Parametric Integrals

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Integrals with Parameters

We consider the general properties of integrals with parameters, specifically focusing on proper integrals. We examine their analytical properties (limits, continuity, differentiability, and integrability), and demonstrate their utility in evaluating definite integrals.

0.1 Proper Integrals with Parameters

We begin by motivating the study with a geometric problem. Consider the ellipse defined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $b > a > 0$. The arc length L of this ellipse is given by the integral:

$$L = 4\int_0^{\pi/2} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}\, dt.$$

By factoring out b^2 and employing the trigonometric identity $\cos^2 t = 1 - \sin^2 t$, we may rewrite this as:

$$L = 4b\int_0^{\pi/2} \sqrt{1 - \frac{b^2 - a^2}{b^2}\sin^2 t}\, dt.$$

Defining the eccentricity $k = \frac{\sqrt{b^2-a^2}}{b}$, the integral becomes:

$$I(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 t}\, dt$$

is the *complete elliptic integral of the second kind*. It cannot be expressed in terms of elementary functions and serves as a prototypical example of an integral with a parameter.

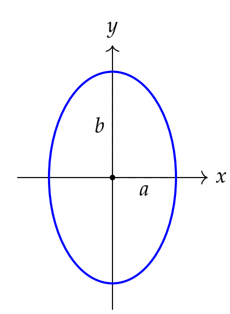

Figure 1: An ellipse with semi-major axis b and semi-minor axis a, where $b > a > 0$.

> **Definition 0.1.** ***Proper Integral with Parameter.***
> Let $D \subset \mathbb{R}$. Let $f(x,t)$ be a function defined on $[a,b] \times D$. If for every $t \in D$, $f(x,t)$ is integrable with respect to x on $[a,b]$, then the function
>
> $$\varphi(t) = \int_a^b f(x,t)\, dx$$

is called a **proper integral with a parameter** defined on D.

定義

We state the fundamental analytic properties of these functions.

Proposition 0.1. ***Limit Property.***
Let t_0 be an accumulation point of D. If $\lim_{t\to t_0} f(x,t) = \psi(x)$ and the convergence is uniform for $x \in [a,b]$, then $\psi(x)$ is bounded and integrable on $[a,b]$, and:

$$\lim_{t\to t_0} \varphi(t) = \lim_{t\to t_0} \int_a^b f(x,t)\,dx = \int_a^b \psi(x)\,dx.$$

命題

Proposition 0.2. ***Continuity.***
If $f(x,t)$ is continuous on $[a,b]\times[c,d]$, then

$$\varphi(t) = \int_a^b f(x,t)\,dx$$

is continuous on $[c,d]$.

命題

Proof

Since f is continuous on the compact set R, it is uniformly continuous. Let $\epsilon > 0$. There exists $\delta > 0$ such that for any $(x,t),(x',t') \in R$:

$$|(x,t)-(x',t')| < \delta \implies |f(x,t)-f(x',t')| < \frac{\epsilon}{b-a}.$$

Let $t_0 \in [c,d]$. For any $t \in [c,d]$ with $|t-t_0| < \delta$, we have:

$$|\varphi(t)-\varphi(t_0)| = \left|\int_a^b [f(x,t)-f(x,t_0)]\,dx\right| \le \int_a^b |f(x,t)-f(x,t_0)|\,dx.$$

Since $|(x,t)-(x,t_0)| = |t-t_0| < \delta$, the integrand is bounded by $\frac{\epsilon}{b-a}$. Thus:

$$|\varphi(t)-\varphi(t_0)| < \int_a^b \frac{\epsilon}{b-a}\,dx = \epsilon.$$

∎

Proposition 0.3. ***Interchanging Order of Integration.***
If $f(x,t)$ is continuous on $[a,b]\times[c,d]$, then

$$\int_c^d dt \int_a^b f(x,t)\,dx = \int_a^b dx \int_c^d f(x,t)\,dt.$$

命題

Proof

Let $\varphi(t) = \int_a^b f(x,t)\,dx$. Since f is continuous, φ is continuous and thus integrable on $[c,d]$. The equality follows from Fubini's Theorem for continuous functions on compact rectangles. Alternatively, it can be proven by showing that both iterated integrals are limits of the same Riemann sums due to uniform continuity.

