in the first argument:

$$\sum_{i=1}^{p} \alpha_i \langle v_i, v_j \rangle = 0.$$

By orthogonality, $\langle v_i, v_j \rangle = 0$ whenever $i \neq j$. The sum collapses to the single term $i = j$:

$$\alpha_j \langle v_j, v_j \rangle = \alpha_j \|v_j\|^2 = 0.$$

Since $v_j \neq 0$, we have $\|v_j\|^2 \neq 0$, implying $\alpha_j = 0$. This holds for all $j \in \{1, \ldots, p\}$.

■

Corollary 0.1. *Orthogonal Bases.* If V has dimension n, any orthogonal family of n non-zero vectors forms a basis for V, called an **orthogonal basis**. If the vectors are orthonormal, it is an **orthonormal basis**.

推論

The primary utility of such bases is the ease of computing coordinates. For a general basis, finding coefficients requires solving a linear system (often via Gaussian elimination). For an orthogonal basis, coefficients are decoupled and given by simple inner products.

Theorem 0.3. ***Decomposition in Orthogonal Bases.***
Let $\{u_1, \ldots, u_n\}$ be an orthogonal basis for V. For any $v \in V$:

$$v = \sum_{i=1}^{n} \frac{\langle v, u_i \rangle}{\|u_i\|^2} u_i.$$

If the basis is orthonormal, this simplifies to:

$$v = \sum_{i=1}^{n} \langle v, u_i \rangle u_i.$$

定理

Proof

Since $\{u_i\}$ is a basis, we may write $v = \sum_{j=1}^{n} \alpha_j u_j$. Taking the inner product with u_i:

$$\langle v, u_i \rangle = \left\langle \sum_{j=1}^{n} \alpha_j u_j, u_i \right\rangle = \sum_{j=1}^{n} \alpha_j \langle u_j, u_i \rangle = \alpha_i \|u_i\|^2,$$

where the sum vanishes for $j \neq i$ due to orthogonality. Solving for α_i yields the result.

■

The decomposition formula above suggests a geometric interpretation: the term $\frac{\langle v, u_i \rangle}{\|u_i\|^2} u_i$ represents the "shadow" or projection of v onto the line spanned by u_i. We generalise this to projections onto subspaces.

Let S be a subspace of V spanned by an orthogonal family of non-zero vectors $F = \{u_1, \ldots, u_m\}$. We define the **orthogonal projection** of a vector v onto S as:

$$P_S(v) = \sum_{i=1}^{m} \frac{\langle v, u_i \rangle}{\|u_i\|^2} u_i.$$

This operator $P_S : V \to S$ has three fundamental geometric properties: idempotence, orthogonality of the residual, and distance minimisation.

Theorem 0.4. ***Properties of Orthogonal Projection.***
Let $S = \text{Span}\{u_1, \ldots, u_m\}$ where $\{u_i\}$ is an orthogonal family of non-zero vectors.

1. **Identity on S:** If $s \in S$, then $P_S(s) = s$.
2. **Orthogonality:** The residual vector $v - P_S(v)$ is orthogonal to the subspace S. That is, for all $s \in S$,

$$\langle v - P_S(v), s \rangle = 0.$$

3. **Best Approximation:** For all $s \in S$,

$$\|v - P_S(v)\| \leq \|v - s\|,$$

with equality if and only if $s = P_S(v)$.

定理

1. Identity

Let $s \in S$. Then $s = \sum_{j=1}^{m} \alpha_j u_j$. Applying the decomposition formula restricted to the basis of S yields $P_S(s) = \sum \alpha_j u_j = s$.

証明終

2. Orthogonality

We first check orthogonality against the basis vectors u_k.

$$\langle P_S(v), u_k \rangle = \sum_{i=1}^{m} \frac{\langle v, u_i \rangle}{\|u_i\|^2} \langle u_i, u_k \rangle = \frac{\langle v, u_k \rangle}{\|u_k\|^2} \|u_k\|^2 = \langle v, u_k \rangle.$$

Thus $\langle v - P_S(v), u_k \rangle = \langle v, u_k \rangle - \langle P_S(v), u_k \rangle = 0$. Since the residual is orthogonal to every basis vector u_k, it is orthogonal to any linear combination $s \in S$.

証明終

3. Minimisation

Let $s \in S$. We write $v - s = (v - P_S(v)) + (P_S(v) - s)$. Note that $P_S(v) - s$ lies in S. By Property 2, $v - P_S(v)$ is orthogonal to S, and hence to $P_S(v) - s$. Applying the Pythagorean identity:

$$\|v - s\|^2 = \|v - P_S(v)\|^2 + \|P_S(v) - s\|^2.$$

Since norms are non-negative, $\|v - s\|^2 \geq \|v - P_S(v)\|^2$, with equality only when $\|P_S(v) - s\| = 0$, i.e., $s = P_S(v)$.

証明終

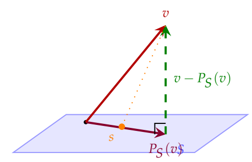


Figure 6: The orthogonal projection $P_S(v)$ is the unique point in S closest to v. The error vector $v - P_S(v)$ is orthogonal to the subspace.

The utility of orthonormal bases prompts a natural question: does every finite-dimensional inner product space possess one? The answer is affirmative and constructive.

Theorem 0.5. ***Gram-Schmidt Process.***
Let $\{v_1, \ldots, v_p\}$ be a linearly independent family in V. There exists an orthonormal family $\{u_1, \ldots, u_p\}$ such that for all $k \leq p$:

$$\text{Span}\{u_1, \ldots, u_k\} = \text{Span}\{v_1, \ldots, v_k\}.$$

定理

We proceed inductively.

Base Step.

Set $u_1 = \frac{v_1}{\|v_1\|}$. Clearly $\text{Span}\{u_1\} = \text{Span}\{v_1\}$.

証明終

Inductive Step.

Suppose we have constructed $\{u_1, \ldots, u_{k-1}\}$. We project v_k onto the subspace spanned by these vectors and subtract the projection to obtain the orthogonal component. Let

$$w_k = v_k - \sum_{j=1}^{k-1} \langle v_k, u_j \rangle u_j.$$

By *theorem* 0.4, w_k is orthogonal to $u_1, \ldots, u_{k-1}$. Since $\{v_i\}$ are linearly independent, $v_k \notin \text{Span}\{v_1, \ldots, v_{k-1}\} = \text{Span}\{u_1, \ldots, u_{k-1}\}$, so $w_k \neq 0$. We normalise to obtain the next vector:

$$u_k = \frac{w_k}{\|w_k\|}.$$

The span is preserved by construction.

証明終

When working with an orthonormal basis $\{u_1, \ldots, u_n\}$, the inner product structure is completely determined by the coefficients.

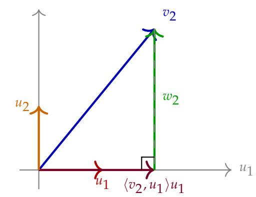


Figure 7: Gram-Schmidt: v_2 is decomposed into its projection onto u_1 and an orthogonal component w_2. Normalising w_2 yields u_2.

Theorem 0.6. ***Isometries of Coefficients.***
Let $\{u_i\}_{i=1}^n$ be an orthonormal basis for V. For any $v, w \in V$:

1. **Parseval's Identity:**

$$\langle v, w\rangle = \sum_{i=1}^{n} \langle v, u_i\rangle \overline{\langle w, u_i\rangle} = \sum_{i=1}^{n} \langle v, u_i\rangle \langle u_i, w\rangle.$$

2. **Plancherel's Identity:**

$$\|v\|^2 = \sum_{i=1}^{n} |\langle v, u_i\rangle|^2.$$

定理

Proof

By the decomposition theorem, $v = \sum \langle v, u_i\rangle u_i$ and $w = \sum \langle w, u_j\rangle u_j$. Using sesquilinearity:

$$\begin{aligned} \langle v, w\rangle &= \left\langle \sum_i \langle v, u_i\rangle u_i, \sum_j \langle w, u_j\rangle u_j \right\rangle \\ &= \sum_{i,j} \langle v, u_i\rangle \overline{\langle w, u_j\rangle} \langle u_i, u_j\rangle. \end{aligned}$$

Since $\langle u_i, u_j\rangle = \delta_{ij}$, the double sum reduces to the single sum over $i = j$. Plancherel's identity follows by setting $w = v$.

■

Plancherel's identity admits a physical interpretation: the total "energy" ($\|v\|^2$) of a signal is the sum of the energies of its harmonic components ($|\langle v, u_i\rangle|^2$). This conservation law is central to Fourier Analysis.

0.2 *Linear Operators and Matrices*

The matrix representation of these operators, particularly those preserving the inner product, is fundamental to the application of Fourier theory. We adopt the standard convention for representing vectors and operators in finite-dimensional spaces.

Notation 0.1. Coordinate Vectors Let V be a vector space of dimension n with a basis $\mathcal{E} = (e_1, \dots, e_n)$. Any vector $v \in V$ admits a unique expansion $v = \sum_{j=1}^{n} c_j e_j$. The **coordinate vector** of v relative to $\mathcal{E}$ is the column vector:

$$[v]_{\mathcal{E}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{F}^n.$$

記法

Consider a linear map $L : V \to W$, where $\dim(V) = n$ and $\dim(W) = m$. Let $\mathcal{E} = (e_j)_{j=1}^{n}$ be a basis for V and $\mathcal{F} = (f_i)_{i=1}^{m}$ be a basis for W. The action of L is completely determined by its action on the basis vectors of V. We decompose the image of each e_j in the basis $\mathcal{F}$:

$$L(e_j) = \sum_{i=1}^{m} a_{ij} f_i, \quad \text{for } j = 1, \dots, n. \tag{1}$$

Theorem 0.7. ***Matrix Representation.***
There exists a unique $m \times n$ matrix $A = (a_{ij})$, called the **representative matrix** of L relative to bases $\mathcal{E}$ and $\mathcal{F}$, such that for all $v \in V$:

$$[L(v)]_{\mathcal{F}} = A[v]_{\mathcal{E}}.$$

We denote this matrix by $A = A_L^{\mathcal{E},\mathcal{F}}$.

定理

Proof

Let $v \in V$ with $[v]_{\mathcal{E}} = (c_1, \dots, c_n)^T$. By linearity and eq. (1):

$$L(v) = L\left(\sum_{j=1}^{n} c_j e_j\right) = \sum_{j=1}^{n} c_j L(e_j) = \sum_{j=1}^{n} c_j \left(\sum_{i=1}^{m} a_{ij} f_i\right).$$

Interchanging the finite sums:

$$L(v) = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} c_j\right) f_i.$$

The coefficient of f_i corresponds exactly to the i-th row of the matrix product $A[v]_{\mathcal{E}}$. Thus $[L(v)]_{\mathcal{F}} = A[v]_{\mathcal{E}}$.
To prove uniqueness, suppose another matrix B satisfies the condition. Then for all v, $(A - B)[v]_{\mathcal{E}} = 0$. Choosing $v = e_j$ (where $[e_j]_{\mathcal{E}}$ is the standard basis vector of $\mathbb{F}^n$) implies the j-th column of $A - B$ is zero. Since this holds for all j, $A = B$.

∎

Definition 0.6. ***Endomorphisms.***
An operator $L : V \to V$ is called an **endomorphism**. The set of all endomorphisms on V forms a vector space, denoted $\mathrm{End}(V)$. In this case, the representative matrix is square ($n \times n$).

定義

We often require the representation of a vector or operator in a different basis to simplify calculations (e.g., diagonalisation). Let $\mathcal{E} = (e_j)_{j=1}^n$ and $\mathcal{F} = (f_i)_{i=1}^n$ be two bases of a vector space V.
We define the **transition matrix** P from $\mathcal{E}$ to $\mathcal{F}$ as the representative matrix of the identity operator id_V, where the domain is equipped with basis $\mathcal{F}$ and the codomain with basis $\mathcal{E}$. That is, $P = A_{id}^{\mathcal{F},\mathcal{E}}$.

Proposition 0.3. ***Properties of the Transition Matrix.***

1. The columns of P are the coordinates of the "new" basis vectors $\mathcal{F}$ expressed in the "old" basis $\mathcal{E}$.
2. The matrix P transforms coordinates from $\mathcal{F}$ to $\mathcal{E}$:
$$[v]_{\mathcal{E}} = P[v]_{\mathcal{F}}. \tag{2}$$
3. P is invertible. Its inverse $Q = P^{-1}$ satisfies $[v]_{\mathcal{F}} = Q[v]_{\mathcal{E}}$ and represents the coordinates of $\mathcal{E}$ expressed in $\mathcal{F}$.

命題

Proof
Property 2 follows directly from the definition $P = A_{id}^{\mathcal{F},\mathcal{E}}$ and the matrix representation theorem: $[id(v)]_{\mathcal{E}} = P[v]_{\mathcal{F}}$. For Property 1, choose $v = f_j$. Then $[f_j]_{\mathcal{F}}$ is the j-th canonical vector of $\mathbb{F}^n$. The product $P[f_j]_{\mathcal{F}}$ yields the j-th column of P. By eq. (2), this equals $[f_j]_{\mathcal{E}}$. Invertibility follows because id is a bijection.

■

Note
Confusion often arises regarding the direction of P. A mnemonic is that P acts on the *new* coordinates to produce the *old* coordinates. Explicitly, if $\mathcal{F}$ are the eigenvectors of an operator, P is the matrix containing these eigenvectors as columns.

The transformation of vectors induces a transformation of operators.

Theorem 0.8. ***Similarity of Matrix Representations.***
Let $L \in \mathrm{End}(V)$. Let A be the matrix of L relative to basis $\mathcal{E}$, and B be the matrix of L relative to basis $\mathcal{F}$. Let P be the transition matrix such that $[v]_{\mathcal{E}} = P[v]_{\mathcal{F}}$. Then:
$$B = P^{-1}AP.$$

定理

Proof

We rely on the commutativity of the mapping diagram. For any $v \in V$:

$$[L(v)]_{\mathcal{F}} = B[v]_{\mathcal{F}}.$$

Alternatively, we can map v through basis $\mathcal{E}$:

$$\begin{aligned} [L(v)]_{\mathcal{F}} &= P^{-1}[L(v)]_{\mathcal{E}} \quad \text{(converting output)} \\ &= P^{-1}(A[v]_{\mathcal{E}}) \quad \text{(applying } L \text{ in } \mathcal{E}) \\ &= P^{-1}A(P[v]_{\mathcal{F}}) \quad \text{(converting input).} \end{aligned}$$

Thus $B[v]_{\mathcal{F}} = (P^{-1}AP)[v]_{\mathcal{F}}$ for all v, implying $B = P^{-1}AP$. ■

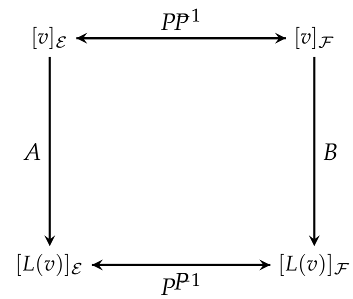


Figure 8: Commutative diagram illustrating the similarity transformation. The operator L can be computed via A in the $\mathcal{E}$ basis or B in the $\mathcal{F}$ basis.

Matrices A and B satisfying $B = P^{-1}AP$ are said to be **similar**. Similar matrices represent the same linear operator under different choices of basis. A primary goal in linear algebra, and essential for Fourier analysis, is selecting a basis $\mathcal{F}$ such that B is diagonal.

Isometries and Unitary Operators

An operator that preserves the inner product preserves lengths and angles, effectively acting as a rigid motion (possibly including reflection) in the vector space.

Definition 0.7. ***Preservation of Inner Product.***
Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. An endomorphism $L \in \mathrm{End}(V)$ is said to **preserve the inner product** if:

$$\forall v, w \in V, \quad \langle Lv, Lw \rangle = \langle v, w \rangle.$$

定義

Such operators are characterised by their action on orthonormal bases.

Theorem 0.9. ***Preservation of Bases.***
Let V be finite-dimensional. An operator L preserves the inner product if and only if it maps every orthonormal basis of V to an orthonormal basis of V.

定理

($\Longrightarrow$)

Assume L preserves the inner product. Let $\{u_i\}_{i=1}^n$ be an orthonormal basis. Then:

$$\langle Lu_i, Lu_j \rangle = \langle u_i, u_j \rangle = \delta_{ij}.$$

Thus, $\{Lu_i\}_{i=1}^n$ is an orthonormal family. Since the dimension is preserved, it forms a basis.

証明終

($\Longleftarrow$)

Assume L maps any orthonormal basis $\{u_i\}$ to an orthonormal basis $\{Lu_i\}$. Let $v, w \in V$. By Parseval's identity (established in the previous chapter), the inner product is determined by coordinates:

$$\langle v, w\rangle = \sum_{i=1}^{n} \langle v, u_i\rangle \overline{\langle w, u_i\rangle}.$$

By linearity, $Lv = \sum_{i=1}^{n} \langle v, u_i\rangle Lu_i$. Since $\{Lu_i\}$ is also an orthonormal basis, we apply Parseval's identity to the images:

$$\langle Lv, Lw\rangle = \sum_{i=1}^{n} \langle v, u_i\rangle \overline{\langle w, u_i\rangle}.$$

Comparing the two sums yields $\langle Lv, Lw\rangle = \langle v, w\rangle$.

証明終

It is immediate that if L preserves the inner product, it preserves the norm: $\|Lv\|^2 = \langle Lv, Lv\rangle = \langle v, v\rangle = \|v\|^2$. Such operators are often called **isometries**. The converse holds due to the polarisation identity.

Proposition 0.4. ***Properties of Isometries.***
Let $L \in \mathrm{End}(V)$ preserve the inner product.

1. **Norm Preservation:** $\|Lv\| = \|v\|$ for all $v \in V$.
2. **Injectivity:** $\mathrm{Ker}(L) = \{0\}$. Since V is finite-dimensional, L is invertible.
3. **Eigenvalues:** If λ is an eigenvalue of L, then $|\lambda| = 1$.

命題

Proof

We prove property 3. Let v be a non-zero eigenvector such that $Lv = \lambda v$. Since L is an isometry:

$$\|v\| = \|Lv\| = \|\lambda v\| = |\lambda|\|v\|.$$

Dividing by $\|v\| \neq 0$ yields $|\lambda| = 1$. Thus, in $\mathbb{C}$, eigenvalues are of the form $e^{i\theta}$.

■

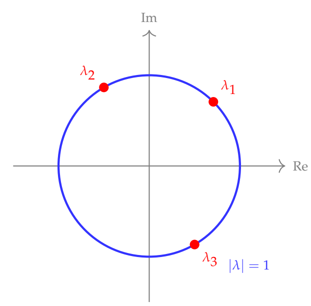


Figure 9: The eigenvalues of an isometry lie on the unit circle in the complex plane. The operator acts as a rotation or reflection.

Matrix Representation: Unitary and Orthogonal

The algebraic characterisation of isometries depends on the base field. Let U be the representative matrix of L relative to an orthonormal basis.

Definition 0.8. ***Unitary Matrices.***
A complex matrix $U \in M_n(\mathbb{C})$ is **unitary** if its inverse is its conjugate transpose (adjoint):

$$U^{-1} = U^{\dagger} = (\overline{U})^T.$$

Equivalently, $U^{\dagger}U = I$.

定義

Definition 0.9. ***Orthogonal Matrices.***
A real matrix $O \in M_n(\mathbb{R})$ is **orthogonal** if its inverse is its transpose:

$$O^{-1} = O^T.$$

Equivalently, $O^T O = I$.

定義

These definitions precisely capture the preservation of the standard Euclidean (or Hermitian) inner product.

Theorem 0.10. ***Equivalence of Definitions.***
Let $U \in M_n(\mathbb{C})$ (resp. $O \in M_n(\mathbb{R})$). The following are equivalent:

1. The matrix preserves the standard inner product: $\langle Ux, Uy \rangle = \langle x, y \rangle$.
2. The matrix is an isometry: $\|Ux\| = \|x\|$.
3. The matrix is Unitary (resp. Orthogonal).
4. The columns (and rows) form an orthonormal basis of $\mathbb{C}^n$ (resp. $\mathbb{R}^n$).

定理

Diagonalisation and Spectral Theory

Diagonalisation is the process of finding a basis in which the action of an operator is a simple scaling of coordinates.

Definition 0.10. ***Diagonalisability.***
An endomorphism L is **diagonalisable** if there exists a basis of V consisting of eigenvectors of L. Equivalently, its representative matrix A is similar to a diagonal matrix D, i.e., $A = PDP^{-1}$.

定義

While not all operators are diagonalisable, those interacting "nicely" with the inner product often are. A central class of such operators are those that are self-adjoint.

Definition 0.11. ***Hermitian Operators.***
An endomorphism L is **Hermitian** (or self-adjoint) if for all $v, w \in V$:

$$\langle Lv, w \rangle = \langle v, Lw \rangle.$$

定義

In terms of matrices, this corresponds to $A = A^\dagger$ (Hermitian matrix) in the complex case, or $A = A^T$ (Symmetric matrix) in the real case. The Spectral Theorem is a cornerstone of linear algebra, guaranteeing that Hermitian operators can be decomposed into independent modes (eigenvectors) that are orthogonal to each other.

Theorem 0.11. ***Spectral Theorem for Hermitian Operators.***
Let L be a Hermitian endomorphism on a finite-dimensional inner product space V. Then:

1. All eigenvalues of L are real.
2. Eigenvectors corresponding to distinct eigenvalues are orthogonal.
3. V admits an orthonormal basis consisting of eigenvectors of L.

Consequently, the representative matrix A can be diagonalised by a unitary matrix P:

$$A = PDP^\dagger,$$

where D is a real diagonal matrix and P is unitary.

定理

This theorem provides the algebraic justification for the decomposition of signals into orthogonal modes, a principle we will exploit extensively in the construction of Fourier series.

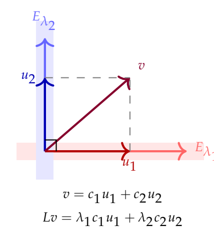


Figure 10: Spectral decomposition: a Hermitian operator acts by scaling along orthogonal eigenspaces E_{λ_i}, each by its real eigenvalue λ_i.

1

Introduction

The theory of Fourier series relies fundamentally on decomposing complex functions into superpositions of elementary basis functions, as introduced in chapter 0. This approach originates from the analysis of partial differential equations governing physical phenomena, specifically the heat and wave equations. Although physically motivated, the resulting theory of trigonometric series necessitates rigorous definitions of convergence, integration, and function spaces.

1.1 The Vibrating String

Consider an ideal elastic string of length l, fixed at endpoints $x = 0$ and $x = l$. At time $t = 0$, the string is displaced from its equilibrium and released. The vertical displacement $u(x,t)$ evolves according to the one-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{1.1}$$

where c^2 is a physical constant relating tension and mass density. The system is subject to Dirichlet boundary conditions:

$$u(0,t) = 0, \quad u(l,t) = 0 \quad \text{for all } t > 0. \tag{1.2}$$

The state of the system is uniquely determined by the initial configuration $f(x)$ and the initial velocity $g(x)$:

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x). \tag{1.3}$$

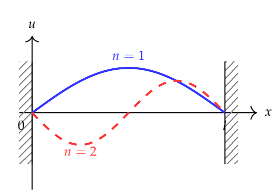


Figure 1.1: Fundamental modes of a vibrating string. The displacement vanishes at the boundaries $x = 0$ and $x = l$.

> *Note*
>
> To maintain the fixed-endpoint conditions for all t, we assume the compatibility conditions $f(0) = f(l) = 0$ and $g(0) = g(l) = 0$.

d'Alembert (1747) provided the first solution using the method of travelling waves. Observing that any function of the form $\phi(x \pm ct)$ satisfies eq. (1.1), he derived:

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy. \tag{1.4}$$

This is the Cauchy solution on the whole line. For a string fixed at $x = 0$ and $x = l$, one extends f and g to $\mathbb{R}$ by odd $2l$-periodic reflection.

Bernoulli's Separation of Variables

Daniel Bernoulli (1753) proposed a distinct method based on the physical observation that strings vibrate in fundamental modes. Separating the solution into spatial and temporal components $u(x,t) = F(x)G(t)$ yields:

$$F(x)G''(t) = c^2F''(x)G(t) \implies \frac{F''(x)}{F(x)} = \frac{1}{c^2}\frac{G''(t)}{G(t)}.$$

Since the left side depends solely on x and the right solely on t, both must equal a common separation constant k. This decouples the PDE into two ordinary differential equations:

$$F''(x) - kF(x) = 0, \tag{1.1}$$

$$G''(t) - kc^2G(t) = 0. \tag{1.2}$$

The boundary conditions (eq. (1.2)) imply $F(0)G(t) = 0$ and $F(l)G(t) = 0$. For non-trivial solutions ($G(t) \not\equiv 0$), we require $F(0) = F(l) = 0$. We now analyse the eigenvalues k.

Proposition 1.1. ***Eigenvalues of the Fixed String.***
The spatial boundary value problem given by eq. (1.1) with $F(0) = F(l) = 0$ admits non-trivial solutions if and only if $k = -(n\pi/l)^2$ for $n \in \mathbb{Z}^+$.

命題

We consider the three possible cases for the real constant k:

Case $k = 0$

The equation reduces to $F''(x) = 0$, with general solution $F(x) = Ax + B$. $F(0) = 0 \implies B = 0$. $F(l) = 0 \implies Al = 0 \implies A = 0$. This yields only the trivial solution.

証明終

Case $k = \mu^2 > 0$

The equation is $F''(x) - \mu^2F(x) = 0$. The general solution is $F(x) = Ae^{\mu x} + Be^{-\mu x}$. $F(0) = 0 \implies A + B = 0$. $F(l) = 0 \implies A(e^{\mu l} - e^{-\mu l}) = 2A\sinh(\mu l) = 0$. Since $\mu \neq 0$ and $l > 0$, $\sinh(\mu l) \neq 0$, forcing $A = 0$ and $B = 0$.

証明終

Case $k = -\lambda^2 < 0$

> The equation is $F''(x) + \lambda^2 F(x) = 0$. The general solution is:
>
> $$F(x) = A\cos(\lambda x) + B\sin(\lambda x).$$
>
> $F(0) = 0 \implies A = 0$. $F(l) = 0 \implies B\sin(\lambda l) = 0$. For a non-trivial solution ($B \neq 0$), we require $\sin(\lambda l) = 0$. This implies $\lambda l = n\pi$ for $n \in \mathbb{Z}$. Since $\sin(-x) = -\sin(x)$, we may restrict n to the positive integers $\mathbb{Z}^+$. Thus $\lambda_n = \frac{n\pi}{l}$.
>
> 証明終

Corresponding to each spatial eigenvalue $\lambda_n = n\pi/l$, the temporal equation (eq. (1.2)) becomes:

$$G_n''(t) + c^2\lambda_n^2 G_n(t) = 0.$$

This is the equation of a simple harmonic oscillator, with solution:

$$G_n(t) = C_n\cos(c\lambda_n t) + D_n\sin(c\lambda_n t).$$

By the principle of superposition for linear differential equations, Bernoulli asserted that the general solution is the sum of these normal modes:

$$u(x,t) = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{l}\right)\left[\tilde{C}_n\cos\left(\frac{n\pi ct}{l}\right) + \tilde{D}_n\sin\left(\frac{n\pi ct}{l}\right)\right]. \tag{1.5}$$

Evaluating this at $t = 0$ against the initial conditions (eq. (1.3)):

$$u(x,0) = \sum_{n=1}^{\infty}\tilde{C}_n\sin\left(\frac{n\pi x}{l}\right) = f(x), \partial_t u(x,0) \quad = \sum_{n=1}^{\infty}(c\lambda_n\tilde{D}_n)\sin\left(\frac{n\pi x}{l}\right) = g(x). \tag{1.1}$$

Bernoulli's solution implied that *any* arbitrary function $f(x)$ describing the initial displacement of a string could be represented as a series of sines. This claim contradicted the prevailing intuition that discontinuous plucks could not be constructed from smooth analytic functions, a debate resolved by Joseph Fourier's 1807 work on heat conduction. Fourier considered the heat equation:

$$\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2}, \tag{1.6}$$

subject to fixed temperatures at the ends $u(0,t) = u(l,t) = 0$ and initial temperature distribution $u(x,0) = f(x)$.
Applying separation of variables yields the same spatial eigenfunctions $\sin(\lambda_n x)$. However, the temporal equation is first-order:

$$G'(t) = -\kappa\lambda_n^2 G(t) \implies G_n(t) = A_n e^{-\kappa\lambda_n^2 t}.$$

Superposing these solutions leads to the series:

$$u(x,t) = \sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi x}{l}\right)e^{-\kappa\lambda_n^2 t}. \tag{1.7}$$

Setting $t = 0$, we recover the same fundamental problem encountered by Bernoulli:

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right).$$

Fourier provided explicit formulas for the coefficients A_n, asserting that for any function f:

$$A_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)\, dx.$$

The validity of the expansion at $t = 0$ requires a rigorous theory of integration and convergence. Throughout this chapter, we work with functions defined on intervals, such as $[0, l]$ for the string or $[-\pi, \pi]$ for the general theory. We briefly recall the relevant notions of integrability.

> **Definition 1.1.** ***Continuous Functions.***
> A complex-valued function f on $[0, L]$ is **continuous** if it is continuous at every point of $[0, L]$ in the usual sense.
>
> 定義

> **Definition 1.2.** ***Piecewise Continuous Functions.***
> A function $f : [0, L] \to \mathbb{C}$ is **piecewise continuous** if:
> · f is bounded on $[0, L]$, and
> · there are only finitely many points in $[0, L]$ where f is discontinuous, and at each such point the one-sided limits exist and are finite.
>
> 定義

Piecewise continuous functions are sufficient for many examples (e.g. step functions, simple waves), but for a clean theory of Fourier coefficients we adopt the more general framework of Riemann integrable functions.

> **Definition 1.3.** ***Riemann Integrable Function.***
> Let $f : [0, L] \to \mathbb{R}$ be bounded. For a subdivision
>
> $$0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = L$$
>
> define the **upper sum** and **lower sum**:
>
> $$U = \sum_{j=1}^{N}\left(\sup_{x_{j-1} \le x \le x_j} f(x)\right)(x_j - x_{j-1}),$$
> $$L = \sum_{j=1}^{N}\left(\inf_{x_{j-1} \le x \le x_j} f(x)\right)(x_j - x_{j-1}).$$
>
> We say f is **(Riemann) integrable** on $[0, L]$ if for every $\epsilon > 0$ there exists a subdivision such that $U - L < \epsilon$. For a complex-valued func-

tion $f : [0, L] \to \mathbb{C}$, we say f is integrable if its real and imaginary parts are both integrable.

定義

Note

It is a standard result that a bounded function is Riemann integrable if and only if its set of discontinuities has measure zero. From now on, unless explicitly stated otherwise, all functions are assumed to be (Riemann) integrable.

1.2 *Introduction to Fourier Series*

The analysis of periodic phenomena is ubiquitous in the physical sciences, appearing in contexts ranging from celestial mechanics to the theory of sound and heat. Such phenomena are characterised by the property that the state of the system repeats after a fixed duration T, the **period**. Mathematically, this motivates the study of periodic functions and their decomposition into elementary oscillatory components.

The simplest periodic function is the harmonic wave:

$$x(t) = a \sin(\omega t + \varphi),$$

where a is the amplitude, $\omega = 2\pi/T$ is the angular frequency, and φ is the initial phase. Linearity governs the superposition of such waves. While the sum of two harmonics with identical frequency remains a simple harmonic, the superposition of differing frequencies yields complex waveforms.

Example 1.1. Superposition of Harmonics. Consider the waves $x_1(t) = \sin t$ and $x_2(t) = \frac{1}{3}\sin 3t$. The superposition $x(t) = x_1(t) + x_2(t)$ exhibits a more intricate structure than its constituents, yet retains periodicity.

範例

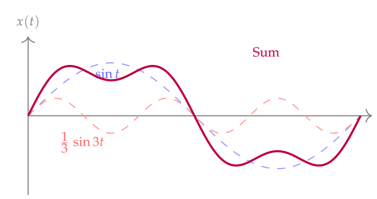


Figure 1.2: The superposition of fundamental and third harmonics approximates a square wave structure.

This observation suggests an inverse hypothesis: can an arbitrary periodic function f be decomposed into a series of simple harmonics?

$$f(t) = \sum_{n=0}^{\infty} A_n \sin(n\omega t + \varphi_n). \tag{1.8}$$

By rescaling the variable $x = (2\pi/T)t$, we may restrict our attention to functions of period 2π (angular frequency $\omega = 1$). Expanding the sine term via the addition formula $\sin(nx + \varphi_n) = \sin nx \cos \varphi_n + \cos nx \sin \varphi_n$, we rewrite eq. (1.8) as:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{1.9}$$

where $a_0 = 2A_0 \sin\varphi_0$, $a_n = A_n \sin\varphi_n$, and $b_n = A_n \cos\varphi_n$. The constant term is conventionally written as $a_0/2$ to unify the coefficient formulas derived below.

The Orthogonality of the Trigonometric System

To determine the coefficients a_n and b_n, we exploit the geometric properties of the trigonometric system $\mathcal{T} = \{1, \cos x, \sin x, \cos 2x, \sin 2x, \dots\}$. In the language of chapter 0, we consider these functions as vectors in an infinite-dimensional space equipped with the inner product $\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$.

Proposition 1.2. ***Orthogonality of Real Trigonometric Functions.*** The system $\mathcal{T}$ is orthogonal on $[-\pi, \pi]$. Specifically, for non-negative integers m, n:

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0 & m \neq n \\ \pi & m = n \neq 0 \\ 2\pi & m = n = 0 \end{cases} \tag{1.1}$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & m \neq n \\ \pi & m = n \neq 0 \\ 0 & m = n = 0 \end{cases} \tag{1.2}$$

$$\int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 0 \quad \text{for all } m, n. \tag{1.3}$$

命题

Proof

We employ the product-to-sum identities:

$$\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x].$$

If $m \neq n$, integration yields terms of the form $\left[\frac{\sin kx}{k}\right]_{-\pi}^{\pi}$ with $k \in \{m-n, m+n\}$, which vanish since $\sin(k\pi) = 0$. If $m = n \neq 0$, the identity becomes $\frac{1}{2}[1 + \cos 2nx]$. The integral is $\frac{1}{2}[x + \frac{\sin 2nx}{2n}]_{-\pi}^{\pi} = \pi$. The case $m = n = 0$ is simply $\int_{-\pi}^{\pi} 1\,dx = 2\pi$.
Similarly for sine:

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x].$$

Integration follows the same logic, yielding $\pi\delta_{mn}$ for $n \geq 1$.
Finally, $\cos mx \sin nx = \frac{1}{2}[\sin(m+n)x - \sin(m-n)x]$. Since the integrand is an odd function (assuming m, n integers) over a symmetric interval, the integral vanishes identically.

■

Assuming the series eq. (1.9) converges uniformly to $f(x)$, we may integrate term-by-term. Multiplying by $\cos kx$ and integrating over $[-\pi, \pi]$:

$$\int_{-\pi}^{\pi} f(x)\cos kx\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi}\cos kx\,dx + \sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos nx\cos kx\,dx + b_n\int_{-\pi}^{\pi}\sin nx\cos kx\,dx\right).$$

By orthogonality, all terms in the sum vanish except when $n = k$. Thus, for $k \geq 1$:

$$\int_{-\pi}^{\pi} f(x)\cos kx\,dx = a_k(\pi).$$

A similar process isolates b_k. This yields the Euler-Fourier formulas:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx. \tag{1.10}$$

1.3 *Formal Fourier Series*

To streamline the theory, we work on the interval $[-\pi, \pi]$ and employ the complex exponential e^{inx}. This transition highlights the connection between Fourier analysis and the geometry of inner product spaces established in chapter 0.

> **Definition 1.4.** ***Periodic Function.***
> Let $f : \mathbb{R} \to \mathbb{C}$. We say that f is **periodic** if there exists $T \neq 0$ such that for all $\theta \in \mathbb{R}$,
> $$f(\theta + T) = f(\theta).$$
> We say that T is a **period** of f. If f admits a smallest period $T > 0$, this is called the **fundamental period**.
>
> 定義

> *Remark.*
> If T is a period for f, then kT is also a period for all $k \in \mathbb{Z} \setminus \{0\}$. Common examples include $\theta \mapsto e^{in\theta}$ (period $2\pi/n$) and $\theta \mapsto \tan\theta$ (period π).

There is a natural identification between 2π-periodic functions on $\mathbb{R}$ and functions on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Every point on $\mathbb{T}$ can be written as $z = e^{i\theta}$ for some real θ, unique up to integer multiples of 2π.

> **Definition 1.5.** ***From the Circle to the Line.***
> Given a function $F : \mathbb{T} \to \mathbb{C}$, define $f : \mathbb{R} \to \mathbb{C}$ by $f(\theta) = F(e^{i\theta})$. Then f is 2π-periodic. We may freely identify functions on $\mathbb{T}$, 2π-periodic functions on $\mathbb{R}$, and functions on any interval of length 2π (e.g. $[-\pi, \pi]$) with matching endpoint values.
>
> 定義

We can always reduce a general T-periodic function to a 2π-periodic one via a simple change of variables.

Proposition 1.3. ***Rescaling.***
Let $T > 0$. A function $\phi : \mathbb{R} \to \mathbb{C}$ is T-periodic if and only if the function $f(\theta) = \phi\left(\frac{T\theta}{2\pi}\right)$ is 2π-periodic.

命題

Proof
If ϕ is T-periodic, then $f(\theta + 2\pi) = \phi\left(\frac{T(\theta+2\pi)}{2\pi}\right) = \phi\left(\frac{T\theta}{2\pi} + T\right) = \phi\left(\frac{T\theta}{2\pi}\right) = f(\theta)$. The converse is analogous.

■

Consequently, we restrict our attention to 2π-periodic functions without loss of generality.

Trigonometric Polynomials and Series

The central idea of Fourier analysis is to decompose a periodic signal into a sum of simple building blocks:

$$e_n(\theta) = e^{in\theta}, \quad n \in \mathbb{Z}. \tag{1.11}$$

The set of finite linear combinations of these functions forms the space of trigonometric polynomials.

Definition 1.6. ***Trigonometric Polynomial.***
A function P is a **trigonometric polynomial** if it is of the form

$$P(\theta) = \sum_{n=-N}^{N} c_n e^{in\theta},$$

where $c_n \in \mathbb{C}$ are constants and $N \in \mathbb{N}$. Using Euler's formula, this can equivalently be written in terms of sines and cosines:

$$P(\theta) = \frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos(n\theta) + b_n \sin(n\theta)).$$

定義

To determine the coefficients c_n for a general function f, we exploit the orthogonality of the basis functions $\{e^{in\theta}\}$ with respect to the standard inner product on the circle. This corresponds to the Hermitian inner product defined in *definition* 0.2, scaled by a factor of 2π.

Proposition 1.4. ***Orthogonality.***

For any integers n, m,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}e^{-im\theta}\,d\theta = \delta_{nm} = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}$$

命題

Proof

If $n = m$, the integrand is $e^0 = 1$, and the integral is 2π. Thus the value is 1. If $n \neq m$, let $k = n - m \neq 0$. The integral is:

$$\int_{-\pi}^{\pi} e^{ik\theta}\,d\theta = \left[\frac{e^{ik\theta}}{ik}\right]_{-\pi}^{\pi} = \frac{e^{ik\pi} - e^{-ik\pi}}{ik} = \frac{(-1)^k - (-1)^k}{ik} = 0.$$

■

This orthogonality property suggests that if f can be written as a uniformly convergent series $f(\theta) = \sum c_n e^{in\theta}$, then the coefficients c_n must be given by projecting f onto e_n.

Definition 1.7. ***Fourier Coefficients.***
Let f be an integrable 2π-periodic function. The **n-th Fourier coefficient** of f is defined as:

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\,d\theta, \quad n \in \mathbb{Z}.$$

定義

Definition 1.8. ***Fourier Series.***
The **Fourier series** of f is the formal series formed by these coefficients:

$$S[f](\theta) \sim \sum_{n=-\infty}^{+\infty} \hat{f}(n)e^{in\theta}.$$

In real form, using the relations $e^{in\theta} = \cos n\theta + i \sin n\theta$, this corresponds to the series eq. (1.9) with:

$$a_n = \hat{f}(n) + \hat{f}(-n), \quad b_n = i(\hat{f}(n) - \hat{f}(-n)).$$

定義

Notation 1.1. The notation $\sim$ indicates that the series is associated with f, but implies nothing about equality or convergence.

記法

The fundamental questions of the theory are:

Convergence. In what sense does the partial sum $S_N[f](\theta) = \sum_{n=-N}^{N} \hat{f}(n)e^{in\theta}$ converge to $f(\theta)$ as $N \to \infty$?

- Pointwise? ($\forall\theta$)
- Uniformly? ($\sup_\theta |S_N - f| \to 0$)
- In the mean? ($\int |S_N - f|^2 \to 0$)

Uniqueness. If $\hat{f}(n) = 0$ for all n, is f identically zero?

These questions depend heavily on the regularity of f (continuity, differentiability) and form the core of the subsequent analysis.

1.4 The Geometry of Periodic Functions

The orthogonality of the exponentials $\{e^{inx}\}$ observed previously hints at a deeper geometric structure. By viewing functions as vectors in an infinite-dimensional space, Fourier coefficients behave like coordinates with respect to an orthonormal basis. To make this precise, we introduce a Hermitian inner product structure, referencing the definitions established in chapter 0.

Inner Product Spaces

We consider the space of "sufficiently regular" 2π-periodic functions, denoted $\mathcal{R}_{2\pi}$. For our purposes, this space consists of Riemann integrable functions on $[-\pi, \pi]$ extended periodically to $\mathbb{R}$.

Definition 1.9. ***Inner Product and Norm.***
For two functions $f, g \in \mathcal{R}_{2\pi}$, we define the **inner product**:

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\overline{g(\theta)}\, d\theta.$$

This induces the L^2**-norm** (or root-mean-square norm) as per *definition* 0.3:

$$\|f\|_2 = \sqrt{\langle f, f \rangle} = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^2\, d\theta \right)^{1/2}.$$

定義

Remark.

Strictly speaking, $\|f\|_2 = 0$ implies $f = 0$ only if f is continuous. For general integrable functions, $\|f\|_2 = 0$ implies $f(\theta) = 0$ almost everywhere. Identifying functions that differ only on a set of measure zero renders this a true inner product space.

The shift-invariance of the integral for periodic functions is a crucial property for calculations.

Proposition 1.5. ***Shift Invariance.***

Let f be a 2π-periodic integrable function. For any $\alpha \in \mathbb{R}$:

$$\int_{\alpha}^{\alpha+2\pi} f(\theta)\, d\theta = \int_{-\pi}^{\pi} f(\theta)\, d\theta.$$

命题

Proof

Let $I(\alpha) = \int_{\alpha}^{\alpha+2\pi} f(\theta)\, d\theta$. By periodicity, we may replace α by $\alpha - 2\pi k$ and assume $\alpha \in [-\pi, \pi]$. We split the integral at π:

$$\int_{\alpha}^{\alpha+2\pi} f(\theta)\, d\theta = \int_{\alpha}^{\pi} f(\theta)\, d\theta + \int_{\pi}^{\alpha+2\pi} f(\theta)\, d\theta.$$

Using the substitution $\phi = \theta - 2\pi$ in the second integral and observing $f(\phi + 2\pi) = f(\phi)$, we obtain:

$$\int_{\pi-2\pi}^{\alpha} f(\phi)\, d\phi = \int_{-\pi}^{\alpha} f(\phi)\, d\phi.$$

Thus the sum recombines to $\int_{-\pi}^{\pi} f(\theta)\, d\theta$.

■

Using this notation, the Fourier coefficient definition becomes simply the projection of f onto the basis vector $e_n(\theta) = e^{in\theta}$:

$$\hat{f}(n) = \langle f, e_n \rangle.$$

The orthogonality relation derived in section 1.2 can be restated as $\langle e_n, e_m \rangle = \delta_{nm}$. Thus, the Fourier series $f \sim \sum \hat{f}(n) e_n$ is formally the expansion of f in the orthonormal system $\{e_n\}_{n \in \mathbb{Z}}$.

Real and Complex Representations

While the complex exponential basis is algebraically superior, physical applications often require the decomposition into sines and cosines. Recall the real Fourier coefficients:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta)\, d\theta, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin(n\theta)\, d\theta.$$

The transition between these forms is given by Euler's formula.

Proposition 1.6. ***Coefficient Relations.***
For $n \geq 1$, the coefficients are related by:

$$\hat{f}(n) = \frac{a_n - ib_n}{2}, \qquad \hat{f}(-n) = \frac{a_n + ib_n}{2},$$
$$a_n = \hat{f}(n) + \hat{f}(-n), \qquad b_n = i(\hat{f}(n) - \hat{f}(-n)).$$

The constant term is $\hat{f}(0) = a_0/2$.

命题

Proof

Using $e^{-in\theta} = \cos(n\theta) - i\sin(n\theta)$, for $n \geq 1$ we have:

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta - \frac{i}{2\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta = \frac{a_n}{2} - \frac{ib_n}{2}.$$

Similarly, $e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$ implies $\hat{f}(-n) = \frac{a_n}{2} + \frac{ib_n}{2}$. Solving this linear system yields the stated relations.

■

These relations reveal symmetries based on the range of the function f.

Proposition 1.7. ***Symmetry Properties.***
Let f be a 2π-periodic function.

1. **Real-Valued:** If $f(\mathbb{R}) \subseteq \mathbb{R}$, then $\hat{f}(-n) = \overline{\hat{f}(n)}$. Consequently, a_n and b_n are real numbers.
2. **Evenness:** If f is even ($f(-\theta) = f(\theta)$), then $b_n = 0$ for all n. The series consists only of cosines.
3. **Oddness:** If f is odd ($f(-\theta) = -f(\theta)$), then $a_n = 0$ for all n. The series consists only of sines.

命題

Proof

For 1, since f is real, $\overline{\hat{f}(n)} = \overline{\langle f, e_n\rangle} = \langle \overline{f}, \overline{e_n}\rangle = \langle f, e_{-n}\rangle = \hat{f}(-n)$. For 2 and 3, consider the integral of odd functions over symmetric intervals. If f is even, $f(\theta)\sin(n\theta)$ is odd, so $b_n = 0$. If f is odd, $f(\theta)\cos(n\theta)$ is odd, so $a_n = 0$.

■

Example 1.2. The Sawtooth Wave. Consider the function $f(x) = x$ for $x \in (-\pi, \pi]$, extended periodically. This function is odd, so $a_n = 0$ for all $n \geq 0$. We compute b_n:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx = \frac{2}{\pi}\int_{0}^{\pi} x\sin(nx)\,dx.$$

Integrating by parts:

$$b_n = \frac{2}{\pi}\left(\left[-\frac{x\cos(nx)}{n}\right]_0^{\pi} + \int_0^{\pi}\frac{\cos(nx)}{n}\,dx\right) = \frac{2}{\pi}\left(-\frac{\pi(-1)^n}{n}\right) = (-1)^{n+1}\frac{2}{n}.$$

Thus, the Fourier series is:

$$x \sim \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin(nx).$$

範例

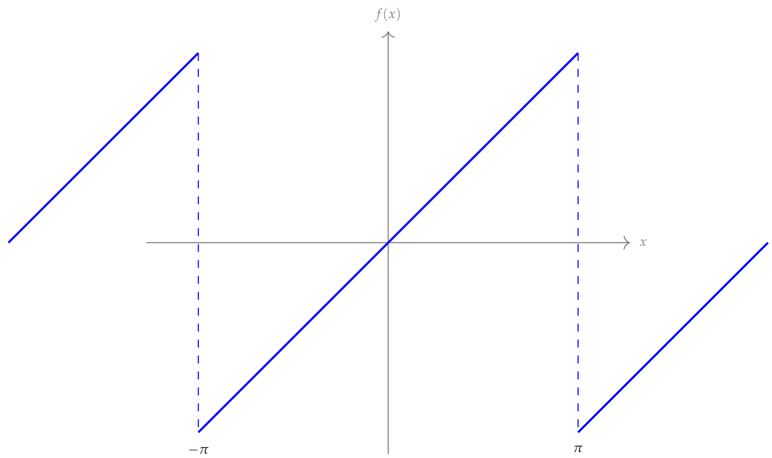


Figure 1.3: The periodic extension of $f(x) = x$ creates a sawtooth wave with discontinuities at odd multiples of π.

1.5 *Regularity and Decay*

A fundamental principle in Fourier analysis is that the smoothness of a function f dictates the rate at which its Fourier coefficients $\hat{f}(n)$ decay as $|n| \to \infty$. Conversely, the decay rate of the coefficients determines the smoothness of the function constructed from the series.

We begin with the Riemann-Lebesgue Lemma, which asserts that high-frequency oscillations "cancel out" when integrated against an integrable function.

Theorem 1.1. ***Riemann-Lebesgue Lemma.***
If f is Riemann integrable on $[-\pi, \pi]$, then

$$\lim_{|n| \to \infty} \hat{f}(n) = 0.$$

定理

Proof

Let $\epsilon > 0$. Since f is Riemann integrable, there exists a step function g such that $\|f - g\|_1 = \int_{-\pi}^{\pi} |f(\theta) - g(\theta)| \, d\theta < 2\pi\epsilon$. By the triangle inequality:

$$|\hat{f}(n)| \leq |\widehat{f-g}(n)| + |\hat{g}(n)|.$$

The first term is bounded by the L^1 norm:

$$|\widehat{f-g}(n)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta) - g(\theta)||e^{-in\theta}| \, d\theta < \epsilon.$$

For the step function $g = \sum_{k=1}^{M} c_k \mathbf{1}_{[a_k, b_k]}$, we compute $\hat{g}(n)$ explicitly:

$$\hat{g}(n) = \frac{1}{2\pi} \sum_{k=1}^{M} c_k \int_{a_k}^{b_k} e^{-in\theta} \, d\theta = \frac{1}{2\pi} \sum_{k=1}^{M} c_k \frac{e^{-inb_k} - e^{-ina_k}}{-in}.$$

Thus $|\hat{g}(n)| \leq \frac{1}{\pi|n|}\sum|c_k| \to 0$ as $|n| \to \infty$. It follows that $\limsup_{|n|\to\infty}|\hat{f}(n)| \leq \epsilon$. Since ϵ was arbitrary, the limit is zero.

■

From the Riemann-Lebesgue Lemma, we immediately obtain that the real coefficients a_n and b_n also tend to zero as $n \to \infty$. If the function admits derivatives, we can obtain stronger decay bounds using integration by parts.

Theorem 1.2. ***Decay of Coefficients for Differentiable Functions.*** Let f be differentiable on $[-\pi, \pi]$ such that f' is integrable. If f satisfies the periodicity condition $f(-\pi) = f(\pi)$, then:

$$a_n = o\left(\frac{1}{n}\right), \quad b_n = o\left(\frac{1}{n}\right) \quad \text{as } n \to \infty.$$

定理

Proof

Let a'_n and b'_n denote the Fourier coefficients of the derivative f'. We integrate the expression for a_n by parts:

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \\ &= \frac{1}{\pi}\left[f(x)\frac{\sin(nx)}{n}\right]_{-\pi}^{\pi} - \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\sin(nx)\,dx. \end{aligned}$$

The boundary term vanishes because $\sin(n\pi) = \sin(-n\pi) = 0$. The remaining integral is proportional to the sine coefficient of f':

$$a_n = -\frac{1}{n}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\sin(nx)\,dx\right) = -\frac{1}{n}b'_n.$$

Similarly for b_n:

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \\ &= \frac{1}{\pi}\left[-f(x)\frac{\cos(nx)}{n}\right]_{-\pi}^{\pi} + \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\cos(nx)\,dx. \end{aligned}$$

Using the periodicity $f(\pi) = f(-\pi)$ and $\cos(n\pi) = \cos(-n\pi)$, the boundary term vanishes. Thus:

$$b_n = \frac{1}{n}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos(nx)\,dx\right) = \frac{1}{n}a'_n.$$

Since f' is integrable, the Riemann-Lebesgue Lemma implies $a'_n \to 0$ and $b'_n \to 0$ as $n \to \infty$. Therefore, $a_n = o(1/n)$ and $b_n = o(1/n)$.

■

We can generalise this result to functions with higher-order derivatives.

Theorem 1.3. ***Decay for C^k Functions.***
Let f have derivatives up to order k on $[-\pi, \pi]$ such that $f^{(k)}$ is integrable. Assume f and its first $k-1$ derivatives satisfy periodic boundary conditions:

$$f^{(j)}(\pi) = f^{(j)}(-\pi) \quad \text{for } j = 0, \dots, k-1.$$

Then:

$$a_n = o\left(\frac{1}{n^k}\right), \quad b_n = o\left(\frac{1}{n^k}\right) \quad \text{as } n \to \infty.$$

定理

Proof

Let $a_n^{(j)}$ and $b_n^{(j)}$ denote the Fourier coefficients of the j-th derivative $f^{(j)}$. Applying the relations derived in the previous theorem iteratively:

$$a_n = -\frac{1}{n} b_n^{(1)} = -\frac{1}{n}\left(\frac{1}{n} a_n^{(2)}\right) = \cdots = \begin{cases} \pm\frac{1}{n^k} b_n^{(k)} & k \text{ odd} \\ \pm\frac{1}{n^k} a_n^{(k)} & k \text{ even} \end{cases}$$

and

$$b_n = \frac{1}{n} a_n^{(1)} = \frac{1}{n}\left(-\frac{1}{n} b_n^{(2)}\right) = \cdots = \begin{cases} \pm\frac{1}{n^k} a_n^{(k)} & k \text{ odd} \\ \pm\frac{1}{n^k} b_n^{(k)} & k \text{ even} \end{cases}.$$

Since $f^{(k)}$ is integrable, its Fourier coefficients $a_n^{(k)}$ and $b_n^{(k)}$ are $o(1)$. Consequently, a_n and b_n are $o(1/n^k)$.

■

This provides a rapid test for the smoothness of a function based on its spectrum:

- Discontinuous functions (e.g., Sawtooth, *figure* 1.3) typically have coefficients decaying as $O(1/n)$.
- Continuous functions with discontinuous derivatives (e.g., triangle wave) decay as $O(1/n^2)$.
- Smooth (C^∞) functions decay faster than any polynomial.

1.6 Exercises

1. **Consistency of the Definition.** Let $T_N(x)$ be a trigonometric polynomial of degree N:

$$T_N(x) = \frac{\alpha_0}{2} + \sum_{k=1}^{N} (\alpha_k \cos kx + \beta_k \sin kx).$$

Prove that the Fourier coefficients of T_N are exactly the coefficients defining it. That is, show that $a_n(T_N) = \alpha_n$ for $0 \le n \le N$ (and 0 otherwise), and $b_n(T_N) = \beta_n$ for $1 \le n \le N$ (and 0 otherwise).

> *Remark.*
>
> This confirms that the operation of taking Fourier coefficients acts as the identity map on the space of trigonometric polynomials.

2. **Symmetry and Periodicity.** Let f be a Riemann integrable function with period 2π.

   (a) **Period Halving.** Suppose f satisfies the condition $f(x+\pi) = f(x)$ for all x. Prove that the odd-indexed Fourier coefficients vanish:
   $$a_{2n-1} = b_{2n-1} = 0 \quad \text{for all } n \ge 1.$$

   (b) **Anti-periodicity.** Suppose f satisfies $f(x+\pi) = -f(x)$ for all x. Prove that the even-indexed Fourier coefficients vanish:
   $$a_{2n} = b_{2n} = 0 \quad \text{for all } n \ge 0.$$

   (c) **Translation.** Let $h \in \mathbb{R}$. Express the Fourier coefficients $\tilde{a}_n, \tilde{b}_n$ of the translated function $g(x) = f(x+h)$ in terms of a_n, b_n of the original function f. Show that:
   $$\tilde{a}_n = a_n \cos nh + b_n \sin nh, \quad \tilde{b}_n = b_n \cos nh - a_n \sin nh.$$

3. **Absolute Convergence.** Prove the converse to the definition of Fourier coefficients in the following sense: If a sequence of coefficients satisfies
   $$\frac{|a_0|}{2} + \sum_{k=1}^{\infty} (|a_k| + |b_k|) < +\infty,$$
   then the trigonometric series
   $$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$
   converges uniformly to a continuous function $f(x)$, and $\{a_k, b_k\}$ are precisely the Fourier coefficients of f.

4. **Spectral Positivity and Decay.** Let f be a 2π-periodic function that is monotonic on the interval $(0, 2\pi)$.

   (a) If f is decreasing on $(0, 2\pi)$, prove that the sine coefficients satisfy $b_n \ge 0$ for all $n \ge 1$. Conversely, if f is increasing, show that $b_n \le 0$.

> *Remark.*
>
> Hint: Use the Second Mean Value Theorem for integrals.

(b) Prove that for such monotonic functions, the coefficients decay as $O(1/n)$. That is, there exists a constant C such that:

$$|a_n| \leq \frac{C}{n}, \quad |b_n| \leq \frac{C}{n}.$$

5. **Asymptotic Mean Values.** The Riemann-Lebesgue Lemma asserts that $\int f(x) \sin nx \, dx \to 0$. Here we investigate the limit when the absolute value is taken. Let f be Riemann integrable on $[a, b]$. Prove that:

$$\lim_{n\to\infty} \int_a^b f(x)|\sin nx| \, dx = \frac{2}{\pi} \int_a^b f(x) \, dx.$$

Show that the same result holds if $|\sin nx|$ is replaced by $|\cos nx|$.

> *Remark.*
>
> Consider $|\sin nx|$ as a periodic function itself. What is the mean value of $|\sin x|$? You may find it helpful to expand $|\sin x|$ as a Fourier series and integrate term-by-term, or approximate f by step functions.

6. **Improper Integrals and Riemann-Lebesgue.**

(a) Let f be absolutely integrable on $(-\infty, \infty)$. Generalise the result of the previous exercise to show:

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} f(x)|\sin nx| \, dx = \frac{2}{\pi} \int_{-\infty}^{+\infty} f(x) \, dx.$$

(b) Calculate the following limit explicitly:

$$\lim_{\lambda\to\infty} \int_0^1 \log x \cos^2(\lambda x) \, dx.$$

> *Remark.*
>
> Hint: Linearise the squared cosine term and check if the Riemann-Lebesgue lemma applies to the improper integral of $\log x$.

7. **A Singular Integral Limit.** Let f be a continuously differentiable function on $[-a, a]$.

(a) Show that the function $g(x) = \frac{f(x)-f(-x)}{x}$ is bounded on $[-a, a]$ (defining $g(0) = 2f'(0)$).

(b) Consider the parity of the kernel $K(x) = \frac{1-\cos \lambda x}{x}$. Show that $\int_{-a}^{a} K(x) f_{\text{even}}(x) \, dx = 0$.

(c) Using the Riemann-Lebesgue lemma on the function $g(x)$, prove that:

$$\lim_{\lambda \to +\infty} \int_{-a}^{a} \frac{1 - \cos \lambda x}{x} f(x)\, dx = \int_{0}^{a} \frac{f(x) - f(-x)}{x}\, dx.$$

2

Partial Sums and Convolution

Having defined Fourier coefficients and their decay properties in chapter 1, we address the inverse problem: recovering f from the sequence $\{\hat{f}(n)\}$. This necessitates the study of the partial sums.

2.1 Partial Sums and Examples

The central object of study is the sequence of trigonometric polynomials formed by truncating the Fourier series.

> **Definition 2.1. *Partial Sums.***
> Let f be an integrable function on $[-\pi, \pi]$. For any $N \in \mathbb{N}$, the N**-th partial sum** is defined as:
>
> $$S_N[f](x) = \sum_{n=-N}^{N} \hat{f}(n) e^{inx}.$$
>
> In terms of the real coefficients a_n and b_n defined in eq. (1.10), this is equivalent to:
>
> $$S_N[f](x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos(nx) + b_n \sin(nx) \right).$$
>
> 定義

We examine the behaviour of these sums through specific examples. These computations illustrate the correlation between the regularity of a function and the rate of decay of its coefficients derived in *theorem* 1.1 and subsequent theorems.

> **Example 2.1.** The Square Wave. Let H be the 2π-periodic function defined on $(-\pi, \pi]$ by:
>
> $$H(x) = \begin{cases} -1 & x \in (-\pi, 0), \\ 1 & x \in (0, \pi), \\ 0 & x = 0, \pi. \end{cases}$$

Since H is odd, $\hat{H}(0) = 0$ and the expansion consists solely of sine terms ($a_n = 0$). We compute the complex coefficients for $n \neq 0$:

$$\hat{H}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(x)e^{-inx}\,dx$$
$$= \frac{1}{2\pi}\left(\int_{-\pi}^{0}(-1)e^{-inx}\,dx + \int_{0}^{\pi}(1)e^{-inx}\,dx\right).$$

Evaluating the second integral:

$$\int_0^{\pi} e^{-inx}\,dx = \left[\frac{e^{-inx}}{-in}\right]_0^{\pi} = \frac{(-1)^n - 1}{-in} = \frac{1-(-1)^n}{in}.$$

By the symmetry of the integrand, the integral from $-\pi$ to 0 contributes an identical value. Thus:

$$\hat{H}(n) = \frac{1}{\pi}\frac{1-(-1)^n}{in}.$$

If n is even, $\hat{H}(n) = 0$. If n is odd, $\hat{H}(n) = \frac{2}{i\pi n}$. Converting to the sine coefficients via $b_n = 2i\hat{H}(n)$ yields $b_n = \frac{4}{n\pi}$ for odd n. The Fourier series is:

$$H(x) \sim \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sin((2k+1)x)}{2k+1}.$$

The coefficients decay as $O(1/n)$, consistent with the jump discontinuities in H.

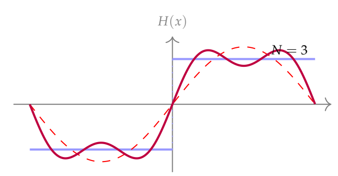


Figure 2.1: The square wave $H(x)$ and its partial sum $S_3[H](x)$. The oscillation near the discontinuity is the Gibbs phenomenon.

範例

Example 2.2. The Triangular Wave. Let T be the 2π-periodic function defined on $[0,\pi]$ by $T(x) = \frac{2}{\pi} - x$, extended as an even function to $[-\pi, 0]$.
Since T is even, $b_n = 0$. The mean value is $a_0 = 0$. For $n \geq 1$, we compute a_n:

$$a_n = \frac{2}{\pi}\int_0^{\pi}\left(\frac{2}{\pi} - x\right)\cos(nx)\,dx.$$

Integrating by parts with $u = \frac{2}{\pi} - x$:

$$a_n = \frac{2}{\pi}\left[\left(\frac{2}{\pi} - x\right)\frac{\sin(nx)}{n}\right]_0^{\pi} + \frac{2}{\pi n}\int_0^{\pi}\sin(nx)\,dx$$
$$= 0 + \frac{2}{\pi n}\left[\frac{-\cos(nx)}{n}\right]_0^{\pi} = \frac{2}{\pi n^2}(1-(-1)^n).$$

If n is even, $a_n = 0$. If n is odd, $a_n = \frac{4}{\pi n^2}$. The series is:

$$T(x) \sim \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\cos((2k+1)x)}{(2k+1)^2}.$$

The decay $O(1/n^2)$ reflects the continuity of T. Note that $T'(x) = -H(x)$ almost everywhere (except at $x = 0, \pi$). By *definition 1.7* (implied), differentiation corresponds to multiplication by in, transforming the $O(1/n^2)$ decay of T into the $O(1/n)$ decay of H.

範例

Example 2.3. The Parabolic Wave. Let $f(x) = x^2$ on $[-\pi, \pi]$. This function is even and continuous on the circle (since $f(-\pi) = f(\pi) = \pi^2$).
The mean value is $\hat{f}(0) = \frac{\pi^2}{3}$. For $n \neq 0$, two integrations by parts yield:

$$\hat{f}(n) = \frac{2(-1)^n}{n^2}.$$

The expansion is:

$$x^2 \sim \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx).$$

範例

Example 2.4. Shifted Poles. We consider a case where the coefficients are rational functions of n. Let $\alpha \in \mathbb{R} \setminus \mathbb{Z}$. Define f on $[0, 2\pi]$ by:

$$f(x) = \frac{\pi}{\sin(\pi\alpha)} e^{i(\pi - x)\alpha}.$$

Computing the coefficients directly:

$$\begin{aligned}\hat{f}(n) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\pi}{\sin(\pi\alpha)} e^{i(\pi-x)\alpha} e^{-inx}\, dx \\ &= \frac{e^{i\pi\alpha}}{2\sin(\pi\alpha)} \int_0^{2\pi} e^{-i(n+\alpha)x}\, dx \\ &= \frac{e^{i\pi\alpha}}{2\sin(\pi\alpha)} \left[\frac{e^{-i(n+\alpha)x}}{-i(n+\alpha)}\right]_0^{2\pi}.\end{aligned}$$

Using $e^{-i2\pi n} = 1$, the bracketed term simplifies to $(1 - e^{-i2\pi\alpha})/(i(n+\alpha))$. Algebraic manipulation confirms:

$$\hat{f}(n) = \frac{1}{n+\alpha}.$$

範例

General Intervals

While the canonical theory is developed on $[-\pi, \pi]$, physical applications often dictate the geometry of the domain. Adapting the machinery to an arbitrary interval $[a, b]$ of length $L = b - a$ is a straightforward rescaling.

Definition 2.2. ***Fourier Coefficients on General Intervals.***
Let $f : [a, b] \to \mathbb{C}$ be an integrable function. The Fourier coefficients adapted to this interval are defined by:

$$\hat{f}(n) = \frac{1}{L} \int_a^b f(x) \, e^{-2\pi i n x / L} \, dx, \quad n \in \mathbb{Z}.$$

The corresponding formal Fourier series is:

$$f(x) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) \, e^{2\pi i n x / L}.$$

定義

Remark.

In the context of the vibrating string discussed in chapter 1, we set $a = 0$ and $b = l$. The basis functions become $\exp(2\pi i n x / l)$.

2.2 *The Dirichlet Kernel and Convolution*

To analyse the convergence of $S_N[f](x)$ to $f(x)$, we seek an integral representation of the partial sum. Substituting the definition of the coefficients into the partial sum yields:

$$\begin{aligned} S_N[f](x) &= \sum_{n=-N}^{N} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} \, dt \right) e^{inx} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \left(\sum_{n=-N}^{N} e^{in(x-t)} \right) dt. \end{aligned}$$

Let $u = x - t$. By the shift invariance of the integral for periodic functions (*proposition* 1.5), we may integrate over any interval of length 2π. The summation term depends only on the difference $x - t$ and acts as a kernel function.

Definition 2.3. ***Dirichlet Kernel.***
For $N \geq 0$, the **Dirichlet kernel** is the trigonometric polynomial:

$$D_N(x) = \sum_{n=-N}^{N} e^{inx}.$$

定義

Using the geometric series summation formula, we derive a closed form for D_N.

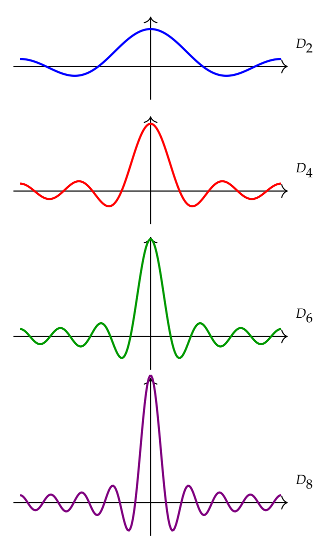


Figure 2.2: Dirichlet kernels $D_N(x)$ for $N = 2, 4, 6, 8$. As N increases, the central peak sharpens while oscillations persist.

Proposition 2.1. ***Closed Form of the Dirichlet Kernel.***
For $x \notin 2\pi\mathbb{Z}$:

$$D_N(x) = \frac{\sin((N+1/2)x)}{\sin(x/2)}.$$

For $x \in 2\pi\mathbb{Z}$, $D_N(x) = 2N+1$.

命题

Proof

The sum is a geometric progression with ratio $w = e^{ix}$.

$$D_N(x) = e^{-iNx}\sum_{k=0}^{2N}(e^{ix})^k = e^{-iNx}\frac{1-e^{i(2N+1)x}}{1-e^{ix}} = \frac{e^{-iNx}-e^{i(N+1)x}}{1-e^{ix}}.$$

To symmetrise the expression, we multiply the numerator and denominator by $e^{-ix/2}$:

$$D_N(x) = \frac{e^{-i(N+1/2)x}-e^{i(N+1/2)x}}{e^{-ix/2}-e^{ix/2}} = \frac{-2i\sin((N+1/2)x)}{-2i\sin(x/2)} = \frac{\sin((N+1/2)x)}{\sin(x/2)}.$$

The value at $x = 0$ follows from L'Hôpital's rule or direct summation of $2N+1$ ones.

■

The partial sum can now be expressed as the **convolution** of f with D_N.

$$S_N[f](x) = (f * D_N)(x) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_N(t)\,dt. \tag{2.1}$$

For the specific purpose of proving convergence theorems, it is often useful to exploit the symmetry of D_N. Since D_N is an even function, we can fold the integral onto $[0,\pi]$. Substituting $u = x-t$ (so $t = x-u$) into eq. (2.1) and shifting the bounds to $[-\pi,\pi]$:

$$\begin{aligned} S_N[f](x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_N(t)\,dt \\ &= \frac{1}{2\pi}\left(\int_{-\pi}^{0} f(x-t)D_N(t)\,dt + \int_{0}^{\pi} f(x-t)D_N(t)\,dt\right). \end{aligned}$$

In the first integral, let $t = -s$. Using $D_N(-s) = D_N(s)$:

$$\int_{-\pi}^{0} f(x-t)D_N(t)\,dt = \int_{\pi}^{0} f(x+s)D_N(s)(-ds) = \int_{0}^{\pi} f(x+s)D_N(s)\,ds.$$

Combining terms leads to the **Dirichlet Integral**:

$$S_N[f](x) = \frac{1}{\pi}\int_0^{\pi}\frac{f(x+t)+f(x-t)}{2}D_N(t)\,dt. \tag{2.2}$$

Or explicitly:

$$S_N[f](x) = \frac{1}{\pi}\int_0^{\pi}\left(\frac{f(x+t)+f(x-t)}{2}\right)\frac{\sin((N+1/2)t)}{\sin(t/2)}\,dt. \tag{2.3}$$

The convergence problem of Fourier series is thus reduced to determining whether the limit of this integral exists as $N \to \infty$. The oscillatory nature of the kernel means that convergence depends on the local behaviour of f. This formulation underpins the convergence proofs in chapter 3.

The Poisson Kernel and the Dirichlet Problem

While the Dirichlet kernel arises from truncation, the **Poisson kernel** arises from solving Laplace's equation on the unit disc $\mathbb{D} = \{z = re^{i\theta} : r < 1\}$. The **Dirichlet Problem** asks: given a continuous function f on the boundary $\partial\mathbb{D}$ (the circle), find a function $u(r, \theta)$ harmonic in $\mathbb{D}$ such that $u(1, \theta) = f(\theta)$.
In polar coordinates, $\Delta u = 0$ is:

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0.$$

Separation of variables yields solutions of the form $r^{|n|}e^{in\theta}$. To match the boundary condition $f(\theta) = \sum \hat{f}(n)e^{in\theta}$, we propose the solution:

$$u(r, \theta) = \sum_{n=-\infty}^{\infty} \hat{f}(n) r^{|n|} e^{in\theta}.$$

This series converges absolutely for $r < 1$ because $|r^{|n|}|$ decays geometrically. Substituting the formula for $\hat{f}(n)$ and swapping sum and integral yields:

$$u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) P_r(\theta - t)\, dt,$$

where $P_r(\theta)$ is the Poisson kernel.

> **Proposition 2.2.** ***Poisson Kernel.***
> For $0 \le r < 1$, the Poisson kernel is:
>
> $$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}.$$
>
> 命題

> *Proof*
> We split the sum into non-negative and negative powers. Let $z = re^{i\theta}$. Note that $|z| < 1$.
>
> $$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \quad \sum_{n=-1}^{-\infty} \bar{z}^{|n|} = \sum_{k=1}^{\infty} \bar{z}^k = \frac{\bar{z}}{1-\bar{z}}.$$
>
> Summing these:
>
> $$P_r(\theta) = \frac{1}{1-z} + \frac{\bar{z}}{1-\bar{z}} = \frac{1 - \bar{z} + \bar{z}(1-z)}{|1-z|^2} = \frac{1 - |z|^2}{1 - 2\mathrm{Re}(z) + |z|^2}.$$

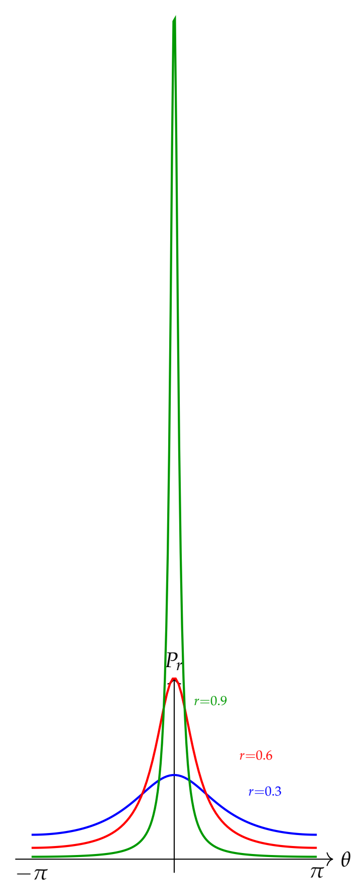


Figure 2.3: Poisson kernel $P_r(\theta)$ for $r = 0.3, 0.6, 0.9$. As $r \to 1^-$, the kernel concentrates at $\theta = 0$, approximating a delta function.

> Substituting $z = re^{i\theta}$ gives the result. ■

Unlike the Dirichlet kernel, the Poisson kernel is strictly positive. This property is crucial for proving that $u(r,\theta) \to f(\theta)$ uniformly as $r \to 1^-$.
Returning to Fourier's original problem, the heat equation $u_t = u_{xx}$ on the circle with initial data $u(x,0) = f(x)$ yields the solution:

$$u(x,t) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{-n^2 t} e^{inx}.$$

This can be expressed as a convolution with the **Heat Kernel**:

$$H_t(x) = \sum_{n=-\infty}^{\infty} e^{-n^2 t} e^{inx}.$$

Unlike the Dirichlet and Poisson kernels, $H_t(x)$ (a Jacobi theta function) does not possess a simple closed form in terms of elementary functions, though it is intimately related to the Gaussian distribution via the Poisson Summation Formula.

2.3 *Convergence of Fourier Series*

The oscillatory nature of the kernel suggests that the behaviour of the integral is dominated by the singularity at $t = 0$. This observation leads to the fundamental principle of localization, which asserts that the convergence of the Fourier series at a point depends solely on the behaviour of the function in the immediate neighbourhood of that point.
To analyse the contribution of different intervals to the integral, we fix $\delta \in (0,\pi)$ and partition the domain of integration into $[0,\delta]$ and $[\delta,\pi]$. We rewrite eq. (2.3) as:

$$S_N[f](x) = \frac{1}{\pi}\int_0^\delta \dots\, dt + \frac{1}{\pi}\int_\delta^\pi \frac{f(x+t)+f(x-t)}{2\sin(t/2)} \sin\left(\left(N+\tfrac{1}{2}\right)t\right) dt. \tag{2.4}$$

Consider the second term. The function

$$g(t) = \frac{f(x+t)+f(x-t)}{2\sin(t/2)},$$

is integrable on $[\delta,\pi]$ because $\sin(t/2)$ is bounded away from zero on this interval. Applying the Riemann-Lebesgue Lemma (*theorem* 1.1), this integral vanishes as $N \to \infty$.
Consequently, the convergence properties are entirely determined by the integral over the arbitrarily small neighbourhood $[0,\delta]$.

Theorem 2.1. ***Riemann's Localization Principle.***
Let f be an integrable periodic function of period 2π. The convergence of the Fourier series $S_N[f](x)$ at a specific point x depends only on the values of f in an arbitrarily small neighbourhood $(x-\delta, x+\delta)$. Specifically, if two functions f and g coincide on an interval $(x-\delta, x+\delta)$, then:

$$\lim_{N\to\infty} (S_N[f](x) - S_N[g](x)) = 0.$$

定理

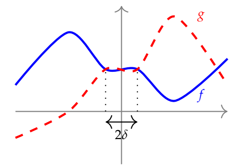


Figure 2.4: The Localization Principle: If f and g agree on $(x - \delta, x + \delta)$, their Fourier series exhibit identical convergence behaviour at x.

This result is somewhat counter-intuitive, as the Fourier coefficients a_n, b_n depend on the values of f over the entire domain $[-\pi, \pi]$. Changes to f far from x will alter every coefficient, yet these changes cancel out perfectly in the sum at x.

Pointwise Convergence Criteria

We now establish sufficient conditions for the Fourier series to converge to a specific value s. Usually, $s = f(x)$, but at points of discontinuity, we expect convergence to the average of the left and right limits.

Let s be a candidate for the limit. Since $\frac{1}{\pi}\int_0^\pi D_N(t)\,dt = \frac{1}{2}$ (integrating over half the symmetric interval of the total mass 2π), we have $\frac{2}{\pi}\int_0^\pi D_N(t)\,dt = 1$. We can write the error as:

$$\begin{aligned} S_N[f](x) - s &= \frac{1}{\pi}\int_0^\pi \frac{f(x+t)+f(x-t)}{2} D_N(t)\,dt - \frac{s}{\pi}\int_0^\pi D_N(t)\,dt \\ &= \frac{1}{\pi}\int_0^\pi \left(\frac{f(x+t)+f(x-t)-2s}{2\sin(t/2)}\right)\sin\left((N+\tfrac{1}{2})t\right)\,dt. \end{aligned}$$

Define the difference function $\varphi_x(t) = f(x+t) + f(x-t) - 2s$. By the Riemann-Lebesgue Lemma, the integral on the right converges to zero provided that the factor multiplying the sine term is integrable. Since $\sin(t/2) \sim t/2$ as $t \to 0$, integrability hinges on the ratio $\varphi_x(t)/t$.

Theorem 2.2. ***Dini's Criterion.***
Let f be integrable on $[-\pi, \pi]$. If for some $s \in \mathbb{C}$ there exists $\delta > 0$ such that

$$\int_0^\delta \left|\frac{f(x+t)+f(x-t)-2s}{t}\right| dt < \infty,$$

then $\lim_{N\to\infty} S_N[f](x) = s$.

定理

Proof
Let $g(t) = \frac{f(x+t)+f(x-t)-2s}{2\sin(t/2)}$. Since $\lim_{t\to 0}\frac{t}{2\sin(t/2)} = 1$, the integrability

of $\varphi_x(t)/t$ on $[0,\delta]$ implies the integrability of $g(t)$ on $[0,\delta]$. Outside this neighbourhood, $g(t)$ is integrable because $\sin(t/2)$ is bounded away from zero. Thus g is integrable on $[0,\pi]$. We have:

$$S_N[f](x) - s = \frac{1}{\pi}\int_0^{\pi} g(t)\sin\left((N+\tfrac{1}{2})t\right)\, dt.$$

By *theorem* 1.1, this integral tends to zero as $N \to \infty$.

■

While Dini's condition is precise, it is often difficult to check directly. A more practical condition involves the smoothness of the function.

Definition 2.4. ***Lipschitz Continuity.***
A function f satisfies a **Lipschitz condition of order** α at x if there exist constants $L, \delta > 0$ such that for all $|h| < \delta$:

$$|f(x+h) - f(x)| \le L|h|^{\alpha}.$$

定義

Proposition 2.3. ***Convergence for Lipschitz Functions.***
If f satisfies a Lipschitz condition of order $\alpha > 0$ at x, then the Fourier series converges to $f(x)$.

命題

Proof

Take $s = f(x)$. Then $|f(x+t) + f(x-t) - 2f(x)| \le |f(x+t) - f(x)| + |f(x-t) - f(x)| \le 2Lt^{\alpha}$. The integrand in Dini's criterion is bounded by $2Lt^{\alpha-1}$. This is integrable near 0 for any $\alpha > 0$.

■

Piecewise Smooth Functions

In physical applications, functions often possess jump discontinuities. We define a class of functions covering most practical cases.

Definition 2.5. ***Piecewise Differentiable.***
A function f on $[a,b]$ is **piecewise differentiable** if there exists a partition $a = t_0 < t_1 < \cdots < t_n = b$ such that f is differentiable on each open interval (t_{i-1}, t_i), and the one-sided limits of f and f' exist at the endpoints t_i.

定義

Such functions satisfy a Lipschitz condition of order 1 everywhere (one-sided at discontinuities). This leads to the classic convergence theorem often attributed to Dirichlet.

Theorem 2.3. ***Dirichlet's Convergence Theorem.***
Let f be a 2π-periodic function that is piecewise differentiable on $[-\pi, \pi]$. Then for every $x \in \mathbb{R}$, the Fourier series converges to the average of the one-sided limits:

$$\lim_{N\to\infty} S_N[f](x) = \frac{f(x^+) + f(x^-)}{2},$$

where $f(x^\pm) = \lim_{h\to 0^+} f(x \pm h)$. In particular, at points of continuity, the series converges to $f(x)$.

定理

Proof

Set $s = \frac{f(x^+)+f(x^-)}{2}$. The numerator in Dini's integrand becomes:

$$\varphi_x(t) = \big(f(x+t) - f(x^+)\big) + \big(f(x-t) - f(x^-)\big)\,.$$

Since f has finite one-sided derivatives at x, the Mean Value Theorem (or the definition of the derivative) implies that $|f(x \pm t) - f(x^\pm)| \le Kt$ for small t. Thus $|\varphi_x(t)|/t \le 2K$, which is bounded and hence integrable. The result follows by Dini's Criterion.

■

The convergence theorems allow us to evaluate the sums of numerical series by substituting specific values of x into Fourier expansions.

Example 2.5. The Parabolic Wave and $\zeta(2)$. Consider $f(x) = x^2$ on $[-\pi, \pi]$, extended periodically. Since f is continuous and piecewise smooth, the Fourier series converges to x^2 for all $x \in [-\pi, \pi]$. Recall the expansion derived in the previous chapter:

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nx).$$

Evaluating at $x = \pi$:

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(n\pi) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n^2}.$$

Simplifying yields $4\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2 - \frac{\pi^2}{3} = \frac{2\pi^2}{3}$. Thus we recover the Basel problem solution:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Evaluating at $x = 0$ gives $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$.

範例

Example 2.6. Expansion of Cosine. Let $f(x) = \cos(ax)$ for $x \in [-\pi, \pi]$ with $a \notin \mathbb{Z}$. The periodic extension is continuous. The coefficients are:

$$a_n = \frac{2}{\pi}\int_0^\pi \cos(ax)\cos(nx)\,dx = (-1)^n \frac{2a\sin(a\pi)}{\pi(a^2-n^2)}.$$

The series is:

$$\cos(ax) = \frac{\sin(a\pi)}{\pi}\left[\frac{1}{a} + \sum_{n=1}^{\infty}(-1)^n \frac{2a}{a^2-n^2}\cos(nx)\right].$$

Setting $x = 0$, we obtain a partial fraction decomposition for the cosecant:

$$\frac{\pi}{\sin(a\pi)} = \frac{1}{a} + \sum_{n=1}^{\infty}(-1)^n \frac{2a}{a^2-n^2}.$$

This formula is instrumental in complex analysis for evaluating residues and infinite sums.

範例

2.4 *Half-Range Expansions*

Often, a function f is defined only on an interval $[0, L]$. To apply Fourier theory, we can extend f to $[-L, L]$ and then periodically to $\mathbb{R}$. The choice of extension determines the nature of the series.

Let $f : [0, \pi] \to \mathbb{R}$.

Even Extension. Define $f_e(x) = f(|x|)$ for $x \in [-\pi, \pi]$. Since f_e is even, $b_n = 0$. The series contains only cosine terms:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx), \quad a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)\,dx.$$

Odd Extension. Define $f_o(x) = \operatorname{sgn}(x) f(|x|)$ for $x \in [-\pi, \pi]$. Since f_o is odd, $a_n = 0$. The series contains only sine terms:

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx), \quad b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,dx.$$

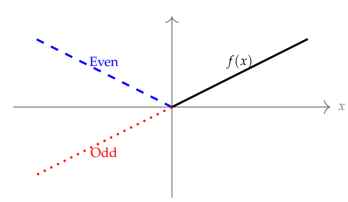


Figure 2.5: A function on $[0, \pi]$ (black) can be extended evenly (blue dashed) or oddly (red dotted), resulting in purely cosine or sine series respectively.

Example 2.7. Expansions of $f(x) = x$. Consider $f(x) = x$ on $(0, \pi)$.

Sine Series. We extend f oddly. This matches the example of the sawtooth wave, yielding:

$$x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\sin(nx), \quad x \in [0, \pi).$$

Note that at $x = \pi$, the series sums to 0, while $f(\pi) = \pi$. The odd extension is discontinuous at π.

Cosine Series. We extend f evenly to $|x|$ on $[-\pi, \pi]$. The coefficients are:

$$a_n = \frac{2}{\pi} \int_0^\pi x \cos(nx)\, dx = \frac{2}{\pi n^2}((-1)^n - 1).$$

This is non-zero only for odd n. Thus:

$$x = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\cos((2k+1)x)}{(2k+1)^2}, \quad x \in [0, \pi].$$

Here, the extension is continuous at $x = \pi$, and the series converges uniformly. Setting $x = 0$ yields $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$.

範例

For a function defined on an interval $[0, L]$, the same principles apply by rescaling to $[0, \pi]$ via the transformation $t = \frac{\pi x}{L}$. The sine series becomes:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

This form is ubiquitous in the solution of boundary value problems where the physical domain length is L, such as the vibrating string discussed in chapter 1.

2.5 *Exercises*

1. **Step Function Series.** Expand the signum function

$$f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 \leq x \leq \pi \end{cases}$$

into a Fourier series on $(-\pi, \pi)$. Use this series to evaluate the sum:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}.$$

2. **Basic Expansions.** Compute the Fourier series for the following functions on the interval $(-\pi, \pi)$:

   (a) $f(x) = |x|$.

   (b) $f(x) = \sin(ax)$ where a is not an integer.

   (c) $f(x) = x \sin x$.

3. **Fractional Part.** Expand the periodic function $f(x) = x - \lfloor x \rfloor$ into a Fourier series. Note that the fundamental interval here is $[0, 1]$, not $[-\pi, \pi]$.

4. **General Intervals.** Expand the following functions into Fourier series on the interval $(-l, l)$:
   (a) $f(x) = x$.
   (b) $f(x) = x + |x|$.
5. **Piecewise Linear Function.** Expand the following function, defined on $[0, 3]$, into a Fourier series:
$$f(x) = \begin{cases} x & 0 \le x \le 1 \\ 1 & 1 < x < 2 \\ 3 - x & 2 \le x \le 3 \end{cases}$$
   Assume the function is extended to an odd function on $[-3, 3]$ (sine series) or generally on $[0, 3]$ with period 3.
6. **Rectified Waves.** Prove the following expansions for the absolute values of sine and cosine:
$$|\cos x| = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1}\cos(2nx) \quad \text{for } x \in \mathbb{R},$$
$$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos(2nx) \quad \text{for } x \in \mathbb{R}.$$
7. **Exponential Expansion.** For $x \in (0, 2\pi)$ and a non-zero constant $a \neq 0$, prove:
$$e^{ax} = \frac{e^{2a\pi} - 1}{\pi}\left(\frac{1}{2a} + \sum_{k=1}^{\infty}\frac{a\cos kx - k\sin kx}{k^2 + a^2}\right).$$
8. **Log-Trigonometric Series.** Establish the following identities by integrating known Fourier series:
   (a) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos nx}{n} = \log\left(2\cos\frac{x}{2}\right)$ for $-\pi < x < \pi$.
   (b) $\sum_{n=1}^{\infty}\frac{\cos nx}{n} = -\log\left(2\sin\frac{x}{2}\right)$ for $0 < x < 2\pi$.
9. **Mean Values of Modulated Functions.** Let f be Riemann integrable on $[a, b]$. Using the expansions for $|\cos x|$ and $|\sin x|$ from Exercise 6, prove:
$$\lim_{\lambda\to+\infty}\int_a^b f(x)|\cos\lambda x|\,dx = \frac{2}{\pi}\int_a^b f(x)\,dx,$$
$$\lim_{\lambda\to+\infty}\int_a^b f(x)|\sin\lambda x|\,dx = \frac{2}{\pi}\int_a^b f(x)\,dx.$$
10. **The Dirichlet Integral.** Let $0 < x < 2\pi$.
   (a) By integrating the Dirichlet kernel identity $\frac{1}{2} + \sum_{k=1}^{n}\cos kt = \frac{\sin(n+1/2)t}{2\sin(t/2)}$, prove that:
$$\sum_{k=1}^{n}\frac{\sin kx}{k} = -\frac{x}{2} + \int_0^x \frac{\sin((n+\frac{1}{2})t)}{2\sin\frac{t}{2}}\,dt.$$

(b) Use the fact that the Fourier series of the sawtooth wave $(\pi - x)/2$ converges to the function on $(0, 2\pi)$ to deduce the value of the improper integral:

$$\int_0^{+\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

11. Localisation for Monotonic Functions. Let g be an increasing function on the interval $[0, h]$ with $h > 0$. Prove:

$$\lim_{\lambda \to +\infty} \int_0^h g(t) \frac{\sin \lambda t}{t}\, dt = \frac{\pi}{2} g(0^+).$$

> *Remark.*
>
> Hint: Use the Second Mean Value Theorem for integrals.

12. Partial Fractions via Fourier Series. Using the Fourier expansion of $\cos(ax)$ on $[-\pi, \pi]$ derived in the text, prove the following partial fraction decompositions valid for $x \notin \pi\mathbb{Z}$:

(a) $\cot x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2x}{x^2 - n^2\pi^2}$.

(b) $\csc x = \frac{1}{x} + \sum_{n=1}^{\infty} (-1)^n \frac{2x}{x^2 - n^2\pi^2}$.

3
Uniqueness and Uniform Convergence

In chapter 2, we analysed pointwise convergence, establishing Dini's criterion (*theorem* 2.2) and Dirichlet's theorem. We now address two fundamental questions: does the set of Fourier coefficients uniquely determine an integrable function, and under what conditions does the series converge uniformly?

3.1 Uniqueness of Fourier Coefficients

If two integrable functions f and g have identical Fourier coefficients, are they necessarily equal? By linearity, this is equivalent to determining whether a function with vanishing Fourier coefficients must itself vanish.

$$\hat{f}(n) = 0 \quad \forall n \in \mathbb{Z} \implies f \equiv 0\,?$$

For Riemann integrable functions, the function can be altered on a finite set of points without changing the integrals defining the coefficients. Thus, we cannot expect strict equality everywhere. However, we can assert equality at all points of continuity.

> **Theorem 3.1. *Uniqueness Theorem.***
> Let f be a 2π-periodic integrable function such that $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$. If f is continuous at θ_0, then $f(\theta_0) = 0$.
>
> 定理

> *Proof*
> We first consider the case where f is real-valued and proceed by contradiction. Suppose there exists a point of continuity θ_0 such that $f(\theta_0) \neq 0$. Without loss of generality: we assume $\theta_0 = 0$ and $f(0) > 0$. Since f is continuous at the origin, we can choose $\delta \in (0, \pi/2]$ such that $f(\theta) > f(0)/2$ whenever $|\theta| < \delta$.
> The proof relies on constructing a family of trigonometric polynomials that "peak" at the origin while remaining small elsewhere (*figure* 3.1). Let $p(\theta) = \epsilon + \cos\theta$, where $\epsilon > 0$ is chosen small

enough so that $|p(\theta)| < 1 - \epsilon/2$ for all $\theta \in [-\pi, \pi] \setminus (-\delta, \delta)$. Since $\cos\theta$ is strictly increasing as $\theta \to 0$ on the interval $[-\delta, \delta]$, we can find $\eta \in (0, \delta)$ such that $p(\theta) \geq 1 + \epsilon/2$ for all $|\theta| < \eta$.

Define $p_k(\theta) = [p(\theta)]^k$ for $k \in \mathbb{N}$. Since the Fourier coefficients of f vanish, and p_k is a finite linear combination of terms $e^{in\theta}$, the orthogonality of the exponentials implies:

$$\int_{-\pi}^{\pi} f(\theta) p_k(\theta)\, d\theta = 0 \quad \text{for all } k \in \mathbb{N}.$$

We estimate this integral by partitioning the domain into three regions. Let $B = \sup|f(\theta)|$.

Case 1: $|\theta| \geq \delta$**.** On this region, $|p_k(\theta)| \leq (1-\epsilon/2)^k$. The contribution to the integral is bounded by:

$$\left| \int_{\delta \leq |\theta| \leq \pi} f(\theta) p_k(\theta)\, d\theta \right| \leq 2\pi B(1-\epsilon/2)^k \to 0 \text{ as } k \to \infty.$$

Case 2: $\eta \leq |\theta| < \delta$**.** In this intermediate region, $f(\theta) > f(0)/2 > 0$ and $p(\theta) > p(\delta) \geq 0$ (since $\epsilon > 0$ and $\delta \leq \pi/2$). Thus, the integrand is non-negative:

$$\int_{\eta \leq |\theta| < \delta} f(\theta) p_k(\theta)\, d\theta \geq 0.$$

Case 3: $|\theta| < \eta$**.** In this neighbourhood of the peak, $f(\theta) > f(0)/2$ and $p(\theta) \geq 1 + \epsilon/2$. The integral is bounded below by:

$$\int_{|\theta| < \eta} f(\theta) p_k(\theta)\, d\theta \geq 2\eta \frac{f(0)}{2}(1+\epsilon/2)^k \to \infty \text{ as } k \to \infty.$$

Combining these estimates, we conclude that $\int_{-\pi}^{\pi} f p_k\, d\theta \to \infty$ as $k \to \infty$, which contradicts the assumption that the integrals vanish for all k. Thus $f(0) = 0$.

For a general complex-valued function $f = u + iv$, the condition $\hat{f}(n) = 0$ implies $\hat{\bar{f}}(n) = \overline{\hat{f}(-n)} = 0$. By linearity, the Fourier coefficients of the real-valued functions $u = (f + \bar{f})/2$ and $v = (f - \bar{f})/(2i)$ also vanish. Applying the previous result to u and v independently yields $f(\theta_0) = 0$.

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An immediate consequence is the injectivity of the Fourier transform on the space of continuous functions.

Corollary 3.1. *Injectivity on Continuous Functions.* Let f and g be continuous 2π-periodic functions. If $\hat{f}(n) = \hat{g}(n)$ for all $n \in \mathbb{Z}$, then $f = g$ everywhere.

推論

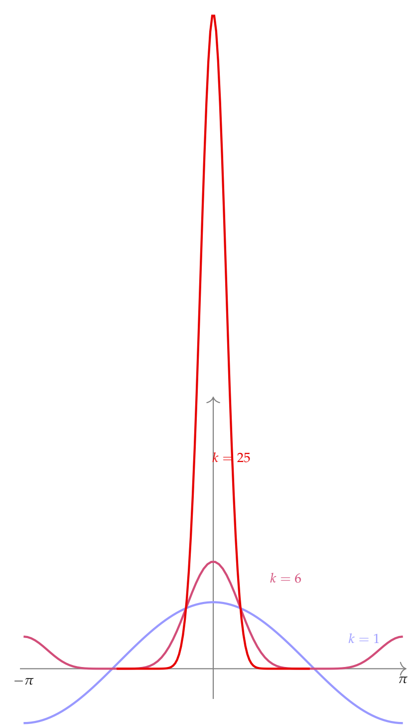

Figure 3.1: The peaking polynomials $p_k(\theta) = (\epsilon + \cos\theta)^k$ concentrate mass at the origin as k increases, acting as an approximate identity.

Proof

Define $h = f - g$. By linearity, $\hat{h}(n) = \hat{f}(n) - \hat{g}(n) = 0$ for all n. Since h is continuous, *theorem* 3.1 implies $h(\theta) = 0$ for all θ, so $f(\theta) = g(\theta)$.

■

This corollary provides a powerful method for verifying identities. If we can verify that two continuous functions share the same Fourier series, they must be identical. This logic was implicit in the solution to the Basel problem in chapter 2, where we equated the function x^2 with its series sum.

3.2 *Uniform Convergence*

While the Uniqueness Theorem links the function to its coefficients, it does not guarantee that the Fourier series converges to the function. It merely states that *if* the series converges to some continuous g, and that series is the Fourier series of f, then $f = g$.

We now identify a condition on the coefficients that guarantees the Fourier series converges uniformly to f.

Theorem 3.2. ***Absolute Convergence Implies Uniform Convergence.*** Let f be a continuous 2π-periodic function. If the Fourier coefficients satisfy

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty,$$

then the partial sums $S_N[f](\theta)$ converge uniformly to $f(\theta)$ as $N \to \infty$.

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Proof

Consider the infinite series $\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{in\theta}$. Since $|e^{in\theta}| = 1$, we have:

$$\left|\hat{f}(n)e^{in\theta}\right| = |\hat{f}(n)|.$$

By the Weierstrass M-test, the condition $\sum |\hat{f}(n)| < \infty$ implies that the series converges absolutely and uniformly to some limit function $g(\theta)$:

$$g(\theta) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{in\theta}.$$

Since each term $e^{in\theta}$ is continuous and convergence is uniform, the limit g is a continuous function.

It remains to show that $g = f$. We compute the Fourier coefficients of g. Due to uniform convergence, we may interchange the sum and the integral:

$$\hat{g}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{n=-\infty}^{\infty}\hat{f}(n)e^{in\theta}\right)e^{-ik\theta}\,d\theta$$
$$= \sum_{n=-\infty}^{\infty}\hat{f}(n)\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{in\theta}e^{-ik\theta}\,d\theta\right).$$

By the orthogonality of the exponentials, the inner integral is δ_{nk}. Thus the sum collapses to $\hat{f}(k)$. Since g and f are continuous functions with identical Fourier coefficients ($\hat{g}(k) = \hat{f}(k)$), *theorem* 3.1 implies $f = g$.

■

This theorem reduces the problem of uniform convergence to the problem of estimating the decay rate of the coefficients. If $\hat{f}(n)$ decays sufficiently fast (e.g., faster than $1/|n|$), the series converges uniformly. Recall from chapter 1 that the smoothness of f dictates this decay. Specifically, we established that if f is C^k, then $\hat{f}(n) = o(n^{-k})$.

Corollary 3.2. *Uniform Convergence for C^2 Functions*. If f is a 2π-periodic function of class C^2 (twice continuously differentiable), then its Fourier series converges absolutely and uniformly to f.

推論

Proof

Since $f \in C^2$, we can apply integration by parts twice to relate the coefficients of f to those of f''. As derived in the previous chapters:

$$\hat{f}(n) = \frac{1}{(in)^2}\widehat{f''}(n) = -\frac{1}{n^2}\widehat{f''}(n) \quad \text{for } n \neq 0.$$

Since f'' is continuous, it is bounded, and thus its coefficients $\widehat{f''}(n)$ are bounded (in fact they tend to zero). Let $C = \sup|\widehat{f''}(n)|$. Then:

$$|\hat{f}(n)| \leq \frac{C}{n^2}.$$

The series $\sum|\hat{f}(n)|$ is dominated by $C\sum\frac{1}{n^2}$, which converges (p-series with $p = 2$). By *theorem* 3.2, the result follows.

■

Remark (Stronger Results).

The C^2 condition is sufficient but not necessary. A more precise analysis shows that $f \in C^1$ is sufficient for absolute convergence. Even weaker, if f satisfies a Hölder condition of order $\alpha > 1/2$ (i.e., $|f(x) - f(y)| \leq C|x-y|^{\alpha}$), the series converges absolutely. For continuous functions that are merely piecewise smooth (like the triangle wave), the decay is $O(1/n^2)$, ensuring uniform con-

> vergence. However, for functions with jump discontinuities (like the square wave), the decay is only $O(1/n)$; the series does not converge absolutely, nor uniformly (due to the Gibbs phenomenon).

3.3 *The Convolution Product*

In chapter 2, we observed that the partial sum $S_N[f]$ is an integral transform of f against the Dirichlet kernel D_N. This is a specific instance of convolution, which generalises the partial sum representation and is central to the theory of approximation.

> **Definition 3.1.** ***Convolution.***
> Let f and g be 2π-periodic integrable functions. The **convolution** of f and g, denoted $f * g$, is the function defined by:
>
> $$(f * g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) g(x - y)\, dy. \tag{3.1}$$
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> 定義

The integral is well-defined for every x because the product of Riemann integrable functions is integrable. Due to the shift-invariance of the integral over a period (*proposition* 1.5), the variable of integration can be shifted, yielding the symmetric form:

$$(f * g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x - y) g(y)\, dy. \tag{3.2}$$

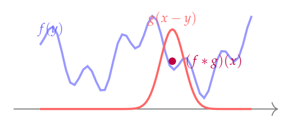

Figure 3.2: Convolution at a point x: the value $(f * g)(x)$ is the integral of the product of f and the reversed, shifted kernel g.

Geometric intuition suggests that if g is a localised "bump" function with unit area (such as the peaking polynomials constructed in the proof of *theorem* 3.1), then $(f * g)(x)$ represents an average of f in the neighbourhood of x, weighted by the profile of g.
Recall that the partial sum of the Fourier series is given by:

$$S_N[f](x) = (f * D_N)(x).$$

Thus, the convergence of Fourier series is essentially a question of how the convolution with the sequence of Dirichlet kernels $\{D_N\}$ behaves as $N \to \infty$.

Algebraic and Analytic Properties

The convolution operation endows the space of integrable functions with a multiplicative structure that interacts gracefully with Fourier coefficients.

> **Proposition 3.1.** ***Properties of Convolution.***
> Let f, g, h be 2π-periodic integrable functions and $c \in \mathbb{C}$.
> 1. **Linearity:** $f * (g + h) = f * g + f * h$ and $(cf) * g = c(f * g)$.

2. **Commutativity:** $f * g = g * f$.
3. **Associativity:** $(f * g) * h = f * (g * h)$.
4. **Regularity:** The function $f * g$ is continuous on $\mathbb{R}$.
5. **Convolution Theorem:** For all $n \in \mathbb{Z}$,

$$\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n).$$

命题

While the Fourier coefficient of a product fg is not simply $\hat{f}\hat{g}$, the Fourier coefficient of a convolution is the product of the coefficients. This transforms convolution into multiplication in the frequency domain.

Proof

Properties 1 (Linearity) follow immediately from the linearity of the integral.

Commutativity. Using the substitution $z = x - y$ (so $y = x - z$ and $dy = -dz$):

$$\begin{aligned}(f * g)(x) &= \frac{1}{2\pi} \int_{x+\pi}^{x-\pi} f(x-z)g(z)(-dz) \\ &= \frac{1}{2\pi} \int_{x-\pi}^{x+\pi} g(z)f(x-z)\, dz.\end{aligned}$$

By periodicity, the interval $[x - \pi, x + \pi]$ is equivalent to $[-\pi, \pi]$. Thus $(f * g)(x) = (g * f)(x)$.

Associativity. This follows by writing out the double integral and interchanging the order of integration (Fubini's theorem), which is justified for bounded Riemann integrable functions.

$$((f * g) * h)(x) = \frac{1}{4\pi^2} \int \int f(y)g(z-y)h(x-z)\, dy\, dz.$$

The substitution $u = z - y$ allows one to regroup terms to obtain $f * (g * h)$.

Regularity. We first prove this for continuous functions. If g is continuous on the circle, it is uniformly continuous. Given $\epsilon > 0$, there exists $\delta > 0$ such that $|s - t| < \delta \implies |g(s) - g(t)| < \epsilon$. Then for any x_1, x_2 with $|x_1 - x_2| < \delta$:

$$\begin{aligned}|(f * g)(x_1) - (f * g)(x_2)| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)[g(x_1 - y) - g(x_2 - y)]\, dy \right| \\ &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(y)| \cdot \epsilon\, dy = \epsilon \|f\|_1.\end{aligned}$$

Thus $f * g$ is continuous.

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Note

For the general case where f and g are merely integrable, we require a density argument. We cite the following standard result from measure theory (adapted for Riemann integration).

Claim 3.1. Approximation Lemma. Let f be Riemann integrable on $[-\pi, \pi]$ and bounded by B. There exists a sequence of continuous functions $\{f_k\}$ bounded by B such that $\|f - f_k\|_1 = \frac{1}{2\pi} \int |f - f_k| \to 0$ as $k \to \infty$.

主張

Proof of Approximation Lemma

Let $\{f_k\}$ and $\{g_k\}$ be continuous approximations of f and g. Then:

$$f * g - f_k * g_k = (f - f_k) * g + f_k * (g - g_k).$$

Estimating the first term:

$$|((f - f_k) * g)(x)| \leq \frac{1}{2\pi} \int |f(y) - f_k(y)| \sup |g| \, dy = \|f - f_k\|_1 \sup |g|.$$

This tends to 0 uniformly in x. Similarly for the second term. Thus $f_k * g_k$ converges uniformly to $f * g$. Since the uniform limit of continuous functions is continuous, $f * g$ is continuous.

証明終

Proof Continuation

Convolution Theorem. Assume f, g are continuous. We define the coefficients:

$$\begin{aligned}
\widehat{f * g}(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) g(x - y) \, dy \right) e^{-inx} \, dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} g(x - y) e^{-in(x-y)} \, dx \right) dy.
\end{aligned}$$

The inner integral (substituting $z = x - y$) is exactly $\hat{g}(n)$. The remaining outer integral yields $\hat{f}(n)$. For integrable functions, we again use the approximation sequence f_k, g_k. Since $f_k * g_k \to f * g$ uniformly, the Fourier coefficients converge: $\widehat{f_k * g_k}(n) \to \widehat{f * g}(n)$. Simultaneously, $\hat{f}_k(n) \to \hat{f}(n)$ and $\hat{g}_k(n) \to \hat{g}(n)$. The identity holds in the limit.

■

Remark.

The smoothing property of convolution (Regularity) is crucial. Convolving an integrable function (which may be discontinuous) with another integrable function yields a continuous function. If g is differentiable, $f * g$ inherits that differentiability.

3.4 *Good Kernels and Approximation*

In the proof of the Uniqueness Theorem (*theorem* 3.1), we constructed a sequence of trigonometric polynomials $\{p_k\}$ that peaked at the origin. This behaviour allowed us to isolate the value of f at a specific point. We now generalise this idea by introducing the concept of a *good kernel*, often referred to as an *approximation to the identity*. These kernels provide a systematic mechanism for recovering a function from its convolutions.

Definition 3.2. ***Good Kernels.***
A sequence of kernels $\{K_n\}_{n=1}^{\infty}$ on the circle is called a **family of good kernels** if it satisfies the following properties:

1. **Normalisation:** For all $n \geq 1$,
$$\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(x)\, dx = 1.$$
2. **Boundedness:** There exists a constant $M > 0$ such that for all $n \geq 1$:
$$\int_{-\pi}^{\pi} |K_n(x)|\, dx \leq M.$$
3. **Concentration:** For every $\delta > 0$, the mass outside the neighbourhood $(-\delta, \delta)$ vanishes as $n \to \infty$:
$$\int_{\delta \leq |x| \leq \pi} |K_n(x)|\, dx \to 0.$$

定義

In many practical cases, we encounter kernels where $K_n(x) \geq 0$. In such instances, the boundedness property follows automatically from the normalisation condition (with $M = 2\pi$). We may interpret such kernels as weight distributions that concentrate their mass near the origin as n increases.
The importance of these kernels lies in their ability to approximate continuous functions via convolution.

Theorem 3.3. ***Convergence of Convolutions.***
Let $\{K_n\}$ be a family of good kernels and let f be an integrable function on the circle.

1. If f is continuous at a point x, then
$$\lim_{n \to \infty} (f * K_n)(x) = f(x).$$

2. If f is continuous everywhere on the circle, then the convergence is uniform:

$$\lim_{n\to\infty} \sup_x |(f * K_n)(x) - f(x)| = 0.$$

定理

Proof

Let $\epsilon > 0$. If f is continuous at x, there exists $\delta > 0$ such that $|y| < \delta$ implies $|f(x-y) - f(x)| < \epsilon$. By the normalisation property, we may write:

$$(f * K_n)(x) - f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(y)[f(x-y) - f(x)]\, dy.$$

We split the integral into the region near the origin ($|y| < \delta$) and the region away from the origin ($\delta \leq |y| \leq \pi$).

$$\begin{aligned} |(f * K_n)(x) - f(x)| \leq \frac{1}{2\pi} \int_{|y|<\delta} |K_n(y)||f(x-y) - f(x)|\, dy \\ + \frac{1}{2\pi} \int_{\delta \leq |y| \leq \pi} |K_n(y)||f(x-y) - f(x)|\, dy. \end{aligned}$$

In the first integral, the difference term is bounded by ϵ. Using the boundedness property of the kernel:

$$\frac{1}{2\pi} \int_{|y|<\delta} |K_n(y)|\epsilon\, dy \leq \frac{\epsilon}{2\pi} \int_{-\pi}^{\pi} |K_n(y)|\, dy \leq \frac{M}{2\pi}\epsilon.$$

In the second integral, since f is integrable and therefore bounded by some B, we have $|f(x-y) - f(x)| \leq 2B$. Thus:

$$\frac{1}{2\pi} \int_{\delta \leq |y| \leq \pi} |K_n(y)| 2B\, dy \leq \frac{B}{\pi} \int_{\delta \leq |y| \leq \pi} |K_n(y)|\, dy.$$

By the concentration property, this integral tends to 0 as $n \to \infty$. Thus, for sufficiently large n, this term is negligible. Combining these estimates, the difference can be made arbitrarily small, proving pointwise convergence.
If f is continuous everywhere, it is uniformly continuous on the compact domain $[-\pi, \pi]$. Thus δ can be chosen independent of x, ensuring uniform convergence.

■

The Dirichlet Kernel Revisited

Recall that the partial sums of a Fourier series can be expressed as a convolution with the Dirichlet kernel: $S_N[f] = f * D_N$. It is natural to ask whether $\{D_N\}$ constitutes a family of good kernels. If so,

theorem 3.3 would imply that the Fourier series of any continuous function converges to the function.
We verify the properties for D_N:

Normalisation.

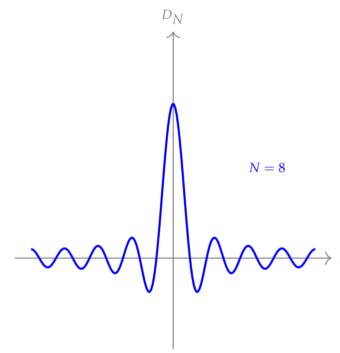
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x)\,dx = 1.$$

This holds.

Boundedness. We examine the integral of the absolute value.

$$\int_{-\pi}^{\pi} |D_N(x)|\,dx \geq c\log N \quad \text{as } N \to \infty.$$

Since the integral of $|D_N|$ grows logarithmically with N, $\{D_N\}$ is **not** a family of good kernels. The signed integral is 1 while the integral of the absolute value diverges due to rapid oscillations (see *figure* 3.3). This failure necessitates alternative summation methods, such as Cesàro means, employing kernels with bounded L^1 norm (chapter 4).



Rapid oscillation and negative lobes cause $\int |D_N|$ to diverge.

Figure 3.3: The Dirichlet kernel takes significant negative values. These negative lobes accumulate area, violating the boundedness condition of good kernels.

3.5 *Exercises*

1. **Identity from Uniqueness.** Let f and g be continuous 2π-periodic functions. Suppose that $\hat{f}(n) = e^{in}\hat{g}(n)$ for all $n \in \mathbb{Z}$. Prove that $f(\theta) = g(\theta + 1)$ for all θ.

   > *Remark.*
   >
   > Hint: Calculate the Fourier coefficients of the function $h(\theta) = g(\theta + 1)$.

2. **Checking Uniform Convergence.** Let $f(x)$ be the 2π-periodic function defined by $f(x) = |x|$ for $x \in [-\pi, \pi]$.

   (a) Does the Fourier series of f converge uniformly to f? Justify your answer using the decay of the coefficients computed in the previous chapter.

   (b) Does the Fourier series of the derivative f' (where it exists) converge uniformly?

3. **Convolution Properties.** Let $f(x) = \cos x$. Compute the convolution $(f * f)(x)$. Verify the Convolution Theorem by comparing the Fourier coefficients of the result with the square of the coefficients of f.

4. **Testing a Kernel.** Consider the sequence of kernels $K_n(x) = n\phi(nx)$, where $\phi(x) = e^{-|x|}$ for $x \in \mathbb{R}$ (and periodised for the circle, or considered locally). Check whether this sequence satisfies the three conditions for a family of good kernels: normalisation (after appropriate scaling), boundedness, and concentration.

4
Cesàro Summation

chapter 3 established that the Fourier series of a continuous function is unique but not necessarily pointwise convergent. Du Bois-Reymond (1876) constructed continuous functions with divergent Fourier series. Since convergence requires additional regularity (e.g., *theorem* 2.2), we introduce Cesàro summation, which recovers the function by averaging partial sums to dampen oscillations.

4.1 *Cesàro Summability*

The concept of convergence for infinite series $\sum a_n$ is rigid: the sequence of partial sums $S_N = \sum_{n=0}^{N} a_n$ must tend to a limit. Many natural series fail this condition despite oscillating around a clearly defined "centre".

> **Definition 4.1.** ***Cesàro Summation.***
> Let $\{S_n\}_{n=0}^{\infty}$ be the sequence of partial sums of a series $\sum a_k$. The series is said to be **Cesàro summable** to σ if the arithmetic means of the partial sums converge to σ:
>
> $$\lim_{N\to\infty} \sigma_N = \sigma, \quad \text{where} \quad \sigma_N = \frac{S_0 + S_1 + \cdots + S_{N-1}}{N}.$$
>
> In this case, we write $\sum_{k=0}^{\infty} a_k = \sigma\,(C)$.
>
> 定義

It is a standard result in analysis that Cesàro summation extends the usual definition of convergence. If $S_N \to s$, then $\sigma_N \to s$. However, σ_N may converge even when S_N diverges.

> **Example 4.1.** Grandi's Series. Consider the series $\sum_{n=1}^{\infty} (-1)^{n-1} = 1 - 1 + 1 - 1 + \ldots$.
> The partial sums are $S_0 = 1, S_1 = 0, S_2 = 1, S_3 = 0$, and so on. The sequence $\{S_n\}$ diverges. However, the means behave as follows:
>
> $$\sigma_{2k} = \frac{k \cdot 1 + k \cdot 0}{2k} = \frac{1}{2}, \quad \sigma_{2k+1} = \frac{(k+1)\cdot 1 + k \cdot 0}{2k+1} \to \frac{1}{2}.$$

Thus, $1 - 1 + 1 - \cdots = \frac{1}{2}\,(C)$.

範例

4.2 The Fejér Kernel

We apply this summation method to the Fourier series of a 2π-periodic integrable function f. Let $S_k[f]$ denote the k-th partial sum. The N-th **Cesàro mean** of the Fourier series is:

$$\sigma_N[f](x) = \frac{1}{N} \sum_{k=0}^{N-1} S_k[f](x).$$

Recall from chapter 2 that $S_k[f] = f * D_k$. By the linearity of convolution (section 3.3), the Cesàro mean is the convolution of f with the average of the Dirichlet kernels.

Definition 4.2. ***Fejér Kernel.***
The N-th **Fejér kernel** F_N is defined as:

$$F_N(t) = \frac{1}{N} \sum_{k=0}^{N-1} D_k(t).$$

Consequently, $\sigma_N[f](x) = (f * F_N)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) F_N(t)\, dt$.

定義

Unlike the Dirichlet kernel, the Fejér kernel possesses a closed form that is strictly non-negative.

Proposition 4.1. ***Closed Form of Fejér Kernel.***
For $t \notin 2\pi\mathbb{Z}$:

$$F_N(t) = \frac{1}{N} \frac{\sin^2(Nt/2)}{\sin^2(t/2)}.$$

For $t \in 2\pi\mathbb{Z}$, $F_N(t) = N$.

命題

Proof

Recall the identity $D_k(t) = \frac{\sin((k+1/2)t)}{\sin(t/2)}$. We sum these terms using the identity $2\sin(A)\sin(B) = \cos(A-B) - \cos(A+B)$.

$$2\sin(t/2) \sum_{k=0}^{N-1} \sin((k+1/2)t) = \sum_{k=0}^{N-1} (\cos(kt) - \cos((k+1)t)).$$

This is a telescoping sum. The terms cancel, leaving:

$$1 - \cos(Nt) = 2\sin^2(Nt/2).$$

Dividing by $2\sin(t/2)$ recovers the sum of the numerators. Dividing further by N (from the definition of the mean) and the remaining $\sin(t/2)$ yields the result. ∎

The positivity of F_N is the decisive factor. We verify that $\{F_N\}$ constitutes a family of **good kernels** as defined in section 3.4:

Normalization. $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(t)\,dt = 1$. This follows because the integral of D_0 is 1 and the integral of D_k for $k \geq 1$ is also 1 (as established in chapter 2).

Positivity. $K_n(t) \geq 0$ for all t. This is immediate from the squared form.

Concentration. For any $\delta \in (0, \pi)$, the integral of the kernel away from the origin vanishes as $n \to \infty$:

$$\lim_{n\to\infty} \int_{\delta \leq |t| \leq \pi} K_n(t)\,dt = 0.$$

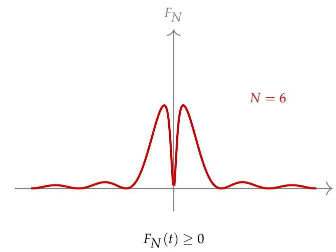


Figure 4.1: The Fejér kernel $F_N(t)$. Unlike D_N, it is non-negative. As N increases, the area concentrates at $t = 0$.

Fejér's Convergence Theorem

Because $\{F_n\}$ constitutes a family of good kernels, we can apply the general theory of approximations. We first establish pointwise convergence at points of continuity or jump discontinuities.

Theorem 4.1. ***Fejér's Theorem.***
Let f be a 2π-periodic integrable function. If the one-sided limits $f(x^+)$ and $f(x^-)$ exist at a point x, then the Fourier series of f is Cesàro summable to the average of these limits:

$$\lim_{n\to\infty} \sigma_n[f](x) = \frac{f(x^+) + f(x^-)}{2}.$$

In particular, if f is continuous at x, then $\sigma_n[f](x) \to f(x)$.

定理

Proof

Let $s = \frac{f(x^+)+f(x^-)}{2}$. By the symmetry $F_n(t) = F_n(-t)$ and the normalisation property, we can write the difference as an integral over $[0, \pi]$:

$$\sigma_n[f](x) - s = \frac{1}{\pi}\int_0^{\pi} \left(\frac{f(x+t) + f(x-t)}{2} - s \right) F_n(t)\,dt.$$

Let $\varphi(t) = f(x+t) + f(x-t) - 2s$. Note that $\varphi(t) \to 0$ as $t \to 0^+$.

$$\sigma_n[f](x) - s = \frac{1}{2\pi}\int_0^{\pi} \varphi(t) F_n(t)\,dt.$$

Given $\epsilon > 0$, there exists $\delta > 0$ such that $|\varphi(t)| < \epsilon$ for $0 < t < \delta$. We split the integral into I_1 (over $[0, \delta]$) and I_2 (over $[\delta, \pi]$).

For I_1, using the positivity of F_n:

$$|I_1| \leq \frac{1}{2\pi}\int_0^\delta |\varphi(t)|F_n(t)\,dt < \frac{\epsilon}{2\pi}\int_0^\delta F_n(t)\,dt \leq \frac{\epsilon}{2\pi}\int_0^\pi F_n(t)\,dt = \frac{\epsilon}{2}.$$

For I_2, let $A = \int_0^\pi |\varphi(t)|\,dt$. Since f is integrable, A is finite. Using the concentration estimate $F_n(t) \leq \frac{1}{n\sin^2(\delta/2)}$ for $t \in [\delta, \pi]$:

$$|I_2| \leq \frac{1}{2\pi}\max_{t\in[\delta,\pi]} F_n(t)\int_\delta^\pi |\varphi(t)|\,dt \leq \frac{A}{2\pi n\sin^2(\delta/2)}.$$

For fixed δ, this term vanishes as $n \to \infty$. Thus, for sufficiently large n, $|\sigma_n[f](x) - s| < \epsilon$.

■

This theorem provides a powerful consistency result for Fourier series: the series cannot converge to an arbitrary value.

Corollary 4.1. *Consistency of Fourier Limits.* Let f be integrable. If the Fourier series $S_n[f](x)$ converges at a point x where the one-sided limits of f exist, it must converge to $\frac{f(x^+)+f(x^-)}{2}$.

推論

Proof

If a sequence S_n converges to L, its arithmetic means σ_n must also converge to L. By *theorem* 4.1, the limit of σ_n is $\frac{f(x^+)+f(x^-)}{2}$. By the uniqueness of limits, L must equal this value.

■

We now consider the case where f is continuous everywhere. Since F_n is a good kernel, we can appeal to the general approximation property established in *theorem* 3.3.

Theorem 4.2. ***Uniform Cesàro Convergence.***
If f is a continuous 2π-periodic function, then the Cesàro means $\sigma_n[f]$ converge uniformly to f on $\mathbb{R}$.

定理

Proof

Since $\{F_n\}$ is a family of good kernels (verified in chapter 4), and f is continuous (and thus uniformly continuous on $[-\pi, \pi]$), the result follows immediately from *theorem* 3.3.

■

Weierstrass Approximation Theorem

An important consequence of Fejér's theorem is the density of trigonometric polynomials in the space of continuous periodic functions.

Theorem 4.3. ***Weierstrass Approximation Theorem (Trigonometric).***
Let f be a continuous 2π-periodic function. For any $\epsilon > 0$, there exists a trigonometric polynomial P such that:

$$\sup_{x \in \mathbb{R}} |f(x) - P(x)| < \epsilon.$$

定理

Proof

The Cesàro mean $\sigma_N[f](x)$ is the arithmetic average of partial sums $S_k[f](x)$. Since each S_k is a trigonometric polynomial of degree at most k, the average σ_N is a trigonometric polynomial of degree at most $N - 1$. By the uniform convergence part of *theorem 4.1*, there exists N sufficiently large such that $\|\sigma_N - f\|_\infty < \epsilon$. We simply take $P(x) = \sigma_N[f](x)$.

■

This constructive proof not only asserts the existence of such polynomials but provides an explicit formula for them via the Fourier coefficients.

Note

This density result is crucial for the spectral theory of operators. It implies that the trigonometric system $\{e^{inx}\}$ is a **complete** basis for the space of continuous functions; no non-zero continuous function is orthogonal to every e^{inx}, reaffirming the result of *theorem 3.1*.

4.3 *Abel Summability and the Dirichlet Problem*

While Cesàro summation provides a robust method for reconstructing a function from its Fourier series using the arithmetic means of partial sums, it is not the only summation method available. Abel developed a method based on power series, originally motivated by the study of boundary value problems.

Abel Means

We first define the concept for numerical series.

Definition 4.3. ***Abel Summability.***
A series of complex numbers $\sum_{k=0}^{\infty} c_k$ is said to be **Abel summable** to s if the power series

$$A(r) = \sum_{k=0}^{\infty} c_k r^k$$

converges for all $0 \leq r < 1$, and the limit as $r \to 1^-$ exists and equals

s:

$$\lim_{r \to 1^-} A(r) = s.$$

定義

Abel summability is a strictly stronger condition than Cesàro summability. It is a standard result (Abel's Theorem) that if a series converges to s, it is Abel summable to s. Furthermore, if a series is Cesàro summable, it is also Abel summable to the same value. The converse, however, does not hold.

Example 4.2. A Divergent Alternating Series. Consider the series

$$\sum_{k=0}^{\infty} (-1)^k (k+1) = 1 - 2 + 3 - 4 + \dots.$$

The partial sums oscillate with increasing amplitude, and the Cesàro means do not converge. However, we consider the associated power series for $|r| < 1$:

$$A(r) = \sum_{k=0}^{\infty} (-1)^k (k+1) r^k = \frac{d}{dr} \sum_{k=0}^{\infty} (-1)^k r^{k+1} = \frac{d}{dr} \left(\frac{r}{1+r} \right).$$

Calculating the derivative:

$$A(r) = \frac{(1+r) - r}{(1+r)^2} = \frac{1}{(1+r)^2}.$$

As $r \to 1^-$, $A(r) \to 1/4$. Thus, the series is Abel summable to $1/4$.

範例

We adapt this definition to Fourier series. Given a function $f \sim \sum \hat{f}(n) e^{in\theta}$, we introduce the radial factor $r^{|n|}$ to dampen the coefficients.

Definition 4.4. ***Abel Means of Fourier Series.***
For an integrable function f and $0 \leq r < 1$, the **Abel mean** $A_r[f]$ is defined by:

$$A_r[f](\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} \hat{f}(n) e^{in\theta}.$$

定義

Since $|\hat{f}(n)|$ is bounded, this series converges absolutely and uniformly for any fixed $r < 1$, yielding a continuous function of θ. Substituting the definition of the Fourier coefficients $\hat{f}(n) = \frac{1}{2\pi} \int f(\phi) e^{-in\phi}\, d\phi$ and interchanging the sum and integral (justified by uniform convergence), we obtain a convolution structure:

$$\begin{aligned}A_r[f](\theta) &= \sum_{n=-\infty}^{\infty} r^{|n|} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi) e^{-in\phi} \, d\phi \right) e^{in\theta} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi) \left(\sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta-\phi)} \right) d\phi \\ &= (f * P_r)(\theta),\end{aligned}$$

where $P_r(\theta)$ is the Poisson kernel introduced in chapter 1.

Convergence via the Poisson Kernel

Recall the explicit formula for the Poisson kernel derived in *figure* 2.3:

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}.$$

To prove that $A_r[f] \to f$, we must verify that $\{P_r\}_{0 \le r < 1}$ acts as an approximation to the identity as $r \to 1^-$. Although our definition of "good kernels" in section 3.4 used a discrete index n, the properties translate directly to the continuous parameter r.

Lemma 4.1. ***The Poisson Kernel is a Good Kernel.***
The family $\{P_r\}_{0 \le r < 1}$ satisfies the following properties as $r \to 1^-$:

1. **Normalisation:** $\frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta) \, d\theta = 1$.
2. **Positivity:** $P_r(\theta) > 0$ for all θ.
3. **Concentration:** For any $\delta > 0$, $\lim_{r \to 1^-} \sup_{\delta \le |\theta| \le \pi} P_r(\theta) = 0$.

引理

Positivity.

Since $r < 1$, $1 - r^2 > 0$. The denominator is $|1 - re^{i\theta}|^2$, which is strictly positive. Thus $P_r > 0$.

証明終

Normalisation.

Integrating the series expansion term-by-term:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} \, d\theta = \sum_{n=-\infty}^{\infty} r^{|n|} \delta_{n0} = 1.$$

証明終

Concentration.

We estimate the denominator for $\delta \le |\theta| \le \pi$.

$$1 - 2r\cos\theta + r^2 = (1 - r)^2 + 2r(1 - \cos\theta).$$

For $|\theta| \ge \delta$, $1 - \cos\theta \ge 1 - \cos\delta > 0$. Let $c_\delta = 1 - \cos\delta$. Then the denominator is bounded below by $2rc_\delta$. Consequently:

$$P_r(\theta) \leq \frac{1-r^2}{2rc_\delta}.$$

As $r \to 1^-$, the numerator vanishes while the denominator remains bounded away from zero. Thus the kernel converges uniformly to 0 outside the interval $(-\delta, \delta)$.

証明終

With these properties established, the convergence theorem for Abel means follows the same logic as Fejér's theorem.

Theorem 4.4. ***Abel Summability of Fourier Series.***
Let f be an integrable function on the circle.

1. At every point θ where f is continuous, $\lim_{r \to 1^-} A_r[f](\theta) = f(\theta)$.
2. If f is continuous everywhere, the convergence is uniform.

定理

Proof

The proof is identical to that of *theorem* 3.3, replacing the sequence index $n \to \infty$ with the parameter $r \to 1^-$.

■

Solution to the Dirichlet Problem

The convergence of Abel means provides the rigorous solution to the Dirichlet problem on the unit disc $\mathbb{D}$, a motivating example discussed in chapter 1. The problem asks for a function $u(r, \theta)$ continuous on the closed disc $\overline{\mathbb{D}}$ and harmonic in the interior, such that $u(1, \theta) = f(\theta)$.
We propose the solution $u(r, \theta) = A_r[f](\theta) = (f * P_r)(\theta)$.

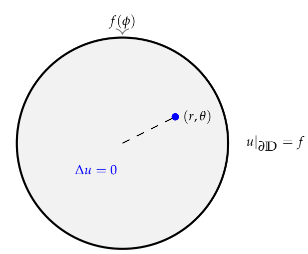


Figure 4.2: The Dirichlet problem: extending boundary data f to a harmonic function u in the interior.

Theorem 4.5. ***Solution to the Dirichlet Problem.***
Let f be a continuous function on the unit circle. The function $u(r, \theta)$ defined by the Poisson integral

$$u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi) P_r(\theta - \phi)\, d\phi$$

satisfies the following:

1. **Harmonicity:** $u \in C^2(\mathbb{D})$ and $\Delta u = 0$ for $r < 1$.
2. **Boundary Continuity:** $\lim_{r \to 1^-} u(r, \theta) = f(\theta)$ uniformly in θ.
3. **Uniqueness:** u is the unique solution to the Dirichlet problem.

定理

(i) Harmonicity.

Recall the series representation $u(r,\theta) = \sum_{n=-\infty}^{\infty} \hat{f}(n) r^{|n|} e^{in\theta}$. For any fixed $\rho < 1$, the series converges uniformly on the disc $r \leq \rho$. Since term-by-term differentiation is valid for power series inside the radius of convergence, u is smooth. In polar coordinates, the Laplacian is $\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2$. Applying this to a single term $v_n = r^{|n|} e^{in\theta}$:

$$\Delta v_n = \left(|n|(|n|-1) r^{|n|-2} + \frac{1}{r}|n| r^{|n|-1} - \frac{n^2}{r^2} r^{|n|} \right) e^{in\theta} = \frac{r^{|n|}}{r^2} \left(|n|^2 - |n| + |n| - n^2 \right) e^{in\theta} = 0.$$

By linearity, $\Delta u = 0$.

証明終

(ii) Boundary Continuity.

This is precisely the statement that the Fourier series of f is uniformly Abel summable to f, which follows from the previous theorem since f is continuous.

証明終

(iii) Uniqueness.

Suppose $v(r,\theta)$ is another solution. Let $r \in (0,1)$ be fixed. Since v is continuous in θ, we can compute its Fourier coefficients with respect to θ:

$$c_n(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} v(r,\theta) e^{-in\theta}\, d\theta.$$

Since $\Delta v = 0$, substituting the Fourier series into the Laplacian equation (justified by smoothness) implies that $c_n(r)$ satisfies the ordinary differential equation:

$$c_n''(r) + \frac{1}{r} c_n'(r) - \frac{n^2}{r^2} c_n(r) = 0.$$

This is an Euler-Cauchy equation. The general solution is of the form $A_n r^{|n|} + B_n r^{-|n|}$ for $n \neq 0$ (and $A_0 + B_0 \ln r$ for $n = 0$). Boundedness of v at the origin requires $B_n = 0$ (and $B_0 = 0$). Thus $c_n(r) = A_n r^{|n|}$.

As $r \to 1^-$, the uniform convergence $v(r,\theta) \to f(\theta)$ implies $c_n(r) \to \hat{f}(n)$. Therefore, $A_n = \hat{f}(n)$, and $c_n(r) = \hat{f}(n) r^{|n|}$. This uniquely determines the Fourier series of $v(r,\theta)$ for every r, and hence v itself.

証明終

Remark.

The uniqueness result implies that a harmonic function on the disc is completely determined by its boundary values. Conversely, if a harmonic function vanishes on the boundary, it must vanish everywhere (a consequence of the Maximum Principle, which this

> Fourier-based proof recovers).

4.4 Exercises

1. **Calculating Cesàro Sums.** Determine the Cesàro sum of the following divergent series.
   (a) The sequence of terms $1, 0, -1, 1, 0, -1, \ldots$ repeated periodically.
   (b) The cosine series $\frac{1}{2} + \sum_{n=1}^{\infty} \cos nx$ for $x \in (0, 2\pi)$.
   (c) The sine series $\sum_{n=1}^{\infty} \sin nx$ for $x \in (0, 2\pi)$.
2. **Cosine Approximation.** Prove that any continuous function on the interval $[0, \pi]$ can be uniformly approximated by polynomials involving only cosines, i.e., of the form $P(x) = \sum_{k=0}^{N} a_k \cos kx$.

   > *Remark.*
   >
   > Hint: Consider the even extension of the function to $[-\pi, \pi]$ and apply Weierstrass's theorem.

3. **Necessary Condition for Cesàro.** Prove that if a series $\sum a_n$ is Cesàro summable, then the terms must satisfy the growth condition $a_n = o(n)$ as $n \to \infty$.
4. **Abel Summability Basics.**
   (a) Verify that the series $\sum_{n=0}^{\infty} (-1)^n$ is Abel summable to $1/2$.
   (b) Prove generally that if $\sum_{n=0}^{\infty} a_n$ converges to s in the standard sense, then it is Abel summable to s.
5. **Hierarchy of Summability.**
   (a) Prove that if $\sum_{n=0}^{\infty} a_n$ is Cesàro summable to s, then it is Abel summable to s.
   (b) Prove that the series $\sum_{n=0}^{\infty} (-1)^n (n+1)$ is Abel summable to $1/4$, but is **not** Cesàro summable.

   > *Remark.*
   >
   > This establishes the strict inclusion: Convergence $\subsetneq$ Cesàro $\subsetneq$ Abel.

6. **Logarithmic Series.** Prove that the series $\sum_{n=2}^{\infty} (-1)^n \log n$ is Cesàro summable to $\frac{1}{2} \log \frac{\pi}{2}$.

   > *Remark.*
   >
   > Consider the derivative of the Dirichlet eta function or appropriate Fourier expansions involving $\log(\sin x)$.

7. **Product of Series.** Let $\sum a_n = A$ and $\sum b_n = B$ be convergent series. Let $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ be their Cauchy product. Prove that $\sum c_n$ is Abel summable to AB, even if it does not converge.

8. **Polynomial Approximation.** Derive the classical Weierstrass Approximation Theorem for algebraic polynomials on $[a, b]$ (every continuous function can be uniformly approximated by polynomials) from the trigonometric version proven in the text.

> *Remark.*
>
> Map the interval $[a, b]$ to $[0, \pi]$ and approximate $f(\cos\theta)$ using cosine polynomials.

5
Mean-Square Convergence

Continuous periodic functions admit uniform approximation by trigonometric polynomials (*theorem* 4.2). However, for general integrable functions, uniform convergence is too restrictive. We instead seek to approximate f "on average" using the mean-square norm.

5.1 Inner Products and Mean-Square Error

We restrict our attention to the set of square-integrable functions on an interval $[a, b]$, denoted by $\mathbf{R}[a, b]$. For bounded functions, we assume Riemann integrability (implying f^2 is integrable). For unbounded functions, we assume f^2 is improperly integrable. From the inequality $|f| \leq \frac{1}{2}(1 + f^2)$, such functions are absolutely integrable.

> **Definition 5.1.** ***Mean-Square Convergence.***
> Let $f \in \mathbf{R}[-\pi, \pi]$. A sequence of trigonometric polynomials $\{T_n\}$ is said to **converge to f in the mean-square sense** if:
>
> $$\lim_{n \to \infty} \int_{-\pi}^{\pi} |f(x) - T_n(x)|^2 \, dx = 0.$$
>
> 定義

To formalise this, we utilise the inner product structure on $\mathbf{R}[a, b]$ introduced in chapter 0.

> **Definition 5.2.** ***Inner Product and Norm on* R.**
> The space of square-integrable functions $\mathbf{R}[a, b]$ inherits an inner product structure from the axioms established in chapter 0. For $f, g \in \mathbf{R}[a, b]$:
>
> $$\langle f, g \rangle = \int_a^b f(x) g(x) \, dx.$$
>
> The induced **norm** is $\|f\| = \sqrt{\langle f, f \rangle} = \left(\int_a^b f^2(x) \, dx \right)^{1/2}$.
>
> 定義

Orthogonal Systems

The geometric notion of orthogonality generalises to this infinite-dimensional space. Two functions f, g are orthogonal if $\langle f, g \rangle = 0$.

Definition 5.3. ***Orthonormal Systems.***
Let $\{\varphi_0, \varphi_1, \dots\}$ be a system of functions in $\mathbf{R}[a, b]$. It is an **orthogonal system** if:

$$\langle \varphi_k, \varphi_l \rangle = \begin{cases} 0 & k \neq l, \\ \lambda_k > 0 & k = l. \end{cases}$$

If $\lambda_k = 1$ for all k, the system is said to be **orthonormal**.

定義

Example 5.1. Trigonometric Systems. The system $\{1, \cos x, \sin x, \dots, \cos nx, \sin nx, \dots\}$ is an orthogonal system on $[-\pi, \pi]$. Normalising these functions yields the orthonormal system:

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \dots, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin nx}{\sqrt{\pi}}, \dots \right\}.$$

範例

Given an orthonormal system $\{\varphi_k\}$ and a function $f \in \mathbf{R}[a, b]$, we define the **Fourier coefficients** of f with respect to $\{\varphi_k\}$ as:

$$c_k = \langle f, \varphi_k \rangle = \int_a^b f(x)\varphi_k(x)\, dx. \tag{5.1}$$

The associated series $f(x) \sim \sum_{k=0}^{\infty} c_k \varphi_k(x)$ is the Generalised Fourier Series.

The Extremal Property of Partial Sums

We now address the central question of approximation: given a function f and a fixed degree n, which linear combination of the basis functions $\{\varphi_0, \dots, \varphi_n\}$ provides the best approximation to f in the mean-square sense?
Let $S_n(x) = \sum_{k=0}^{n} c_k \varphi_k(x)$ be the partial sum of the Fourier series using the coefficients defined above. Let $T_n(x)$ be an arbitrary polynomial of degree n formed by the system:

$$T_n(x) = \sum_{k=0}^{n} \alpha_k \varphi_k(x),$$

where α_k are arbitrary real numbers. We seek to minimise the error $\|f - T_n\|$.

Theorem 5.1. ***Extremal Property.***
Let $\{\varphi_k\}$ be an orthonormal system. For any $f \in \mathbf{R}[a, b]$ and any co-

efficients $\alpha_0, \ldots, \alpha_n$:

$$\left\| f - \sum_{k=0}^{n} c_k \varphi_k \right\| \leq \left\| f - \sum_{k=0}^{n} \alpha_k \varphi_k \right\|,$$

where $c_k = \langle f, \varphi_k \rangle$. Equality holds if and only if $\alpha_k = c_k$ for all k.

定理

Proof

We compute the squared norm of the difference using the properties of the inner product and the orthonormality of $\{\varphi_k\}$.

$$\begin{aligned} \|f - T_n\|^2 &= \langle f - T_n, f - T_n \rangle \\ &= \left\langle f - \sum_{k=0}^{n} \alpha_k \varphi_k, f - \sum_{k=0}^{n} \alpha_k \varphi_k \right\rangle. \end{aligned}$$

Expanding the inner product by linearity:

$$\|f - T_n\|^2 = \langle f, f \rangle - 2 \left\langle f, \sum_{k=0}^{n} \alpha_k \varphi_k \right\rangle + \left\langle \sum_{k=0}^{n} \alpha_k \varphi_k, \sum_{l=0}^{n} \alpha_l \varphi_l \right\rangle.$$

Using $\langle f, \varphi_k \rangle = c_k$ and $\langle \varphi_k, \varphi_l \rangle = \delta_{kl}$:

$$\begin{aligned} \|f - T_n\|^2 &= \|f\|^2 - 2 \sum_{k=0}^{n} \alpha_k \langle f, \varphi_k \rangle + \sum_{k=0}^{n} \sum_{l=0}^{n} \alpha_k \alpha_l \delta_{kl} \\ &= \|f\|^2 - 2 \sum_{k=0}^{n} \alpha_k c_k + \sum_{k=0}^{n} \alpha_k^2. \end{aligned}$$

We complete the square with respect to α_k:

$$\begin{aligned} \|f - T_n\|^2 &= \|f\|^2 - \sum_{k=0}^{n} c_k^2 + \sum_{k=0}^{n} c_k^2 - 2 \sum_{k=0}^{n} \alpha_k c_k + \sum_{k=0}^{n} \alpha_k^2 \\ &= \left(\|f\|^2 - \sum_{k=0}^{n} c_k^2 \right) + \sum_{k=0}^{n} (c_k - \alpha_k)^2. \end{aligned}$$

Since $(c_k - \alpha_k)^2 \geq 0$, the expression is minimised if and only if each term in the final summation vanishes, i.e., $\alpha_k = c_k$. In this case, $T_n = S_n$.

■

The minimum error is given explicitly by the remaining terms:

$$\|f - S_n\|^2 = \|f\|^2 - \sum_{k=0}^{n} c_k^2. \tag{5.2}$$

5.2 *Bessel's Inequality and Parseval's Identity*

From the identity derived in eq. (5.2), we observe that $\|f - S_n\|^2 \geq 0$. This immediately implies a bound on the sum of the squared coefficients.

> **Theorem 5.2.** ***Bessel's Inequality.***
> Let $\{c_k\}$ be the Fourier coefficients of f with respect to an orthonormal system $\{\varphi_k\}$. Then for any n:
>
> $$\sum_{k=0}^{n} c_k^2 \leq \|f\|^2.$$
>
> Since the right-hand side is independent of n, letting $n \to \infty$ yields:
>
> $$\sum_{k=0}^{\infty} c_k^2 \leq \|f\|^2.$$
>
> 定理

This inequality ensures that the series $\sum c_k^2$ converges. A necessary consequence is that $\lim_{k\to\infty} c_k = 0$, recovering the Riemann-Lebesgue lemma in this general context.
The question of mean-square convergence reduces to determining when the inequality becomes an equality. If $\|f - S_n\| \to 0$ as $n \to \infty$, then eq. (5.2) implies:

> **Theorem 5.3.** ***Parseval's Identity.***
> The Fourier series of f converges to f in the mean-square sense if and only if:
>
> $$\sum_{k=0}^{\infty} c_k^2 = \|f\|^2.$$
>
> 定理

Geometrically, this is the infinite-dimensional analogue of the Pythagorean theorem. If we view $\{\varphi_k\}$ as a basis, Parseval's identity asserts that the squared length of the vector f equals the sum of the squared lengths of its components. This occurs precisely when the orthogonal system is *complete* (or *closed*) in $\mathbf{R}[a, b]$.

Completeness of the Space

While the vector spaces $\mathbb{R}^d$ and $\mathbb{C}^d$ are complete (every Cauchy sequence converges to a limit within the space), the space of Riemann integrable functions $\mathbf{R}$ equipped with the mean-square norm is not. A sequence $\{f_n\}$ in $\mathbf{R}$ may satisfy $\|f_n - f_m\| \to 0$ as $n, m \to \infty$ (a Cauchy sequence), yet there may be no function $f \in \mathbf{R}$ such that $\|f_n - f\| \to 0$.

Example 5.2. Incompleteness of **R**. Consider the function f on $[0, 2\pi]$ defined by:

$$f(\theta) = \begin{cases} 0 & \theta = 0, \\ \log(1/\theta) & 0 < \theta \leq 2\pi. \end{cases}$$

This function is unbounded, so it does not belong to the space of bounded Riemann integrable functions. However, consider the sequence of truncations $\{f_n\}$:

$$f_n(\theta) = \begin{cases} 0 & 0 \leq \theta \leq 1/n, \\ f(\theta) & 1/n < \theta \leq 2\pi. \end{cases}$$

Each f_n is bounded and integrable. It can be shown that $\{f_n\}$ forms a Cauchy sequence in the mean-square norm. However, this sequence cannot converge to an element in **R**. Any such limit would have to equal f almost everywhere, but f is not square-integrable in the Riemann sense (it requires improper integration).

範例

This difficulty motivates the completion of the space **R** to the Lebesgue space, where such limits exist. Within the context of Riemann integration, however, we can prove that Parseval's identity holds (and thus mean-square convergence is achieved) for all f where the integral is defined. We first formalise the condition for completeness.

Definition 5.4. ***Completeness of an Orthonormal System.*** Let $\{\varphi_k\}$ be an orthonormal system in $\mathbf{R}[a, b]$. The system is said to be **complete** if for any $f \in \mathbf{R}[a, b]$, Parseval's identity holds:

$$\sum_{k=0}^{\infty} |c_k|^2 = \|f\|^2.$$

定義

From the Extremal Property (*theorem* 5.1), we immediately obtain the following equivalence.

Corollary 5.1. *Approximation Equivalence.* A necessary and sufficient condition for the orthonormal system $\{\varphi_k\}$ to be complete is that for any $f \in \mathbf{R}[a, b]$:

$$\lim_{n \to \infty} \left\| f - \sum_{k=0}^{n} c_k \varphi_k \right\|^2 = 0.$$

That is, f can be approximated in the mean square by the partial sums of its Fourier series.

推論

Proof

From the identity derived in eq. (5.2), we have:

$$\|f - S_n\|^2 = \|f\|^2 - \sum_{k=0}^{n} c_k^2.$$

Taking the limit as $n \to \infty$, the left-hand side vanishes if and only if the right-hand side becomes $\|f\|^2 - \sum_{k=0}^{\infty} c_k^2 = 0$, which is exactly Parseval's identity.

■

We now prove that the trigonometric system is indeed complete.

Theorem 5.4. ***Completeness of the Trigonometric System.***
Let $f \in \mathbf{R}[-\pi, \pi]$, and let a_n, b_n be its Fourier coefficients. Then Parseval's identity holds:

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x)\, dx.$$

定理

The proof proceeds in three steps, extending the class of functions from continuous to Riemann integrable, and finally to improperly integrable.

Step 1: Continuous Functions.

Let f be a continuous function on $[-\pi, \pi]$ with $f(-\pi) = f(\pi)$. By the property of uniform approximation (*theorem* 4.2), for any $\epsilon > 0$, there exists a trigonometric polynomial $T_{n_0}(x)$ of degree n_0 such that $|f(x) - T_{n_0}(x)| < \sqrt{\epsilon/2\pi}$ for all x. Consequently:

$$\|f - T_{n_0}\|^2 = \int_{-\pi}^{\pi} |f(x) - T_{n_0}(x)|^2\, dx < \epsilon.$$

By the Extremal Property (*theorem* 5.1), the Fourier partial sum S_{n_0} provides an even better approximation:

$$\|f - S_{n_0}\|^2 \le \|f - T_{n_0}\|^2 < \epsilon.$$

Since $\|f - S_n\|^2$ is non-increasing with n, for all $n > n_0$, $\|f - S_n\|^2 < \epsilon$. Thus, $\lim\limits_{n\to\infty} \|f - S_n\| = 0$.

証明終

Step 2: Riemann Integrable Functions.

Let f be Riemann integrable on $[-\pi, \pi]$. For any $\epsilon > 0$, there exists a partition $-\pi = x_0 < \cdots < x_m = \pi$ such that the lower and upper Darboux sums satisfy $\sum \omega_i \Delta x_i < \epsilon/(4\Omega)$, where ω_i is the oscillation on $[x_{i-1}, x_i]$ and Ω is the total oscillation.

We construct a continuous polygonal approximation $g(x)$ by connecting the points $(x_i, f(x_i))$ linearly, ensuring $g(-\pi) = g(\pi)$. On each subinterval, $|f(x) - g(x)| \leq \omega_i$. Thus:

$$\|f-g\|^2 = \sum_{i=1}^{m} \int_{x_{i-1}}^{x_i} (f(x) - g(x))^2 \, dx \leq \sum_{i=1}^{m} \omega_i^2 \Delta x_i \leq \Omega \sum \omega_i \Delta x_i < \frac{\epsilon}{4}.$$

By Step 1, there exists a trigonometric polynomial $T(x)$ such that $\|g - T\|^2 < \epsilon/4$. Using the inequality $\|A + B\|^2 \leq 2(\|A\|^2 + \|B\|^2)$:

$$\|f - T\|^2 \leq 2\|f - g\|^2 + 2\|g - T\|^2 < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

As before, this implies $\|f - S_n\| \to 0$.

証明終

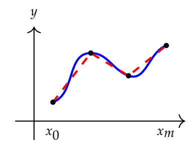


Figure 5.1: A continuous function (blue) approximated by a polygonal chain (red).

Step 3: Improperly Integrable Functions.

Assume f^2 is integrable. Suppose π is the only singular point. For $\epsilon > 0$, choose $\eta > 0$ such that $\int_{\pi-\eta}^{\pi} f^2(x) \, dx < \epsilon/4$. decompose f into f_1 (bounded on $[-\pi, \pi - \eta]$, zero elsewhere) and f_2 (zero on $[-\pi, \pi - \eta]$, f elsewhere). Since f_1 is Riemann integrable, there exists a polynomial T such that $\|f_1 - T\|^2 < \epsilon/4$. Then:

$$\|f - T\|^2 \leq 2\|f_1 - T\|^2 + 2\|f_2\|^2 < \frac{\epsilon}{2} + 2\left(\frac{\epsilon}{4}\right) = \epsilon.$$

証明終

This concludes the proof.

Example 5.3. Evaluation of $\zeta(2)$. Consider the expansion of $f(x) = x/2$ on $(-\pi, \pi)$, which has coefficients $b_n = (-1)^{n-1}/n$ and $a_n = 0$.

$$\frac{x}{2} \sim \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin nx.$$

Applying Parseval's identity:

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{x}{2}\right)^2 dx = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^2.$$

Evaluating the integral:

$$\frac{1}{4\pi} \left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{4\pi} \frac{2\pi^3}{3} = \frac{\pi^2}{6}.$$

Thus, we recover the famous identity $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

範例

From Parseval's identity, two fundamental corollaries follow immediately.

Corollary 5.2. *Completeness and Zero Function*. If a continuous function f on $[-\pi, \pi]$ is orthogonal to every function in the trigonometric system $\{1, \cos x, \sin x, \ldots\}$, then $f \equiv 0$.

推論

Proof

By assumption, all Fourier coefficients a_n and b_n are zero. Parseval's identity implies $\int_{-\pi}^{\pi} f^2(x)\,dx = 0$. Since f is continuous and f^2 is non-negative, f must be identically zero.

■

Corollary 5.3. *Uniqueness Theorem*. If two continuous functions have the same Fourier series, they must be identically equal.

推論

Proof

Let f and g be continuous functions with identical Fourier series. By linearity, the Fourier coefficients of $h = f - g$ are all zero. By *corollary* 5.2, $h \equiv 0$, so $f = g$.

■

5.3 *Generalised Parseval Identity*

The isometric nature of the Fourier transform established in *theorem* 5.3 extends beyond the norm to the inner product itself. By considering the interaction between two different functions, we obtain the Generalised Parseval Identity.

Theorem 5.5. ***Generalised Parseval Identity.***
Let $f, g \in \mathcal{R}$ be 2π-periodic integrable functions with Fourier coefficients $\{c_n\}$ and $\{d_n\}$ respectively. Then:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx = \sum_{n=-\infty}^{\infty} c_n \overline{d_n}.$$

In terms of the real coefficients (where $f \sim \frac{a_0}{2} + \sum a_n \cos nx + b_n \sin nx$ and $g \sim \frac{\alpha_0}{2} + \sum \alpha_n \cos nx + \beta_n \sin nx$), this reads:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx = \frac{a_0\alpha_0}{2} + \sum_{n=1}^{\infty}(a_n\alpha_n + b_n\beta_n).$$

定理

Proof

We employ the polarisation identity, which reconstructs the inner product from the norm. For any complex inner product space:

$$4\langle f, g\rangle = \|f+g\|_2^2 - \|f-g\|_2^2 + i\|f+ig\|_2^2 - i\|f-ig\|_2^2.$$

Applying Parseval's Identity (*theorem* 5.3) to each norm term on the right-hand side, we substitute terms like $\|f+g\|_2^2 = \sum |c_n + d_n|^2$. By the linearity of the coefficients, the algebraic expansion of the sums mirrors the expansion of the norms, yielding $4\sum c_n \overline{d_n}$. Alternatively, for real-valued functions, we may simply consider:

$$\|f+g\|_2^2 = \|f\|_2^2 + \|g\|_2^2 + 2\langle f, g\rangle.$$

Substituting the series sums for the squared norms yields the result immediately.

■

This theorem reinforces the geometric perspective: the Fourier transform preserves the angle between vectors as well as their lengths.

5.4 *Integration of Fourier Series*

One of the most powerful features of Fourier series is their robustness under integration. While term-by-term *differentiation* of a Fourier series requires strict conditions on the smoothness of the function (to ensure the coefficients decay fast enough to counteract the n factor), *integration* improves convergence (introducing a $1/n$ factor). Consequently, term-by-term integration is valid for *any* integrable function, regardless of whether the original Fourier series converges pointwise.

Theorem 5.6. ***Term-by-Term Integration.***
Let f be a 2π-periodic integrable function with Fourier coefficients c_n. For any interval $[a, b] \subseteq [-\pi, \pi]$:

$$\int_a^b f(x)\,dx = \sum_{n=-\infty}^{\infty} c_n \int_a^b e^{inx}\,dx.$$

Explicitly:

$$\int_a^b f(x)\,dx = c_0(b-a) + \sum_{n\neq 0} \frac{c_n}{in}(e^{inb} - e^{ina}).$$

定理

Proof

We apply the Generalised Parseval Identity. Let g be the characteristic function of the interval $[a, b]$, extended periodically:

$$g(x) = \begin{cases} 1 & x \in [a, b], \\ 0 & x \in [-\pi, \pi) \setminus [a, b]. \end{cases}$$

The Fourier coefficients d_n of g are:

$$d_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x) e^{-inx} \, dx = \frac{1}{2\pi} \int_a^b e^{-inx} \, dx.$$

By *theorem 5.5*:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{g(x)} \, dx = \sum_{n=-\infty}^{\infty} c_n \overline{d_n}.$$

The left-hand side evaluates to:

$$\frac{1}{2\pi} \int_a^b f(x) \, dx.$$

The right-hand side becomes:

$$\sum_{n=-\infty}^{\infty} c_n \overline{\left(\frac{1}{2\pi} \int_a^b e^{-inx} \, dx \right)} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} c_n \int_a^b e^{inx} \, dx.$$

Cancelling the factor of $1/2\pi$ from both sides yields the result.

■

This theorem allows us to integrate series that may diverge. For instance, the Fourier series of the Dirac delta function (conceptually $\sum e^{inx}$) does not converge, but its integral yields the step function, whose Fourier series ($\sum \frac{1}{in} e^{inx}$) is well-behaved.
Parseval's identity is particularly effective for evaluating the sums of numerical series.

Example 5.4. Evaluation of $\zeta(4)$. Consider the function $f(x) = x^2$ on $[-\pi, \pi]$.
Its Fourier coefficients were found in chapter 2 to be $c_0 = \pi^2/3$ and $c_n = \frac{2(-1)^n}{n^2}$ for $n \neq 0$. We compute the square of the norm of f:

$$\|f\|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x^2|^2 \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^4 \, dx = \frac{1}{2\pi} \left[\frac{x^5}{5} \right]_{-\pi}^{\pi} = \frac{\pi^4}{5}.$$

Applying Parseval's identity:

$$\|f\|_2^2 = |c_0|^2 + \sum_{n \neq 0} |c_n|^2.$$

Substituting the coefficients:

$$\frac{\pi^4}{5} = \left(\frac{\pi^2}{3} \right)^2 + \sum_{n \neq 0} \left| \frac{2(-1)^n}{n^2} \right|^2 = \frac{\pi^4}{9} + \sum_{n \neq 0} \frac{4}{n^4}.$$

Rearranging the terms:

$$\sum_{n\neq 0}\frac{1}{n^4}=\frac{1}{4}\left(\frac{\pi^4}{5}-\frac{\pi^4}{9}\right)=\frac{\pi^4}{4}\left(\frac{4}{45}\right)=\frac{\pi^4}{45}.$$

Since the sum over $n\neq 0$ is twice the sum over positive integers:

$$2\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{45}\implies \zeta(4)=\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}.$$

範例

5.5 *Exercises*

1. **Autocorrelation and Parseval.** Let f be a continuous 2π-periodic function. Define its autocorrelation function $F(x)$ by:

$$F(x)=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)f(x+t)\,dt.$$

Let $\{a_n, b_n\}$ and $\{A_n, B_n\}$ denote the Fourier coefficients of f and F respectively.

(a) Prove that $A_0 = a_0^2$.

(b) Prove that $A_n = a_n^2 + b_n^2$ and $B_n = 0$ for $n \geq 1$.

(c) Use the convergence of the Fourier series of F at $x = 0$ to deduce Parseval's identity for f.

2. **Wirtinger's Inequality.** Let f be a continuously differentiable 2π-periodic function with zero mean:

$$\int_{-\pi}^{\pi} f(x)\,dx = 0.$$

Prove the inequality:

$$\int_{-\pi}^{\pi}(f'(x))^2\,dx \geq \int_{-\pi}^{\pi} f^2(x)\,dx.$$

Show that equality holds if and only if $f(x) = a\cos x + b\sin x$.

> *Remark.*
>
> Hint: Use Parseval's identity on f and f'. Recall the relationship between their coefficients.

3. **Poincaré Inequality on an Interval.** Let f be continuously differentiable on $[0,1]$ satisfying the boundary conditions $f(0) = f(1) = 0$ and the symmetry condition $f(\frac{1}{2}-x) = -f(\frac{1}{2}+x)$. Prove that:

$$\int_0^1 f^2(x)\,dx \leq \frac{1}{4\pi^2}\int_0^1 (f'(x))^2\,dx.$$

Determine the class of functions for which equality holds.

> *Remark.*
>
> Hint: Extend f to a periodic function and analyse which Fourier modes are permitted by the symmetry.

4. **Completeness via Fejér.** Provide an alternative proof of the Completeness Corollary using Fejér's Theorem. Specifically, prove that if a continuous 2π-periodic function f is orthogonal to all trigonometric polynomials, then $f \equiv 0$.

> *Remark.*
>
> Consider the integral of $|f|^2$ and approximate one factor by a trigonometric polynomial.

5. **Rademacher Functions.** Define the system of functions $\{\varphi_n\}_{n=1}^{\infty}$ on $[0, 1]$ by:

$$\varphi_n(t) = \text{sgn}(\sin(2^n \pi t)).$$

Prove that this system is orthonormal on $[0, 1]$.

> *Remark.*
>
> Consider the binary expansion of t. Is this system complete?

6
Pointwise Convergence and Divergence

This chapter establishes the conditions for pointwise convergence. While local differentiability guarantees convergence, reinforcing the Localisation Principle, continuity alone is insufficient. This is demonstrated via an explicit counterexample relying on "symmetry breaking" within the partial sums.

6.1 A Local Convergence Result

The relationship between smoothness and convergence is strengthened here. While Dini's Criterion (*theorem* 2.2) offers a sufficient condition, the following proof demonstrates directly that differentiability ensures convergence.

Theorem 6.1. ***Convergence for Differentiable Functions.***
Let f be an integrable function on the circle. If f is differentiable at a point x_0, then

$$\lim_{N\to\infty} S_N[f](x_0) = f(x_0).$$

定理

Proof

The Dirichlet integral representation of the partial sum is:

$$S_N[f](x_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x_0 - t)D_N(t)\,dt.$$

Since $\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(t)\,dt = 1$, we can express the error as:

$$S_N[f](x_0) - f(x_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(f(x_0 - t) - f(x_0)\right) D_N(t)\,dt.$$

Define the difference quotient function $g(t)$:

$$g(t) = \begin{cases} \frac{f(x_0-t)-f(x_0)}{t} & t \neq 0,\ t \in [-\pi, \pi] \\ -f'(x_0) & t = 0. \end{cases}$$

Since f is differentiable at x_0, g is bounded in a neighbourhood of 0. Away from the origin ($|t| > \delta$), t is bounded away from zero,

so the integrability of f implies the integrability of g. Thus g is integrable on $[-\pi, \pi]$.
Substituting the explicit form of the Dirichlet kernel $D_N(t) = \frac{\sin((N+1/2)t)}{\sin(t/2)}$:

$$S_N[f](x_0) - f(x_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t)\, t \,\frac{\sin((N+1/2)t)}{\sin(t/2)}\, dt$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) \left(\frac{t}{\sin(t/2)} \right) \sin((N+1/2)t)\, dt.$$

The function $h(t) = g(t)\frac{t}{\sin(t/2)}$ is the product of the integrable function g and the continuous (and bounded) function $t/\sin(t/2)$. Therefore, h is integrable. By the Riemann-Lebesgue Lemma (*theorem* 1.1), the integral of $h(t)$ against the oscillatory term $\sin((N+1/2)t)$ tends to zero as $N \to \infty$.

■

Remark.

The proof relies only on the boundedness of the difference quotient. Consequently, *theorem* 6.1 holds under the weaker assumption that f satisfies a Lipschitz condition at x_0, i.e., $|f(x) - f(x_0)| \leq M|x - x_0|$.

This result provides a rigorous justification for Riemann's Localisation Principle.

Corollary 6.1. *Localisation Principle*. Let f and g be two integrable functions. If $f(x) = g(x)$ for all x in an open interval I containing x_0, then

$$\lim_{N\to\infty} (S_N[f](x_0) - S_N[g](x_0)) = 0.$$

推論

Proof

The difference $h = f - g$ is identically zero on I. Consequently, h is differentiable at x_0 (with derivative 0). By *theorem* 6.1, $S_N[h](x_0) \to h(x_0) = 0$. By the linearity of the partial sum operator, $S_N[f](x_0) - S_N[g](x_0) \to 0$.

■

This confirms that the convergence of a Fourier series at a point is entirely determined by the local behaviour of the function. The "global" information contained in the Fourier coefficients cancels out perfectly via interference at points where the functions agree.

6.2 *Symmetry Breaking and Divergence*

Does continuity imply convergence? The partial sum $S_N[f](x) = \sum_{n=-N}^{N} \hat{f}(n)e^{inx}$ truncates the spectrum symmetrically. However, "breaking" this symmetry by summing over only positive or negative indices reveals unbounded behaviour.
Consider the sawtooth function f (similar to *figure 1.3*), which is odd and defined by $f(x) = i(\pi - x)$ for $x \in (0, 2\pi)$. Its Fourier series is:

$$f(x) \sim \sum_{n\neq 0} \frac{1}{n} e^{inx}.$$

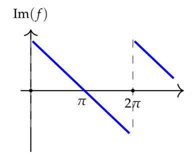


Figure 6.1: The sawtooth function $f(x) = i(\pi - x)$. The imaginary part decreases linearly from π to $-\pi$, with jump discontinuities at multiples of 2π.

While f is bounded, the "half-series" $\sum_{n=1}^{\infty} \frac{e^{inx}}{n}$ behaves like the harmonic series at $x = 0$, which diverges logarithmically. This is formalised by defining truncated blocks.

Definition 6.1. ***Sawtooth Blocks.***
For $N \geq 1$, define the trigonometric polynomials:

$$f_N(x) = \sum_{1\leq |n| \leq N} \frac{e^{inx}}{n}, \qquad \tilde{f}_N(x) = \sum_{-N \leq n \leq -1} \frac{e^{inx}}{n}.$$

定義

The polynomial f_N represents the symmetric partial sum of the bounded sawtooth function. The polynomial $\tilde{f}_N$ represents the asymmetric "negative half". Two properties are crucial:

1. At the origin, the asymmetric sum grows logarithmically: $|\tilde{f}_N(0)| \geq c \log N$.
2. The symmetric sums $f_N(x)$ are uniformly bounded in N and x.

The first property follows directly from the definition: $\tilde{f}_N(0) = \sum_{n=1}^{N} \frac{1}{-n} = -H_N$, where H_N is the harmonic number. Thus $|\tilde{f}_N(0)| \sim \log N$.
The second property requires a more refined estimate. We prove it using a comparison with Abel means.

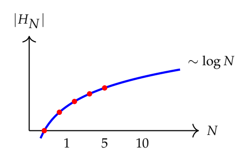


Figure 6.2: The harmonic sum $H_N = \sum_{n=1}^{N} \frac{1}{n}$ grows like $\log N$. This unbounded growth drives the divergence.

Lemma 6.1. ***Uniform Boundedness of f_N.***
There exists a constant C such that $|f_N(x)| \leq C$ for all N and all $x \in [-\pi, \pi]$.

引理

Proof

A Tauberian-style argument suffices. Let $S_N(x)$ be the partial sums of a series $\sum c_n e^{inx}$, and let $A_r(x)$ be its Abel means. Suppose the coefficients satisfy $c_n = O(1/n)$ and the Abel means are uniformly bounded, i.e., $|A_r(x)| \leq M$. Estimating the difference $|S_N - A_r|$

where $r = 1 - 1/N$:

$$|S_N - A_r| = \left| \sum_{|n| \le N} c_n (1 - r^{|n|}) e^{inx} - \sum_{|n| > N} c_n r^{|n|} e^{inx} \right|$$
$$\le \sum_{|n| \le N} |c_n| (1 - r^{|n|}) + \sum_{|n| > N} |c_n| r^{|n|}.$$

Using $1 - r^{|n|} \le |n|(1 - r)$ and $|c_n| \le K/|n|$, the first sum is bounded by

$$\sum_{n=1}^{N} \frac{K}{n} n \frac{1}{N} = K.$$

The second sum is bounded by

$$\frac{K}{N} \sum_{n=N+1}^{\infty} r^n \le \frac{K}{N} \frac{r^{N+1}}{1 - r} \le K.$$

Thus, S_N is bounded if A_r is bounded.

For the sawtooth function $f(x) \sim \sum_{n \ne 0} \frac{e^{inx}}{n}$, the coefficients are $O(1/n)$. The Abel means are $A_r[f] = f * P_r$ (*proposition* 2.2). Since f is bounded and P_r has unit mass, $\|A_r[f]\|_\infty \le \|f\|_\infty$. Therefore, the partial sums $f_N(x)$ are uniformly bounded.

■

6.3 *Counterexample Construction*

We define a continuous function whose Fourier series diverges at $x = 0$ by summing scaled and shifted versions of f_N. Shifting the spectrum of f_N allows specific partial sums to isolate the logarithmic growth of $\tilde{f}_N$.

Definition 6.2. ***Shifted Polynomials.***
Let $P_N(x)$ be the polynomial obtained by shifting the frequencies of f_N by $2N$:

$$P_N(x) = e^{i(2N)x} f_N(x) = \sum_{1 \le |n| \le N} \frac{1}{n} e^{i(2N+n)x}.$$

定義

The frequencies of f_N lie in $[-N, -1] \cup [1, N]$. The frequencies of P_N lie in $[2N - N, 2N - 1] \cup [2N + 1, 2N + N] = [N, 2N - 1] \cup [2N + 1, 3N]$. Crucially, the "centre" of P_N is at the frequency $2N$.

If we compute the partial sum S_{2N} of P_N, we sum all frequencies up to $2N$. This captures exactly the lower block $[N, 2N - 1]$ and discards the upper block $[2N + 1, 3N]$. Note that the lower block corresponds

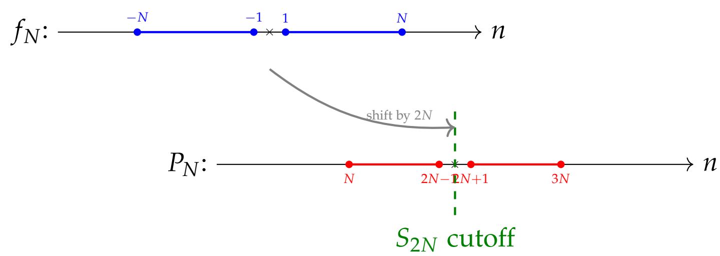


Figure 6.3: Spectral shifting. f_N is symmetric around 0. P_N is shifted to be symmetric around $2N$. A partial sum S_{2N} cuts P_N in half, isolating the logarithmically divergent part.

to the negative indices of the original f_N shifted by $2N$.

$$S_{2N}[P_N](x) = \sum_{k=N}^{2N-1} \hat{P}_N(k)e^{ikx} = \sum_{n=-N}^{-1} \frac{1}{n} e^{i(2N+n)x} = e^{i(2N)x}\tilde{f}_N(x).$$

At $x = 0$, we have $|S_{2N}[P_N](0)| = |\tilde{f}_N(0)| \geq c \log N$. Conversely, if we take $S_M[P_N]$ for $M \geq 3N$, we capture the entire polynomial P_N. Since $|P_N(x)| = |f_N(x)|$, this is uniformly bounded by *lemma* 6.1.
These blocks are now assembled into a single series.

Theorem 6.2. ***Existence of a Divergent Continuous Function.*** There exists a continuous 2π-periodic function f such that the sequence of partial sums $\{S_k[f](0)\}_{k=1}^{\infty}$ is unbounded.

定理

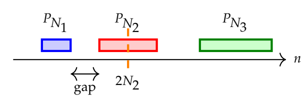


Figure 6.4: Disjoint spectral blocks. Each P_{N_k} occupies $[N_k, 3N_k]$. Since $N_{k+1} > 3N_k$, blocks don't overlap. The cutoff S_{2N_2} splits only P_{N_2}.

Proof

Choose a sequence of integers N_k increasing rapidly enough to separate the spectra of the shifted polynomials, and a sequence of scaling factors α_k to ensure continuity. Let $N_k = 3^{2^k}$ and $\alpha_k = 1/k^2$. Define:

$$f(x) = \sum_{k=1}^{\infty} \alpha_k P_{N_k}(x).$$

1. ***Continuity.*** Since $|P_{N_k}(x)| = |f_{N_k}(x)| \leq C$ (*lemma* 6.1), the series is dominated by $C \sum 1/k^2$, which converges. By the Weierstrass M-test, the series defines a continuous function f.

2. ***Divergence.*** Consider the partial sum of the Fourier series of f at index $M_m = 2N_m$. Because the spectra of P_{N_k} are supported on $[N_k, 3N_k]$ and $N_{k+1} > 3N_k$, the spectral blocks are disjoint. The partial sum operator is linear.

$$S_{2N_m}[f](0) = \sum_{k=1}^{m-1} \alpha_k S_{2N_m}[P_{N_k}](0) + \alpha_m S_{2N_m}[P_{N_m}](0) + \sum_{k=m+1}^{\infty} \alpha_k S_{2N_m}[P_{N_k}](0).$$

Analysing the terms:

- For $k < m$, $2N_m > 3N_k$. The partial sum captures the *entire* polynomial P_{N_k}. The value is $\alpha_k P_{N_k}(0) = 0$ (since $f_N(0) = 0$). Even if non-zero, it is bounded.

- For $k > m$, $2N_m < N_k$. The partial sum captures *none* of P_{N_k}. The value is 0.
- For $k = m$, the partial sum cuts P_{N_m} exactly in the middle.

$$|S_{2N_m}[P_{N_m}](0)| = |\tilde{f}_{N_m}(0)| \geq c \log N_m.$$

Thus, the total sum is dominated by the m-th term:

$$|S_{2N_m}[f](0)| \geq c\alpha_m \log N_m - O(1).$$

Substituting our choices for α_m and N_m:

$$\alpha_m \log N_m = \frac{1}{m^2} \log(3^{2^m}) = \frac{2^m \log 3}{m^2}.$$

As $m \to \infty$, this quantity tends to infinity. Therefore, the Fourier series of f diverges at $x = 0$.

■

Remark.

To construct a function diverging at an arbitrary point x_0, one simply considers $f(x - x_0)$. Using the Baire Category Theorem (a topic for a course on Functional Analysis), one can show that the set of continuous functions with divergent Fourier series is, in a topological sense, "generic" or typical, while those that converge are rare.

6.4 *Exercises*

1. **Lipschitz Convergence.** Prove that if a function f satisfies a Lipschitz condition of order $\alpha \in (0,1)$ at a point x_0, then its Fourier series converges to $f(x_0)$. Specifically, check that the integral condition in Dini's Criterion is satisfied.
2. **Symmetry Breaking.** Calculate the value of the asymmetric sum $\tilde{f}_N(0) = \sum_{n=-N}^{-1} \frac{1}{n}$ explicitly. Compare it with the symmetric sum $f_N(0)$. Why does the symmetric sum vanish while the asymmetric sum diverges?
3. **Constructing Boundedness.** Let $g(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n}$. Prove that the partial sums of this series are uniformly bounded, i.e., there exists M such that $|\sum_{n=1}^{N} \frac{\sin nx}{n}| \leq M$ for all N, x.
4. **Failure of Convergence.** Consider the function constructed in the proof of divergence. Is this function differentiable at $x = 0$? Why or why not? Reconcile this with the convergence theorem for differentiable functions.

Use the integral representation $\int \frac{\sin t}{t}\, dt$ or summation by parts with the Dirichlet kernel.

7
Applications of Fourier Series

While the theory of pointwise convergence reveals certain subtleties (as seen in chapter 6), the robustness of the L^2 theory allows us to solve significant problems in geometry, number theory, and analysis.

7.1 The Riemann Zeta Function and Bernoulli Polynomials

In chapter 5, we utilized Parseval's identity to evaluate $\sum n^{-2}$ and $\sum n^{-4}$. To generalize this to all even positive integers, we introduce a recursive family of functions known as Bernoulli polynomials.

> **Definition 7.1.** ***Bernoulli Polynomials.***
> The **Bernoulli polynomials** $B_n(x)$ for $n \geq 0$ are defined recursively by the conditions:
> 1. $B_0(x) = 1$.
> 2. For $n \geq 1$, $B_n'(x) = nB_{n-1}(x)$.
> 3. For $n \geq 1$, $\int_0^1 B_n(x)\,dx = 0$.
>
> The **Bernoulli numbers** B_n are the values at the origin: $B_n = B_n(0)$.
>
> 定義

We compute the first few polynomials explicitly. Since $B_0(x) = 1$, condition (2) implies $B_1'(x) = 1$, so $B_1(x) = x + c$. Condition (3) fixes the constant:

$$\int_0^1 (x+c)\,dx = \frac{1}{2} + c = 0 \implies c = -\frac{1}{2}.$$

Thus $B_1(x) = x - 1/2$. Continuing this process, one obtains $B_2(x) = x^2 - x + 1/6$, yielding the Bernoulli numbers $B_0 = 1, B_1 = -1/2, B_2 = 1/6$.

We restrict our attention to the interval $[0,1]$. Since these polynomials are not periodic, we consider their periodic extensions (which may have discontinuities at the endpoints) to apply Fourier analysis.

Proposition 7.1. ***Fourier Series of Bernoulli Polynomials.***
For $n \geq 1$, the Fourier coefficients of the 1-periodic extension of $B_n(x)$ are given by:

$$\hat{B}_n(k) = -\frac{n!}{(2\pi ik)^n} \quad \text{for } k \neq 0,$$

and $\hat{B}_n(0) = 0$. Consequently, for $x \in (0,1)$:

$$B_n(x) = -n! \sum_{k \neq 0} \frac{e^{2\pi ikx}}{(2\pi ik)^n}.$$

命题

Proof

We proceed by induction on n. For $n = 1$, $B_1(x) = x - 1/2$. The constant term is zero by definition. For $k \neq 0$:

$$\hat{B}_1(k) = \int_0^1 \left(x - \frac{1}{2}\right) e^{-2\pi ikx}\, dx.$$

Integrating by parts with $u = x - 1/2$ and $dv = e^{-2\pi ikx}dx$:

$$\hat{B}_1(k) = \left[\frac{(x-1/2)e^{-2\pi ikx}}{-2\pi ik}\right]_0^1 - \int_0^1 \frac{e^{-2\pi ikx}}{-2\pi ik}\, dx.$$

The boundary term evaluates to $\frac{(1/2)(1)-(-1/2)(1)}{-2\pi ik} = \frac{1}{-2\pi ik}$. The integral term vanishes. Thus the formula holds for $n = 1$.
Assume the formula holds for $n - 1$. For $B_n(x)$, we have $B_n'(x) = nB_{n-1}(x)$. We relate the coefficients using integration by parts:

$$\hat{B}_n(k) = \int_0^1 B_n(x)e^{-2\pi ikx}\, dx = \left[\frac{B_n(x)e^{-2\pi ikx}}{-2\pi ik}\right]_0^1 + \frac{1}{2\pi ik}\int_0^1 B_n'(x)e^{-2\pi ikx}\, dx.$$

For $n \geq 2$, $B_n(1) = B_n(0)$, so the boundary term vanishes. Substituting $B_n' = nB_{n-1}$:

$$\hat{B}_n(k) = \frac{n}{2\pi ik}\widehat{B}_{n-1}(k).$$

Using the inductive hypothesis:

$$\hat{B}_n(k) = \frac{n}{2\pi ik}\left(-\frac{(n-1)!}{(2\pi ik)^{n-1}}\right) = -\frac{n!}{(2\pi ik)^n}.$$

■

This explicit expansion allows us to relate the Bernoulli numbers to the values of the Riemann zeta function, $\zeta(s) = \sum_{m=1}^{\infty} m^{-s}$.

Theorem 7.1. ***Values of*** $\zeta(2m)$**.**
For any integer $m \geq 1$:

$$\zeta(2m) = \frac{(-1)^{m+1}(2\pi)^{2m}}{2(2m)!} B_{2m}.$$

定理

Proof

Consider the Fourier series of $B_{2m}(x)$ evaluated at $x = 0$. Since $B_{2m}(x)$ is continuous on the circle for $2m \geq 2$, the series converges pointwise.

$$B_{2m}(0) = -(2m)! \sum_{k \neq 0} \frac{1}{(2\pi i k)^{2m}}.$$

Using $i^{2m} = (-1)^m$, we simplify the summand:

$$B_{2m} = -\frac{(2m)!}{(2\pi)^{2m}(-1)^m} \sum_{k \neq 0} \frac{1}{k^{2m}}.$$

The sum over non-zero integers is twice the sum over positive integers: $\sum_{k \neq 0} k^{-2m} = 2\zeta(2m)$.

$$B_{2m} = (-1)^{m+1} \frac{(2m)!}{(2\pi)^{2m}} (2\zeta(2m)).$$

Rearranging for $\zeta(2m)$ yields the result.

■

Example 7.1. Calculation of $\zeta(2)$ and $\zeta(4)$. We previously computed $B_2 = 1/6$. Applying the theorem with $m = 1$:

$$\zeta(2) = \frac{(-1)^2(2\pi)^2}{2(2!)} \left(\frac{1}{6}\right) = \frac{4\pi^2}{24} = \frac{\pi^2}{6}.$$

To find $\zeta(4)$, we compute B_4. By recursion:

$$B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x \implies B_3 = 0.$$
$$B_4(x) = x^4 - 2x^3 + x^2 + C.$$

Using $\int_0^1 B_4 = 0$, we find $1/5 - 2/4 + 1/3 + C = 0$, so $C = -1/30$. Thus $B_4 = -1/30$.

$$\zeta(4) = \frac{(-1)^3(2\pi)^4}{2(24)} \left(-\frac{1}{30}\right) = \frac{16\pi^4}{1440} = \frac{\pi^4}{90}.$$

範例

7.2 *Infinite Products and Wallis' Formula*

Fourier series can also be used to derive infinite product expansions for elementary functions. We return to the interval $[-\pi, \pi]$ and consider the function $f(x) = \cos(px)$, where $p \in \mathbb{R} \setminus \mathbb{Z}$.
The Fourier coefficients are given by:

$$\begin{aligned}\hat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(px)e^{-inx}\,dx \\ &= \frac{1}{4\pi}\int_{-\pi}^{\pi} \left(e^{i(p-n)x} + e^{-i(p+n)x}\right)\,dx.\end{aligned}$$

Evaluating the integrals yields:

$$\hat{f}(n) = \frac{1}{4\pi}\left[\frac{e^{i(p-n)\pi} - e^{-i(p-n)\pi}}{i(p-n)} + \frac{e^{-i(p+n)\pi} - e^{i(p+n)\pi}}{-i(p+n)}\right].$$

Using $e^{in\pi} = (-1)^n$, this simplifies to:

$$\hat{f}(n) = \frac{(-1)^n \sin(p\pi)}{\pi}\left(\frac{1}{p-n} + \frac{1}{p+n}\right)\frac{1}{2} = \frac{(-1)^n p\sin(p\pi)}{\pi(p^2-n^2)}.$$

Since $\sum |\hat{f}(n)| < \infty$ (decay is $O(n^{-2})$), the Fourier series converges uniformly to $\cos(px)$ on $[-\pi, \pi]$:

$$\cos(px) = \frac{\sin(p\pi)}{\pi}\left[\frac{1}{p} + \sum_{n\neq 0}\frac{(-1)^n p}{p^2-n^2}e^{inx}\right].$$

Grouping positive and negative n, we obtain the cosine series:

$$\cos(px) = \frac{2p\sin(p\pi)}{\pi}\left[\frac{1}{2p^2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{p^2-n^2}\cos(nx)\right]. \tag{7.1}$$

Setting $x = \pi$ in eq. (7.1), and noting $\cos(n\pi) = (-1)^n$:

$$\cos(p\pi) = \frac{2p\sin(p\pi)}{\pi}\left[\frac{1}{2p^2} + \sum_{n=1}^{\infty}\frac{1}{p^2-n^2}\right].$$

Dividing by $\sin(p\pi)$ (valid since $p \notin \mathbb{Z}$), we derive the partial fraction decomposition of the cotangent:

$$\pi\cot(p\pi) = \frac{1}{p} + \sum_{n=1}^{\infty}\frac{2p}{p^2-n^2} = \frac{1}{p} + \sum_{n=1}^{\infty}\left(\frac{1}{p-n} + \frac{1}{p+n}\right). \tag{7.2}$$

This identity holds for all $p \in \mathbb{R} \setminus \mathbb{Z}$. To obtain the product formula for the sine function, we integrate eq. (7.2) with respect to p from 0 to x. For small p, $\pi\cot(p\pi) - 1/p \approx \pi(1/(\pi p) - \pi p/3) - 1/p = -\pi^2 p/3$, which is bounded. Thus we integrate:

$$\int_0^x \left(\pi\cot(\pi p) - \frac{1}{p}\right)\,dp = \sum_{n=1}^{\infty}\int_0^x \frac{2p}{p^2-n^2}\,dp.$$

The left side is $[\ln(\sin(\pi p)) - \ln p]_0^x = \ln\left(\frac{\sin(\pi x)}{\pi x}\right)$ (taking the limit at 0). The right side is $\sum_{n=1}^{\infty} \ln\left(1 - \frac{x^2}{n^2}\right)$. Exponentiating both sides yields Euler's infinite product formula:

$$\frac{\sin(\pi x)}{\pi x} = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2}\right).$$

Theorem 7.2. ***Wallis' Product Formula.***

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots$$

定理

Proof

Set $x = 1/2$ in Euler's product formula.

$$\frac{\sin(\pi/2)}{\pi/2} = \frac{1}{\pi/2} = \prod_{n=1}^{\infty}\left(1 - \frac{1}{4n^2}\right) = \prod_{n=1}^{\infty} \frac{4n^2-1}{4n^2}.$$

Inverting the expression:

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{4n^2}{4n^2-1} = \prod_{n=1}^{\infty} \frac{(2n)^2}{(2n-1)(2n+1)}.$$

This matches the stated product.

■

7.3 *The Isoperimetric Inequality*

We now turn to a classical problem in geometry: among all simple closed curves of a fixed length L, which one encloses the maximal area? Intuition suggests the circle is the unique solution. We prove this using the orthogonality of the trigonometric system.

Geometric Preliminaries

We define a **parameterised curve** γ as a C^1 map $\gamma : [a,b] \to \mathbb{R}^2$, denoted $\gamma(t) = (x(t), y(t))$. The curve is **simple** if it does not intersect itself (except at the endpoints) and **closed** if $\gamma(a) = \gamma(b)$.
The **length** L of the curve is given by:

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

We may always reparameterise the curve by its **arc length** s. If we scale the domain such that $s \in [0, 2\pi]$, the constant speed condition implies:

$$x'(s)^2 + y'(s)^2 = \left(\frac{L}{2\pi}\right)^2. \quad (7.3)$$

The **area** A enclosed by γ is determined by Green's Theorem. A convenient symmetric form is:

$$A = \frac{1}{2}\int_0^{2\pi} (x(s)y'(s) - y(s)x'(s))\, ds. \tag{7.4}$$

Theorem 7.3. ***The Isoperimetric Inequality.***
Let Γ be a simple closed C^1 curve of length L enclosing an area A. Then:

$$A \leq \frac{L^2}{4\pi}.$$

Equality holds if and only if Γ is a circle.

定理

Proof

1. ***Rescaling.*** Let us define a scaling factor $\lambda = 2\pi/L$. The map $(x, y) \mapsto (\lambda x, \lambda y)$ scales length by λ and area by λ^2. If we prove the inequality for a curve of length 2π (where $A' \leq \pi$), the general case follows:

$$\lambda^2 A \leq \pi \implies \left(\frac{2\pi}{L}\right)^2 A \leq \pi \implies A \leq \frac{L^2}{4\pi}.$$

   Thus, without loss of generality, assume $L = 2\pi$.

2. ***Fourier Representation.*** Let $\gamma(s) = (x(s), y(s))$ be parameterised by arc length on $[0, 2\pi]$. The arc length condition $x'(s)^2 + y'(s)^2 = 1$ implies:

$$\frac{1}{2\pi}\int_0^{2\pi} (x'(s)^2 + y'(s)^2)\, ds = 1. \tag{7.5}$$

   We expand $x(s)$ and $y(s)$ in Fourier series:

$$x(s) \sim \sum a_n e^{ins}, \quad y(s) \sim \sum b_n e^{ins}.$$

   Since x, y are real, $a_{-n} = \overline{a_n}$ and $b_{-n} = \overline{b_n}$. The derivatives have coefficients ina_n and inb_n. Applying Parseval's Identity (*theorem* 5.3) to eq. (7.5):

$$\sum_{n=-\infty}^{\infty} |n|^2(|a_n|^2 + |b_n|^2) = 1. \tag{7.6}$$

3. ***Area Estimation.*** Using the Generalised Parseval Identity (chap-

ter 5) on the area formula eq. (7.4):

$$\begin{aligned} A &= \frac{1}{2}\int_0^{2\pi}(xy' - yx')\,ds \\ &= \pi \sum_{n=-\infty}^{\infty} \left(a_n\overline{(inb_n)} - b_n\overline{(ina_n)}\right) \\ &= \pi \sum_{n=-\infty}^{\infty} (-ina_n\overline{b_n} + inb_n\overline{a_n}) \\ &= \pi \sum_{n=-\infty}^{\infty} ni(b_n\overline{a_n} - a_n\overline{b_n}). \end{aligned}$$

Observe that $b_n\overline{a_n} - a_n\overline{b_n}$ is purely imaginary (it is $z - \bar{z} = 2i\text{Im}(z)$). Thus the sum yields a real value, as expected. We apply the algebraic inequality $|z - \bar{z}| \leq 2|z|$ and $2|a_nb_n| \leq |a_n|^2 + |b_n|^2$:

$$|ni(b_n\overline{a_n} - a_n\overline{b_n})| \leq 2|n||a_n||b_n| \leq |n|(|a_n|^2 + |b_n|^2).$$

Therefore:

$$A \leq \pi \sum_{n=-\infty}^{\infty} |n|(|a_n|^2 + |b_n|^2). \tag{7.7}$$

4. ***The Inequality.*** We compare the series for the length constraint (eq. (7.6)) and the area bound (eq. (7.7)).

$$\frac{L^2}{4\pi} - A = \pi - A \geq \pi \sum_{n=-\infty}^{\infty} (n^2 - |n|)(|a_n|^2 + |b_n|^2).$$

Since $n^2 - |n| = |n|(|n| - 1) \geq 0$ for all integers n, the right-hand side is non-negative. Thus $\pi - A \geq 0$, or $A \leq \pi$.

5. ***Equality Case.*** For $A = \pi$, we require the term $(n^2 - |n|)(|a_n|^2 + |b_n|^2)$ to vanish for all n.

- For $|n| \geq 2$, $n^2 - |n| > 0$, so we must have $a_n = b_n = 0$.
- For $n = 0$, the term vanishes automatically.
- For $|n| = 1$, the term vanishes.

Thus, $x(s)$ and $y(s)$ must be trigonometric polynomials of degree 1:

$$x(s) = a_0 + a_1e^{is} + a_{-1}e^{-is}, \quad y(s) = b_0 + b_1e^{is} + b_{-1}e^{-is}.$$

Since x, y are real, $a_{-1} = \overline{a_1}$. This implies:

$$x(s) = a_0 + 2\text{Re}(a_1e^{is}) = a_0 + \alpha\cos(s) + \beta\sin(s).$$

Similarly for $y(s)$. From the constraint eq. (7.6), only $n = \pm 1$ terms contribute (since $a_n = b_n = 0$ for $|n| \geq 2$):

$$1^2(|a_1|^2 + |b_1|^2) + (-1)^2(|a_{-1}|^2 + |b_{-1}|^2) = 1 \implies 2(|a_1|^2 + |b_1|^2) = 1.$$

Using the area equality condition $2|a_1||b_1| = |a_1|^2 + |b_1|^2$ (from the arithmetic-geometric mean inequality used in step 3), we find $|a_1| = |b_1| = 1/2$. The geometric constraints imply $x(s)$ and $y(s)$ define a circle parameterised by arc length.

■

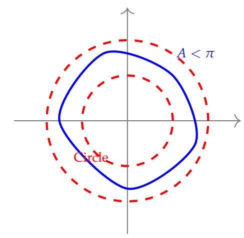


Figure 7.1: The Isoperimetric Inequality: For a fixed perimeter L, the circle maximizes the enclosed area A.

This proof relies entirely on the fact that the Fourier coefficients diagonalize the derivative operator ($\frac{d}{ds} \mapsto in$), allowing algebraic comparison of the "energy" of the derivative (Length) and the "correlation" of coordinates (Area).

Wirtinger's Inequality

The core analytic engine driving the isoperimetric proof is the spectral gap between the constant function ($n = 0$) and the first harmonic ($n = 1$). This principle is encapsulated independently as Wirtinger's Inequality.

Proposition 7.2. ***Wirtinger's Inequality.***
Let f be a 2π-periodic C^1 function with mean zero, i.e., $\int_{-\pi}^{\pi} f(x)\,dx = 0$. Then:

$$\int_{-\pi}^{\pi} |f(x)|^2\,dx \leq \int_{-\pi}^{\pi} |f'(x)|^2\,dx.$$

Equality holds if and only if $f(x) = A\cos x + B\sin x$.

命题

Proof

Since f has mean zero, the Fourier coefficient $\hat{f}(0) = 0$. By Parseval's identity:

$$\|f\|_2^2 = 2\pi \sum_{n \neq 0} |\hat{f}(n)|^2.$$

The Fourier series of the derivative f' has coefficients $in\hat{f}(n)$. Thus:

$$\|f'\|_2^2 = 2\pi \sum_{n \neq 0} |in\hat{f}(n)|^2 = 2\pi \sum_{n \neq 0} n^2 |\hat{f}(n)|^2.$$

Since n is a non-zero integer, $n^2 \geq 1$. It follows immediately that:

$$\sum_{n \neq 0} |\hat{f}(n)|^2 \leq \sum_{n \neq 0} n^2 |\hat{f}(n)|^2.$$

Equality holds if and only if coefficients for $|n| > 1$ are zero, meaning f contains only frequencies $n = \pm 1$.

■

7.4 Weyl's Equidistribution Theorem

Consider the sequence formed by the multiples of a real number γ: $\gamma, 2\gamma, 3\gamma, \ldots$. We are interested in the behaviour of this sequence modulo the integers.

> **Definition 7.2.** ***Fractional Part.***
> For any $x \in \mathbb{R}$, we define the **integer part** of x, denoted $[x]$, as the greatest integer less than or equal to x. The **fractional part** of x is defined as:
> $$\langle x \rangle = x - [x].$$
> By definition, $\langle x \rangle \in [0, 1)$ for all $x \in \mathbb{R}$.
>
> 定義

Reducing a sequence modulo $\mathbb{Z}$ isolates its fractional parts. If we define the equivalence relation $x \equiv y \pmod{\mathbb{Z}}$ if $x - y \in \mathbb{Z}$, then every real number is congruent to a unique number in $[0, 1)$.
The sequence of fractional parts $\langle n\gamma \rangle$ exhibits a dichotomy based on the rationality of γ:

1. If $\gamma \in \mathbb{Q}$, say $\gamma = p/q$ in lowest terms, the sequence is periodic with period q. The sequence visits exactly q distinct points: $\langle p/q \rangle, \langle 2p/q \rangle, \ldots, \langle (q-1)p/q \rangle, 0$.
2. If $\gamma \notin \mathbb{Q}$, the elements $\langle n\gamma \rangle$ are distinct for all n. If $\langle n_1\gamma \rangle = \langle n_2\gamma \rangle$, then $(n_1 - n_2)\gamma \in \mathbb{Z}$, which implies $\gamma \in \mathbb{Q}$, a contradiction.

Leopold Kronecker proved that for irrational γ, the sequence is not only distinct but *dense* in $[0, 1)$. In 1916, Hermann Weyl significantly strengthened this result by showing the sequence is not merely dense, but perfectly uniform. To formalise this, we introduce the concept of equidistribution.

> **Definition 7.3.** ***Equidistributed Sequence.***
> A sequence of real numbers $\{\xi_n\}_{n=1}^{\infty}$ in $[0, 1)$ is said to be **equidistributed** if for every sub-interval $(a, b) \subset [0, 1)$,
> $$\lim_{N \to \infty} \frac{\#\{1 \le n \le N : \xi_n \in (a, b)\}}{N} = b - a.$$
>
> 定義

In other words, the proportion of terms falling into any interval converges to the length of that interval. The sequence sweeps out the interval evenly.

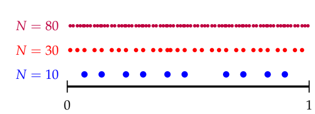


Figure 7.2: The sequence $\langle n\sqrt{2} \rangle$ for $n = 1, \ldots, N$. As N increases, the points fill the interval $[0, 1)$ with remarkable uniformity.

Weyl's Criterion and Ergodicity

To analyse the counting condition in the definition of equidistribution, we rephrase it analytically. Let $\chi_{(a,b)}(x)$ be the characteristic function of the interval (a, b), extended periodically to $\mathbb{R}$ with period 1. The number of terms in (a, b) is exactly the sum of the characteristic function evaluated at the sequence points. Thus, the condition becomes:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{(a,b)}(n\gamma) = \int_0^1 \chi_{(a,b)}(x)\,dx. \tag{7.8}$$

This equation states that the "time average" (the arithmetic mean along the sequence) equals the "space average" (the integral over the domain). This equivalence is the foundation of ergodic theory.
The strategy is to prove eq. (7.8) for the simplest periodic functions (trigonometric polynomials) using Fourier series, extend it to continuous functions via the Weierstrass Approximation Theorem (chapter 4), and finally to characteristic functions via Riemann integrability.

Lemma 7.1. ***Ergodicity for Continuous Functions.***
If f is continuous and periodic of period 1, and γ is irrational, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(n\gamma) = \int_0^1 f(x)\,dx.$$

引理

We proceed in three steps. Note that since the period is 1, the fundamental exponentials are of the form $e^{2\pi ikx}$ rather than e^{inx}.

Step 1: The Exponential Monomials.

Let $f(x) = e^{2\pi ikx}$ for some $k \in \mathbb{Z}$. If $k = 0$, $f(x) = 1$. The sum is $\frac{1}{N}\sum_{n=1}^{N} 1 = 1$, and the integral is $\int_0^1 1\,dx = 1$. The identity holds trivially. If $k \neq 0$, the integral is $\int_0^1 e^{2\pi ikx}\,dx = 0$. We must show the sum vanishes. The sum is a geometric series with ratio $r = e^{2\pi ik\gamma}$. Since γ is irrational and $k \neq 0$, $k\gamma$ is not an integer, so $r \neq 1$.

$$\frac{1}{N} \sum_{n=1}^{N} e^{2\pi ikn\gamma} = \frac{1}{N} e^{2\pi ik\gamma} \frac{1 - e^{2\pi ikN\gamma}}{1 - e^{2\pi ik\gamma}}.$$

The numerator of the fraction is bounded in modulus by 2. The denominator is a non-zero constant independent of N. Therefore, the entire expression is $O(1/N)$ and converges to 0 as $N \to \infty$.

証明終

Step 2: Trigonometric Polynomials.

By the linearity of the limit and the integral, the result holds for any finite linear combination of the form $P(x) = \sum_{k=-M}^{M} c_k e^{2\pi ikx}$.

証明終

Step 3: Continuous Functions.

Let f be continuous and periodic, and let $\epsilon > 0$. By the Weierstrass

Approximation Theorem, trigonometric polynomials are dense in the space of continuous periodic functions. Hence, there exists a trigonometric polynomial $P(x)$ such that $\sup_x |f(x) - P(x)| < \epsilon/3$. Using the triangle inequality, we bound the error:

$$\left| \frac{1}{N} \sum_{n=1}^{N} f(n\gamma) - \int_0^1 f(x)\,dx \right| \leq \frac{1}{N} \sum_{n=1}^{N} |f(n\gamma) - P(n\gamma)| + \left| \frac{1}{N} \sum_{n=1}^{N} P(n\gamma) - \int_0^1 P(x)\,dx \right| + \int_0^1 |P(x) - f(x)|\,dx.$$

The first and third terms are each strictly bounded by $\epsilon/3$ due to the uniform approximation. By Step 2, the middle term converges to 0, so for sufficiently large N, it is also less than $\epsilon/3$. Thus the total difference is less than ϵ, establishing the limit.

証明終

The logic of Step 1 provides a complete characterisation of equidistribution known as **Weyl's Criterion**. A sequence ξ_n is equidistributed if and only if for all non-zero integers k:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k \xi_n} = 0.$$

This transforms a counting problem in number theory into the estimation of "exponential sums", a cornerstone technique in modern analytic number theory.

Approximation of Characteristic Functions

We now extend *lemma 7.1* to prove the main theorem by approximating the discontinuous characteristic function $\chi_{(a,b)}$ with continuous functions.

Theorem 7.4. ***Weyl's Equidistribution Theorem.***
If γ is an irrational number, then the sequence of fractional parts $\langle n\gamma \rangle$ is equidistributed in $[0, 1)$.

定理

Proof

Let $(a, b) \subset [0, 1)$ be a fixed interval. We approximate $\chi_{(a,b)}$ from above and below by continuous functions f_ϵ^+ and f_ϵ^-.
Choose $\epsilon > 0$ small enough such that $a + \epsilon < b - \epsilon$. Define f_ϵ^- to be 1 on $[a + \epsilon, b - \epsilon]$, 0 outside (a, b), and linear on the boundary intervals $[a, a + \epsilon]$ and $[b - \epsilon, b]$. Similarly, define f_ϵ^+ to be 1 on $[a, b]$, 0 outside $[a - \epsilon, b + \epsilon]$, and linear on the transition intervals.

By construction:

$$f_\epsilon^-(x) \le \chi_{(a,b)}(x) \le f_\epsilon^+(x) \quad \text{for all } x \in [0,1).$$

Furthermore, calculating the areas of these trapezoidal functions:

$$\int_0^1 f_\epsilon^-(x)\,dx = (b-a) - \epsilon, \quad \int_0^1 f_\epsilon^+(x)\,dx = (b-a) + \epsilon.$$

Let $S_N = \frac{1}{N}\sum_{n=1}^N \chi_{(a,b)}(n\gamma)$. Due to the pointwise inequalities:

$$\frac{1}{N}\sum_{n=1}^N f_\epsilon^-(n\gamma) \le S_N \le \frac{1}{N}\sum_{n=1}^N f_\epsilon^+(n\gamma).$$

Since $f_\epsilon^\pm$ are continuous, *lemma* 7.1 applies. Taking the limit superior and limit inferior as $N \to \infty$:

$$(b-a) - \epsilon \le \liminf_{N\to\infty} S_N \le \limsup_{N\to\infty} S_N \le (b-a) + \epsilon.$$

Since ϵ is arbitrary, the limits converge to $b-a$.

■

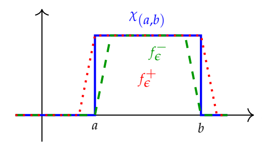


Figure 7.3: Approximation of the characteristic function $\chi_{(a,b)}$ from above (f_ϵ^+) and below (f_ϵ^-) by continuous trapezoidal functions.

Corollary 7.1. *Ergodicity for Riemann Integrable Functions.* The conclusion of *lemma* 7.1 holds for any function f that is Riemann integrable on $[0,1]$ and periodic of period 1.

推論

Proof

A Riemann integrable function can be approximated from above and below by step functions (Darboux sums). Since characteristic functions of intervals satisfy the equidistribution property, linear combinations of them (step functions) do as well. The result follows by the same squeezing argument used in *theorem* 7.4.

■

Geometric Interpretation: Billiards in a Square

The equidistribution theorem possesses a natural geometric interpretation in the theory of dynamical billiards. Consider a square table with sides of length 1, acting as perfect reflecting mirrors. A ray of light is emitted from an internal point at a trajectory with slope γ. By "unfolding" the reflections, the trajectory of the light ray can be represented as a straight line $y = \gamma x + c$ passing through a grid of unit squares in the plane. The position of the ray modulo 1 corresponds exactly to the sequence $\langle n\gamma \rangle$.

If the slope γ is rational, the line will eventually pass through equivalent points on the grid, meaning the trajectory is periodic and forms a closed loop. If γ is irrational, Kronecker's theorem ensures the trajectory never closes and eventually passes arbitrarily close to every point in the square. Weyl's Equidistribution Theorem provides a significantly deeper statement: the ray of light does not just visit every region, it spends an amount of time in any region of the square precisely proportional to the area of that region.

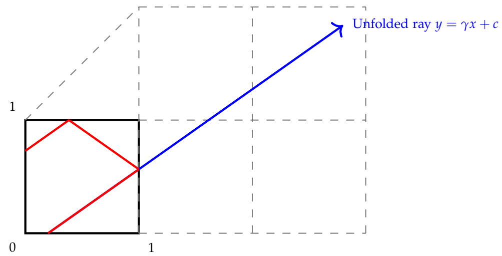


Figure 7.4: Reflection of a light ray in a square. Unfolding the reflections into a grid translates the physical bouncing into the sequence of fractional parts $\langle x \rangle$. An irrational slope guarantees the ray is ergodic.

7.5 *A Continuous but Nowhere Differentiable Function*

Riemann proposed the function $R(x) = \sum_{n=1}^{\infty} \frac{\sin(n^2 x)}{n^2}$ as a candidate for a nowhere differentiable function, though he did not provide a proof. Weierstrass subsequently provided the first rigorous counterexample in 1872, constructing the function $W(x) = \sum_{n=0}^{\infty} b^n \cos(a^n x)$ for parameters satisfying $ab > 1 + 3\pi/2$.
In this section, we construct a similar function using the complex exponential, which simplifies the algebraic manipulations. We prove that for a specific decay rate of coefficients, the function is continuous everywhere but differentiable nowhere.

> **Theorem 7.5.** ***Weierstrass-Type Function.***
> Let $0 < \alpha < 1$. The function
>
> $$f(x) = \sum_{n=0}^{\infty} 2^{-n\alpha} e^{i2^n x}$$
>
> is continuous on $\mathbb{R}$ but differentiable at no point.
>
> 定理

The continuity of f follows immediately from the Weierstrass M-test (*theorem 3.2*), as the series is dominated by $\sum (2^{-\alpha})^n$, a convergent geometric series. The lack of differentiability arises from the *lacunary* nature of the series: the frequencies 2^n increase rapidly, leaving large gaps in the spectrum.
To analyse the differentiability, we introduce a summation method tailored to these spectral gaps: the delayed means.

Delayed Means

Recall the Cesàro means of a Fourier series, $\sigma_N[g] = g * F_N$, where F_N is the Fejér kernel. The coefficients of $\sigma_N[g]$ are obtained by multiplying the Fourier coefficients $\hat{g}(k)$ by the weight $(1 - |k|/N)^+$.

Definition 7.4. ***Delayed Means.***
For a function g and integer N, the **delayed mean** $\Delta_N[g]$ is defined as:

$$\Delta_N[g](x) = 2\sigma_{2N}[g](x) - \sigma_N[g](x).$$

In terms of convolution, $\Delta_N[g] = g * (2F_{2N} - F_N)$.

定義

We determine the spectral weights of Δ_N. Let the Fourier series of g be $\sum c_k e^{ikx}$. The operator Δ_N multiplies c_k by a weight λ_k:

$$\lambda_k = 2\left(1 - \frac{|k|}{2N}\right)^+ - \left(1 - \frac{|k|}{N}\right)^+.$$

This piecewise linear function takes the following shape:

$$\lambda_k = \begin{cases} 1 & |k| \leq N, \\ 2(1 - \frac{|k|}{2N}) & N < |k| \leq 2N, \\ 0 & |k| > 2N. \end{cases}$$

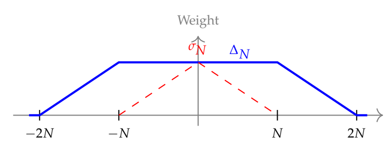


Figure 7.5: The spectral weights of the delayed mean Δ_N. It acts as the identity on frequencies up to N, then decays linearly to 0 at $2N$.

This forms a trapezoidal filter in the frequency domain.
Crucially, if the Fourier series of g has gaps (is lacunary), the partial sums S_N and the delayed means Δ_N may coincide.

Lemma 7.2. ***Lacunary Identity.***
Let f be the function defined in *theorem 7.5*. For any integer $k \geq 0$, let $N = 2^k$. Then:

$$\Delta_N[f](x) = S_N[f](x) = \sum_{j=0}^{k} 2^{-j\alpha} e^{i2^j x}.$$

引理

Proof

The frequencies present in f are powers of 2: $1, 2, 4, \ldots, 2^k, 2^{k+1}, \ldots$. The weight function of Δ_N (where $N = 2^k$) is 1 for frequencies up to 2^k. The decay region of the weight function is $(2^k, 2^{k+1})$. The next frequency in the series is 2^{k+1}, which lies exactly at the point where the weight becomes 0. Thus, Δ_N preserves all terms up to 2^k with weight 1, and suppresses all terms 2^{k+1} and higher with weight 0.

∎

Proof of Nowhere Differentiability

The proof relies on establishing a bound for the derivative of the delayed means of any differentiable function, and then showing that our specific function f violates this bound.

Lemma 7.3. ***Logarithmic Derivative Bound.***
Let g be a continuous function. If g is differentiable at a point x_0, then

$$\frac{d}{dx}\sigma_N[g](x_0) = O(\log N) \quad \text{as } N \to \infty.$$

Consequently, $\frac{d}{dx}\Delta_N[g](x_0) = O(\log N)$.

引理

Proof

Differentiation of the convolution integral yields:

$$\frac{d}{dx}\sigma_N[g](x_0) = \int_{-\pi}^{\pi} g(x_0 - t)F_N'(t)\,dt.$$

Since $\int F_N'(t)\,dt = 0$ (integral of a derivative of a periodic function), we may subtract $g(x_0)\int F_N' = 0$:

$$\frac{d}{dx}\sigma_N[g](x_0) = \int_{-\pi}^{\pi} [g(x_0 - t) - g(x_0)]F_N'(t)\,dt.$$

Since g is differentiable at x_0, there exists $C > 0$ such that $|g(x_0 - t) - g(x_0)| \leq C|t|$ for all t. Thus:

$$\left|\frac{d}{dx}\sigma_N[g](x_0)\right| \leq C\int_{-\pi}^{\pi} |t||F_N'(t)|\,dt.$$

We require estimates for the derivative of the Fejér kernel $F_N(t) = \frac{1}{N}\frac{\sin^2(Nt/2)}{\sin^2(t/2)}$.

Polynomial Bound: F_N is a trigonometric polynomial of degree N bounded by N. By Bernstein's inequality (or direct differentiation of coefficients), $|F_N'(t)| \leq 2N^2$.

Decay Bound: For $t \neq 0$, differentiation of the explicit formula yields $|F_N'(t)| \leq \frac{A}{t^2}$ for some constant A.

We split the integral at $1/N$:

$$\begin{aligned}\int_{-\pi}^{\pi} |t||F_N'(t)|\,dt &= \int_{|t|\le 1/N} |t||F_N'(t)|\,dt + \int_{1/N\le|t|\le\pi} |t||F_N'(t)|\,dt \\ &\le \int_{|t|\le 1/N} |t|(2N^2)\,dt + \int_{1/N\le|t|\le\pi} |t|\frac{A}{t^2}\,dt \\ &\le 2N^2\left[\frac{t^2}{2}\right]_{-1/N}^{1/N} + A\int_{1/N}^{\pi}\frac{1}{t}\,dt \\ &= 2N^2\frac{1}{N^2} + A(\log\pi - \log(1/N)) \\ &= 2 + A\log\pi + A\log N = O(\log N).\end{aligned}$$

The result for Δ_N follows by linearity: $\Delta_N' = 2\sigma_{2N}' - \sigma_N' = O(\log 2N) + O(\log N) = O(\log N)$.

■

We now complete the proof of *theorem* 7.5.

Proof of theorem 7.5

Suppose, for the sake of contradiction, that f is differentiable at some point x_0. By *lemma* 7.2, for $N = 2^k$, the difference between consecutive delayed means isolates a single term of the series:

$$\Delta_{2N}[f](x) - \Delta_N[f](x) = S_{2N}[f](x) - S_N[f](x) = 2^{-(k+1)\alpha}e^{i2^{k+1}x}.$$

Let us differentiate this identity at x_0. On the left side, using the hypothesis that f is differentiable at x_0 and *lemma* 7.3:

$$\left|\frac{d}{dx}\Delta_{2N}[f](x_0) - \frac{d}{dx}\Delta_N[f](x_0)\right| \le C\log N.$$

On the right side, direct differentiation yields:

$$\frac{d}{dx}\left(2^{-(k+1)\alpha}e^{i2^{k+1}x}\right) = i2^{k+1}2^{-(k+1)\alpha}e^{i2^{k+1}x}.$$

Taking the modulus of this derivative:

$$\left|i2^{(k+1)(1-\alpha)}e^{i2^{k+1}x_0}\right| = 2^{(k+1)(1-\alpha)}.$$

Since $N = 2^k$, we have $2^{k+1} = 2N$. The term grows as $(2N)^{1-\alpha}$. Combining these estimates, we arrive at the inequality:

$$(2N)^{1-\alpha} \le C\log N.$$

Since $0 < \alpha < 1$, the exponent $1-\alpha$ is positive. A power function $N^{1-\alpha}$ grows strictly faster than $\log N$ as $N \to \infty$. This yields a contradiction for sufficiently large N. Thus, f is not differentiable at x_0.

■

Remark.

While we used the complex exponential form for simplicity, the real part of this function, $\sum 2^{-n\alpha}\cos(2^n x)$, is also nowhere differentiable. The proof requires a modification of *lemma* 7.3 to bound the derivative at $x_0 + h$ and a strategic choice of h to maximize the cosine term, but the underlying mechanism—the spectral gap allows the high frequencies to dominate the local geometry—remains the same.

7.6 *The Heat Equation on the Circle*

We conclude this chapter by returning to the physical problem that motivated Fourier's original work: the diffusion of heat. While Fourier initially considered propagation in solid bodies, we analyse the simplest periodic case: heat distribution on a thin circular ring.

The ring is modelled as the unit interval $[0, 1]$ with endpoints identified (or equivalently, the real line modulo 1). Let $u(x, t)$ denote the temperature at position $x \in [0, 1)$ and time $t \geq 0$. The evolution of the temperature is governed by the heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}. \tag{7.9}$$

(We have normalized the thermal diffusivity constant to 1 by rescaling time). We are given an initial temperature distribution $u(x, 0) = f(x)$, where f is a periodic function of period 1.

Using the method of separation of variables (as introduced in chapter 1), we seek solutions of the form $u(x, t) = A(x)B(t)$. This leads to the coupled equations:

$$\frac{B'(t)}{B(t)} = \frac{A''(x)}{A(x)} = \lambda.$$

The periodicity of $A(x)$ restricts the separation constant λ to the values $-4\pi^2 n^2$ for $n \in \mathbb{Z}$, with eigenfunctions $e^{2\pi i n x}$. Solving for $B(t)$ yields $e^{-4\pi^2 n^2 t}$. By linearity, the general formal solution is:

$$u(x, t) = \sum_{n=-\infty}^{\infty} c_n e^{-4\pi^2 n^2 t} e^{2\pi i n x}. \tag{7.10}$$

Setting $t = 0$, we identify c_n as the Fourier coefficients of the initial data f:

$$c_n = \hat{f}(n) = \int_0^1 f(y) e^{-2\pi i n y}\, dy.$$

The Heat Kernel

Just as the solution to the Dirichlet problem on the disc was expressed as the convolution of the boundary data with the Poisson kernel, the solution to the heat equation is the convolution of the initial data with the Heat kernel.
We can rewrite eq. (7.10) by interchanging the sum and the integral (justified for $t > 0$ by the rapid decay of the Gaussian factor):

$$\begin{aligned} u(x,t) &= \sum_{n=-\infty}^{\infty} \left(\int_0^1 f(y) e^{-2\pi i n y} \, dy \right) e^{-4\pi^2 n^2 t} e^{2\pi i n x} \\ &= \int_0^1 f(y) \left(\sum_{n=-\infty}^{\infty} e^{-4\pi^2 n^2 t} e^{2\pi i n (x-y)} \right) dy \\ &= (f * H_t)(x). \end{aligned}$$

Definition 7.5. ***Heat Kernel.***
The **periodic heat kernel** $H_t(x)$ for $t > 0$ is defined by the series:

$$H_t(x) = \sum_{n=-\infty}^{\infty} e^{-4\pi^2 n^2 t} e^{2\pi i n x}.$$

定義

This kernel shares the fundamental "smoothing" properties of the Poisson kernel, but with a stronger decay rate.

Proposition 7.3. ***Properties of the Heat Solution.***

1. **Smoothness:** For any $t > 0$, the function $u(x,t)$ is C^∞ in x and t, even if the initial data f is discontinuous.
2. **Convergence:** If f is continuous, $\lim_{t \to 0^+} u(x,t) = f(x)$ uniformly.
3. **Mean-Square:** If f is merely square-integrable, the convergence holds in the L^2 norm.

命題

Proof
(1) The term $n^k e^{-4\pi^2 n^2 t}$ tends to 0 as $|n| \to \infty$ for any k. Thus, the series of derivatives converges uniformly.
(2) This follows if $\{H_t\}_{t>0}$ forms a family of good kernels as $t \to 0^+$. While the normalization ($\int H_t = \hat{H}_t(0) = 1$) is immediate, the positivity and concentration properties are non-trivial to prove from the Fourier series definition alone. We will rigorously establish these properties in the next chapter using the Poisson Summation Formula.

(3) By Parseval's identity:

$$\|u(\cdot,t) - f\|_2^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 |e^{-4\pi^2 n^2 t} - 1|^2.$$

Since each term vanishes as $t \to 0$ and is dominated by $4|\hat{f}(n)|^2$, the result follows from the Dominated Convergence Theorem for series.

■

Remark.

The positivity of $H_t(x)$ is physically intuitive. Heat flows from hot to cold. If we start with a non-negative temperature distribution $f \geq 0$, the temperature $u(x,t)$ should remain non-negative for all time. Since $u = f * H_t$, if H_t were negative in some region, one could construct an initial f concentrated in that region that produces a negative temperature, violating physical principles. This heuristic is confirmed mathematically: $H_t(x)$ is strictly positive everywhere.

7.7 *Exercises*

1. **Calculating Bernoulli Polynomials.** Using the recursive definition, compute the explicit form of the Bernoulli polynomials $B_3(x)$ and $B_4(x)$. Verify that $\int_0^1 B_n(x)\,dx = 0$ for these cases.

2. **Sums of Reciprocals.** Use the formula for $\zeta(2m)$ to evaluate the following series:

   (a) $\sum_{n=1}^{\infty} \frac{1}{n^6}$.

   (b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$.

3. **Product Expansions.**

   (a) By integrating the cotangent series, derive the product formula for $\cos(\pi x)$:

   $$\cos(\pi x) = \prod_{n=1}^{\infty} \left(1 - \frac{4x^2}{(2n-1)^2}\right).$$

   (b) Use this product to calculate $\prod_{n=1}^{\infty} \left(1 - \frac{1}{(2n-1)^2}\right)$.

4. **Geometric Optimisation.**

   (a) Use the isoperimetric inequality to prove that among all rectangles of a fixed perimeter P, the square has the maximum area.

   (b) Can you use Wirtinger's inequality to prove that if $f(0) = f(\pi) = 0$ and $\int_0^{\pi} (f')^2\,dx = 1$, then $\int_0^{\pi} f^2\,dx \leq 1$?

5. **Weyl's Criterion Practice.**

   (a) Prove that if α is rational, say p/q, the sequence $\{n\alpha\}$ is *not* equidistributed in $[0, 1)$.

   (b) Let α be irrational. Prove that the sequence of points $(\{n\alpha\}, \{n^2\alpha\})$ is equidistributed in the unit square $[0, 1)^2$.

   > *Remark.*
   >
   > This requires the multidimensional version of Weyl's criterion involving exponentials $e^{2\pi i(k_1 x + k_2 y)}$.

6. **Spectral Gaps.** Let $f(x) = \sum_{k=1}^{\infty} a_k \sin(3^k x)$ with $\sum |a_k| < \infty$. Show that the Fourier series of f has large gaps. Can you determine if f is differentiable at $x = 0$ if $a_k = 2^{-k}$?

7. **Heat Evolution.** Let the initial temperature distribution on the circle be $f(x) = \cos(2\pi x)$.

   (a) Write down the solution $u(x, t)$ to the heat equation.

   (b) At what time $t > 0$ does the maximum temperature drop to $1/e$ of its initial value?

8
The Fourier Integral

We established that if a function f satisfies specific regularity conditions (such as differentiability or the Dini criterion), it can be represented as a discrete superposition of sinusoids:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{l} x + b_n \sin \frac{n\pi}{l} x \right). \tag{8.1}$$

If f is defined on the entire real line $\mathbb{R}$ and is absolutely integrable, we may attempt to apply this theory by restricting f to $[-l, l]$. However, as $l \to \infty$, the frequency spacing π/l tends to zero, suggesting a transition from a discrete summation to a continuous integral.
To obtain a unified representation for non-periodic functions on $(-\infty, \infty)$, we introduce the Fourier Integral. This transition effectively replaces the integer index n with a continuous frequency parameter u, and the coefficients a_n, b_n with continuous functions $a(u), b(u)$.

8.1 Integral Representation

Let f be a function defined on $\mathbb{R}$ that is absolutely integrable, i.e., $f \in L^1(\mathbb{R})$. Motivated by the coefficients of the Fourier series, we define the following integral transforms for any real number u:

$$a(u) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(t) \cos(ut)\, dt, \quad b(u) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(t) \sin(ut)\, dt. \tag{8.2}$$

Since f is absolutely integrable and $|\cos(ut)| \le 1$, these integrals are absolutely convergent. Analogous to the Fourier series, we form the **Fourier integral** of f:

$$f(x) \sim \int_0^{+\infty} (a(u) \cos(ux) + b(u) \sin(ux))\, du. \tag{8.3}$$

The convergence of this integral to $f(x)$ is not guaranteed by the definition alone. To establish rigorous convergence criteria, we first analyse the analytic properties of the coefficient functions $a(u)$ and $b(u)$.

Theorem 8.1. ***Uniform Continuity of Coefficients.***
Let f be absolutely integrable on $(-\infty, +\infty)$. Then the functions $a(u)$ and $b(u)$ defined in eq. (8.2) are uniformly continuous on $\mathbb{R}$.

定理

Proof

We provide the proof for $a(u)$; the proof for $b(u)$ is identical. Let $\epsilon > 0$. Since $f \in L^1(\mathbb{R})$, there exists a sufficiently large $A > 0$ such that the tails of the integral are negligible:

$$\int_{-\infty}^{-A} |f(t)|\, dt + \int_{A}^{+\infty} |f(t)|\, dt < \frac{\pi\epsilon}{4}.$$

We now consider the integral over the compact interval $[-A, A]$. The function $\cos(x)$ is uniformly continuous on $\mathbb{R}$. Therefore, there exists $\eta > 0$ such that $|z_1 - z_2| < \eta$ implies $|\cos z_1 - \cos z_2| < \delta'$, where δ' satisfies:

$$\delta' \left(\frac{1}{\pi} \int_{-A}^{A} |f(t)|\, dt \right) < \frac{\epsilon}{2}.$$

Let $\delta = \eta / A$. For any $u', u'' \in \mathbb{R}$ with $|u' - u''| < \delta$, and for any $t \in [-A, A]$, we have:

$$|u't - u''t| = |t||u' - u''| \leq A\delta = \eta.$$

Consequently, $|\cos(u't) - \cos(u''t)| < \delta'$.
We estimate the difference $|a(u') - a(u'')|$ by splitting the domain of integration:

$$\begin{aligned} |a(u') - a(u'')| &= \left| \frac{1}{\pi} \int_{-\infty}^{+\infty} f(t)(\cos(u't) - \cos(u''t))\, dt \right| \\ &\leq \frac{1}{\pi} \int_{-\infty}^{-A} |f(t)| \cdot 2\, dt + \frac{1}{\pi} \int_{A}^{+\infty} |f(t)| \cdot 2\, dt \\ &\quad + \frac{1}{\pi} \int_{-A}^{A} |f(t)||\cos(u't) - \cos(u''t)|\, dt. \end{aligned}$$

Using our bounds, the tail contributions sum to less than $\frac{2}{\pi}(\frac{\pi\epsilon}{4}) = \frac{\epsilon}{2}$. The central integral is bounded by $\frac{\epsilon}{2}$. Thus, $|a(u') - a(u'')| < \epsilon$, proving uniform continuity.

■

8.2 Convergence of the Fourier Integral

To study the pointwise convergence, we consider the partial integral over frequencies $[0, \lambda]$:

$$S(\lambda, x) = \int_0^{\lambda} (a(u)\cos(ux) + b(u)\sin(ux))\, du. \tag{8.4}$$

Substituting the definitions of $a(u)$ and $b(u)$ and utilizing the identity $\cos(ut)\cos(ux) + \sin(ut)\sin(ux) = \cos(u(x-t))$, we obtain:

$$S(\lambda, x) = \frac{1}{\pi}\int_0^{\lambda} \left\{ \int_{-\infty}^{+\infty} f(t)\cos(u(x-t))\, dt \right\} du. \tag{8.5}$$

We aim to express this in a form similar to the Dirichlet integral for Fourier series. This requires interchanging the order of integration.

Theorem 8.2. ***Dirichlet Form of the Fourier Integral.***
Let f be absolutely integrable on $(-\infty, +\infty)$. For any $\lambda > 0$:

$$S(\lambda, x) = \frac{1}{\pi}\int_0^{+\infty} (f(x+t) + f(x-t))\frac{\sin(\lambda t)}{t}\, dt.$$

定理

Proof

The crucial step is to justify the exchange of integration order in eq. (8.5). We must show:

$$\int_0^{\lambda}\int_{-\infty}^{+\infty} f(t)\cos(u(x-t))\, dt\, du = \int_{-\infty}^{+\infty}\int_0^{\lambda} f(t)\cos(u(x-t))\, du\, dt. \tag{8.6}$$

Let $A > 0$. On the compact rectangle $[0, \lambda] \times [-A, A]$, Fubini's theorem (or standard calculus of double integrals) guarantees:

$$\int_0^{\lambda}\int_{-A}^{A} f(t)\cos(u(x-t))\, dt\, du = \int_{-A}^{A}\int_0^{\lambda} f(t)\cos(u(x-t))\, du\, dt.$$

Since $f \in L^1(\mathbb{R})$, for any $\epsilon > 0$ there exists A_0 such that for $A > A_0$, the tails $\int_{|t|>A} |f(t)|\, dt < \epsilon/\lambda$. The difference between the integral over $(-\infty, \infty)$ and $[-A, A]$ for the left-hand side of eq. (8.6) is bounded by:

$$\int_0^{\lambda}\int_{|t|>A} |f(t)||\cos(u(x-t))|\, dt\, du \leq \int_0^{\lambda} \frac{\epsilon}{\lambda}\, du = \epsilon.$$

Thus, as $A \to \infty$, the integrals converge uniformly. This justifies eq. (8.6).
Evaluating the inner integral with respect to u:

$$\int_0^{\lambda} \cos(u(x-t))\, du = \frac{\sin(\lambda(x-t))}{x-t}.$$

Substituting this back:

$$S(\lambda, x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(t) \frac{\sin(\lambda(x-t))}{x-t} \, dt.$$

Making the substitution $v = t - x$ (and exploiting the symmetry of the kernel):

$$S(\lambda, x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(x+v) \frac{\sin(\lambda v)}{-v} \, dv = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(x+v) \frac{\sin(\lambda v)}{v} \, dv.$$

Splitting the integral into $(-\infty, 0)$ and $(0, \infty)$ and combining terms yields the result.

■

We are now in a position to state the Localization Principle for the entire real line, which mirrors **??**.

Theorem 8.3. ***Localization Theorem.***
Let f be absolutely integrable on $\mathbb{R}$. The convergence of the Fourier integral at a point x and its limit depend solely on the values of f in an arbitrarily small neighbourhood of x.

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Proof

Consider the expression for $S(\lambda, x)$ derived above. For any $A_0 > 0$, we split the integration domain into $[0, A_0]$ and $[A_0, \infty)$. For the tail integral, we observe that for $t > A_0$, the kernel is bounded: $|\frac{\sin \lambda t}{t}| \leq \frac{1}{A_0}$. Since $f \in L^1$, the function $g(t) = \frac{f(x+t)+f(x-t)}{t}$ is absolutely integrable on $[A_0, \infty)$. By the Riemann-Lebesgue Lemma (*theorem* 1.1), as $\lambda \to \infty$:

$$\int_{A_0}^{+\infty} (f(x+t) + f(x-t)) \frac{\sin \lambda t}{t} \, dt \to 0.$$

Thus, the limit of $S(\lambda, x)$ depends entirely on the behaviour of the integral over $[0, A_0]$.

■

This localization allows us to transplant Dini's convergence test directly from the theory of Fourier series (*theorem* 2.2).

Theorem 8.4. ***Convergence of the Fourier Integral.***
Let $f \in L^1(\mathbb{R})$. If f is differentiable at x, or more generally satisfies the Dini condition at x, then the Fourier integral converges to the value of f at x:

$$f(x) = \frac{1}{\pi} \int_0^{+\infty} du \int_{-\infty}^{+\infty} f(t) \cos(u(x-t)) \, dt.$$

If f has a jump discontinuity at x but satisfies the one-sided Dini conditions, the integral converges to $\frac{f(x^+)+f(x^-)}{2}$.

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8.3 *Sine and Cosine Transforms*

The general Fourier integral formula simplifies significantly if f possesses symmetry. From eq. (8.2):

1. If f is an **even function**, $b(u) = 0$ and $a(u) = \frac{2}{\pi}\int_0^\infty f(t)\cos(ut)\,dt$. The integral representation becomes the **Fourier Cosine Formula**:

$$f(x) = \frac{2}{\pi}\int_0^{+\infty} \cos(ux)\left(\int_0^{+\infty} f(t)\cos(ut)\,dt\right) du. \tag{8.7}$$

2. If f is an **odd function**, $a(u) = 0$ and $b(u) = \frac{2}{\pi}\int_0^\infty f(t)\sin(ut)\,dt$. This yields the **Fourier Sine Formula**:

$$f(x) = \frac{2}{\pi}\int_0^{+\infty} \sin(ux)\left(\int_0^{+\infty} f(t)\sin(ut)\,dt\right) du. \tag{8.8}$$

These formulas allow us to define the Fourier transform for functions defined only on $[0,\infty)$ by extending them evenly or oddly to the whole line.

Example 8.1. The Dirichlet Integral. Consider the box function (or rectangular pulse) defined by:

$$f(x) = \begin{cases} 1, & |x| \le 1, \\ 0, & |x| > 1. \end{cases}$$

This function is even and absolutely integrable. We compute its cosine coefficient $a(u)$:

$$a(u) = \frac{2}{\pi}\int_0^{+\infty} f(t)\cos(ut)\,dt = \frac{2}{\pi}\int_0^{1}\cos(ut)\,dt = \frac{2}{\pi}\frac{\sin u}{u}.$$

The coefficient $b(u)$ is identically zero. For any point x where f is continuous (i.e., $|x| \ne 1$), the convergence theorem implies:

$$f(x) = \int_0^{+\infty} \frac{2}{\pi}\frac{\sin u}{u}\cos(ux)\,du.$$

Rearranging this, we obtain the value of the integral:

$$\int_0^{+\infty} \frac{\sin u\cos(ux)}{u}\,du = \frac{\pi}{2}f(x) = \begin{cases} \frac{\pi}{2}, & |x| < 1, \\ 0, & |x| > 1. \end{cases} \tag{8.9}$$

At the points of discontinuity $x = \pm 1$, the integral converges to the average $\frac{1}{2}(f(1^+) + f(1^-)) = \frac{1}{2}$. Thus:

$$\int_0^{+\infty} \frac{\sin u\cos u}{u}\,du = \frac{\pi}{4}.$$

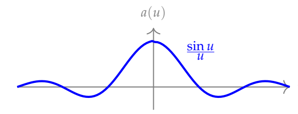
Setting $x = 0$, we recover the classical Dirichlet integral:

$$\int_0^{+\infty} \frac{\sin u}{u}\, du = \frac{\pi}{2}.$$

範例

Remark (Symmetric Form).

Frequently, the factors of π are redistributed to obtain a symmetric form. If we define the transform $g(u) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\cos(ut)\, dt$, then the inversion formula becomes $f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(u)\cos(ux)\, du$.



The transform of the box function.

Figure 8.1: The coefficient function $a(u)$ for the box function decays as $1/u$, illustrating the duality between spatial confinement and spectral decay.

8.4 *Reciprocity of the Sine and Cosine Transforms*

The formulas derived in eq. (8.7) and eq. (8.8) exhibit a striking structural symmetry. If we distribute the normalization factor $2/\pi$ symmetrically as $\sqrt{2/\pi}$, we obtain a pair of reciprocal transformations.

Definition 8.1. ***Fourier Cosine and Sine Transforms.***
Let f be an integrable function on $[0, \infty)$. The **Fourier Cosine Transform**, denoted $\mathcal{F}_c[f]$ or $g(u)$, is defined by:

$$g(u) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(t)\cos(ut)\, dt.$$

The **Fourier Sine Transform**, denoted $\mathcal{F}_s[f]$ or $h(u)$, is defined by:

$$h(u) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} f(t)\sin(ut)\, dt.$$

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From the integral representations established in the previous section, the inversion formulas are identical to the forward transforms:

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} g(u)\cos(xu)\, du, \tag{8.1}$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} h(u)\sin(xu)\, du. \tag{8.2}$$

This duality implies that applying the transformation twice (with the appropriate variable substitution) recovers the original function.

Example 8.2. Transforms of the Exponential Decay. Consider the function $f(x) = e^{-\beta x}$ for $x > 0$, where $\beta > 0$. We compute the cosine transform $g(u)$:

$$g(u) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} e^{-\beta t}\cos(ut)\, dt.$$

Using integration by parts or the real part of $\int e^{(-\beta+iu)t}\, dt$, we obtain:

$$\int_0^{+\infty} e^{-\beta t} \cos(ut)\, dt = \frac{\beta}{\beta^2 + u^2}.$$

Thus:

$$\mathcal{F}_c[e^{-\beta x}](u) = \sqrt{\frac{2}{\pi}} \frac{\beta}{\beta^2 + u^2}.$$

Similarly, the sine transform $h(u)$ is:

$$h(u) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} e^{-\beta t} \sin(ut)\, dt = \sqrt{\frac{2}{\pi}} \frac{u}{\beta^2 + u^2}.$$

Applying the inverse formulas yields two fundamental definite integrals (Laplace integrals):

$$\int_0^{+\infty} \frac{\cos(xu)}{\beta^2 + u^2}\, du = \frac{\pi}{2\beta} e^{-\beta x}, \quad \int_0^{+\infty} \frac{u \sin(xu)}{\beta^2 + u^2}\, du = \frac{\pi}{2} e^{-\beta x}.$$

These identities are valid for $x > 0, \beta > 0$.

範例

Example 8.3. Solution of an Integral Equation. Consider the problem of finding a function $g(u)$ that satisfies the integral equation:

$$\int_0^{+\infty} g(u) \sin(xu)\, du = f(x),$$

where $f(x)$ is defined by:

$$f(x) = \begin{cases} \frac{\pi}{2} \sin x & 0 \leq x \leq \pi, \\ 0 & x > \pi. \end{cases}$$

We observe that the integral equation can be rewritten as a Fourier sine inversion:

$$\sqrt{\frac{2}{\pi}} \int_0^{+\infty} g(u) \sin(xu)\, du = \sqrt{\frac{2}{\pi}} f(x).$$

Thus, $\sqrt{2/\pi} f(x)$ is the sine transform of $g(u)$. By the reciprocity of the sine transform, $g(u)$ is the sine transform of $\sqrt{2/\pi} f(x)$:

$$g(u) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \left(\sqrt{\frac{2}{\pi}} f(t) \right) \sin(ut)\, dt = \frac{2}{\pi} \int_0^{\pi} \frac{\pi}{2} \sin t \sin(ut)\, dt.$$

Calculating the integral using the identity $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$:

$$\begin{aligned} g(u) &= \int_0^{\pi} \sin t \sin(ut)\, dt \\ &= \frac{1}{2} \int_0^{\pi} (\cos(t(1-u)) - \cos(t(1+u)))\, dt \\ &= \frac{1}{2} \left[\frac{\sin(\pi(1-u))}{1-u} - \frac{\sin(\pi(1+u))}{1+u} \right]. \end{aligned}$$

Using $\sin(\pi - \pi u) = \sin(\pi u)$ and $\sin(\pi + \pi u) = -\sin(\pi u)$:

$$g(u) = \frac{\sin(\pi u)}{2}\left(\frac{1}{1-u} + \frac{1}{1+u}\right) = \frac{\sin(\pi u)}{1-u^2}.$$

範例

8.5 *The Complex Fourier Transform*

The separation into sine and cosine transforms is natural for functions with definite parity, but for general functions on $\mathbb{R}$, a unified complex notation is more efficient.

Recall the Fourier Integral Formula derived in *theorem* 8.4:

$$f(x) = \frac{1}{\pi}\int_0^{+\infty} du \int_{-\infty}^{+\infty} f(t)\cos(u(x-t))\,dt. \tag{8.10}$$

The inner integral $\phi(u) = \int_{-\infty}^{+\infty} f(t)\cos(u(x-t))\,dt$ is an even function of u. Consequently, we may extend the outer integral to $(-\infty, +\infty)$ by dividing by 2:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} f(t)\cos(u(x-t))\,dt. \tag{8.11}$$

Similarly, consider the integral involving the sine term:

$$\psi(u) = \int_{-\infty}^{+\infty} f(t)\sin(u(x-t))\,dt.$$

This is an odd function of u. Therefore, its integral over the symmetric domain $(-\infty, +\infty)$ vanishes:

$$0 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} f(t)\sin(u(x-t))\,dt. \tag{8.12}$$

Multiplying eq. (8.12) by the imaginary unit i and adding it to eq. (8.11), we utilise Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ to obtain the **Complex Form of the Fourier Integral**:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} f(t)e^{iu(x-t)}\,dt. \tag{8.13}$$

Rearranging the exponentials as $e^{iu(x-t)} = e^{iux}e^{-iut}$, we can split this double integral into a transformation pair. We adopt the symmetric normalization $1/\sqrt{2\pi}$.

Definition 8.2. ***The Fourier Transform.***
Let f be absolutely integrable on $\mathbb{R}$. The **Fourier Transform** of f, denoted by $\hat{f}$ or $\mathcal{F}[f]$, is the complex-valued function:

$$\hat{f}(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(t)e^{-iut}\,dt. \tag{8.14}$$

The **Inverse Fourier Transform** recovers the function:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(u) e^{iux}\, du. \tag{8.15}$$

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Remark.

Here u represents the frequency variable (often denoted by ξ, ω, or k in physics). Note that $\hat{f}(u)$ is complex-valued even if $f(t)$ is real.

Analogy with Fourier Series

The definition of the Fourier transform is the natural limit of the Fourier series coefficients as the period tends to infinity. Recall the complex form of the Fourier series for a function on $[-\pi, \pi]$:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad \text{where} \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx.$$

Comparing the discrete pair $(c_n, f(x))$ with the continuous pair $(\hat{f}(u), f(x))$:

- The discrete coefficient c_n corresponds to the spectral density $\hat{f}(u)$.
- The sum $\sum_n$ is replaced by the integral $\int du$.
- The harmonic frequencies n become the continuous variable u.

This analogy suggests that the Fourier transform decomposes a non-periodic signal into a continuous spectrum of exponential waves, just as the series decomposes a periodic signal into a discrete spectrum.

8.6 *Operational Properties*

The power of the Fourier transform lies in its ability to convert analytic operations (differentiation, integration) into algebraic operations (multiplication, division).

Theorem 8.5. ***Derivative Property.***
Let f be continuous and absolutely integrable on $\mathbb{R}$, and suppose $\lim_{t\to\pm\infty} f(t) = 0$. If f' is absolutely integrable, then:

$$\widehat{f'}(u) = iu\hat{f}(u).$$

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Proof

We apply the definition of the transform and integrate by parts:

$$\widehat{f'}(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f'(t) e^{-iut}\, dt$$
$$= \frac{1}{\sqrt{2\pi}} \left(\left[f(t) e^{-iut} \right]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} f(t)(-iu) e^{-iut}\, dt \right).$$

By the decay assumption $\lim_{t \to \pm\infty} f(t) = 0$, the boundary terms vanish. The remaining integral is exactly $iu\hat{f}(u)$.

■

By induction, if f and its derivatives up to order $n-1$ vanish at infinity and are absolutely integrable, we obtain the general formula:

$$\widehat{f^{(n)}}(u) = (iu)^n \hat{f}(u). \tag{8.16}$$

This property makes the Fourier transform an indispensable tool for solving linear differential equations with constant coefficients. Consider the differential equation:

$$a_n f^{(n)}(t) + \cdots + a_1 f'(t) + a_0 f(t) = g(t).$$

Applying the Fourier transform to both sides transforms the differential operator into a polynomial in iu:

$$\left(a_n (iu)^n + \cdots + a_1 (iu) + a_0 \right) \hat{f}(u) = \hat{g}(u).$$

The solution $\hat{f}(u)$ is found by algebraic division:

$$\hat{f}(u) = \frac{\hat{g}(u)}{P(iu)},$$

where $P(z) = \sum a_k z^k$ is the characteristic polynomial. The solution $f(t)$ is then recovered via the inverse transform.

8.7 *Exercises*

1. **Calculating Fourier Integrals.** Express the following functions as Fourier integrals (either real form or complex form):

   (a) The signed pulse function:

   $$f(x) = \begin{cases} \operatorname{sgn} x & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$

   (b) The truncated sine wave:

   $$f(x) = \begin{cases} \sin x & |x| \le \pi \\ 0 & |x| > \pi \end{cases}$$

(c) The symmetric exponential decay:

$$f(x) = e^{-a|x|}, \quad a > 0.$$

2. **Solving Integral Equations.** Determine the function $f(t)$ defined on $(0, \infty)$ that satisfies the following integral equations:

   (a) A sine transform equation:

   $$\int_0^{+\infty} f(t) \sin(xt)\, dt = e^{-x}, \quad x > 0.$$

   (b) A cosine transform equation:

   $$\int_0^{+\infty} f(t) \cos(xt)\, dt = \frac{1}{1+x^2}.$$

3. **Verification of Integral Identities.** Evaluate the following integral to prove the equality:

$$\frac{2}{\pi} \int_0^{+\infty} \frac{\sin^2 t}{t^2} \cos(2xt)\, dt = \begin{cases} 1-x & 0 \leq x \leq 1 \\ 0 & x > 1 \end{cases}$$

> *Remark.*
>
> Hint: Consider the Fourier cosine transform of the triangular function on the right-hand side.

4. **Inverse Fourier Transforms.** Compute the inverse Fourier transform $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(u) e^{iux}\, du$ for the following spectral functions:

   (a) $F(u) = ue^{-\beta|u|}$ with $\beta > 0$.

   (b) The Gaussian spectrum $F(u) = e^{-u^2/2}$.

> *Remark.*
>
> For part (b), use the result $\int_{-\infty}^{\infty} e^{-ax^2}\, dx = \sqrt{\pi/a}$.