■

Proposition 0.4. ***Differentiability (Leibniz's Rule).***
If $f(x,t)$ and the partial derivative $f_t(x,t)$ are continuous on $[a,b] \times [c,d]$, then $\varphi(t) = \int_a^b f(x,t)\,dx$ is differentiable on $[c,d]$, and

$$\varphi'(t) = \int_a^b f_t(x,t)\,dx.$$

命题

Proof

Let $g(x,t) = f_t(x,t)$. Since g is continuous on a compact set, it is uniformly continuous. Let $\epsilon > 0$. There exists $\delta > 0$ such that $|\Delta t| < \delta$ implies $|g(x,t+\Delta t) - g(x,t)| < \epsilon/(b-a)$. Consider the difference quotient:

$$\frac{\varphi(t+\Delta t) - \varphi(t)}{\Delta t} = \int_a^b \frac{f(x,t+\Delta t) - f(x,t)}{\Delta t}\,dx.$$

By the Mean Value Theorem, there exists $\theta \in (0,1)$ such that the integrand equals $f_t(x,t+\theta\Delta t)$. Thus:

$$\left|\frac{\varphi(t+\Delta t) - \varphi(t)}{\Delta t} - \int_a^b f_t(x,t)\,dx\right| \le \int_a^b |f_t(x,t+\theta\Delta t) - f_t(x,t)|\,dx.$$

The RHS is bounded by $\int_a^b \frac{\epsilon}{b-a}\,dx = \epsilon$. As $\Delta t \to 0$, the limit holds.

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If the limits of integration also depend on the parameter, we have the following generalisation.

Proposition 0.5. ***General Leibniz Rule.***
Let $f(x,t)$ be continuous on $[a,b] \times [c,d]$. Let $\alpha(t)$ and $\beta(t)$ be continuous on $[c,d]$ such that $a \le \alpha(t), \beta(t) \le b$ for all $t \in [c,d]$. Then

$$\varphi(t) = \int_{\alpha(t)}^{\beta(t)} f(x,t)\,dx$$

is continuous on $[c,d]$. Furthermore, if $f_t(x,t)$ is continuous on $[a,b] \times [c,d]$ and $\alpha(t), \beta(t)$ are differentiable on $[c,d]$, then $\varphi(t)$ is differentiable and:

$$\varphi'(t) = f(\beta(t),t)\beta'(t) - f(\alpha(t),t)\alpha'(t) + \int_{\alpha(t)}^{\beta(t)} f_t(x,t)\,dx.$$

命題

Proof

Let $\Phi(u,v,t) = \int_u^v f(x,t)\,dx$. Then $\varphi(t) = \Phi(\alpha(t),\beta(t),t)$. By the Chain Rule:

$$\frac{d\varphi}{dt} = \frac{\partial\Phi}{\partial u}\alpha'(t) + \frac{\partial\Phi}{\partial v}\beta'(t) + \frac{\partial\Phi}{\partial t}.$$

By the Fundamental Theorem of Calculus, $\frac{\partial\Phi}{\partial v} = f(v,t)$ and $\frac{\partial\Phi}{\partial u} = -f(u,t)$. By proposition 0.4, $\frac{\partial\Phi}{\partial t} = \int_u^v f_t(x,t)\,dx$. Substituting $u = \alpha(t)$ and $v = \beta(t)$ yields the result.

■

Example 0.1. Continuity Analysis. Let $f(x)$ be continuous on $[0,1]$. Investigate the continuity of

$$F(t) = \int_0^1 \frac{t}{x^2+t^2} f(x)\,dx.$$

範例

Solution

$F(t)$ is defined on $(-\infty,+\infty)$. For any $t_0 \neq 0$, the function $h(x,t) = \frac{tf(x)}{x^2+t^2}$ is continuous on $[0,1]\times[t_0/2, 2t_0]$. By proposition 0.2, $F(t)$ is continuous at t_0.
We examine the point $t=0$. Consider the limit as $t \to 0^+$:

$$\int_0^1 \frac{t}{x^2+t^2} f(x)\,dx = \int_0^{t^{1/3}} \frac{t}{x^2+t^2} f(x)\,dx + \int_{t^{1/3}}^1 \frac{t}{x^2+t^2} f(x)\,dx.$$

As $t \to 0^+$:

- For the first integral, by the Mean Value Theorem for integrals, there exists $\xi \in [0, t^{1/3}]$ such that:

$$\int_0^{t^{1/3}} \frac{t}{x^2+t^2} f(x)\,dx = f(\xi)\arctan\frac{t^{1/3}}{t} \to f(0)\frac{\pi}{2}.$$

- For the second integral:

$$\left|\int_{t^{1/3}}^1 \frac{t}{x^2+t^2} f(x)\,dx\right| \le \max_{x\in[0,1]} |f(x)| \cdot \frac{t}{t^{2/3}+t^2} \to 0.$$

Thus $\lim_{t\to 0^+} F(t) = f(0)\frac{\pi}{2}$. Similarly, $\lim_{t\to 0^-} F(t) = -f(0)\frac{\pi}{2}$. Since $F(0) = 0$, $F(t)$ is continuous at $t=0$ if and only if $f(0)=0$.

■

Example 0.2. Differentiation with Variable Limits. Let $F(t) = \int_0^{t^2} dx \int_{x-t}^{x+t} \sin(x^2+y^2-t^2)\,dy$. Find $F'(t)$.

範例

Solution

Let $g(x,t) = \int_{x-t}^{x+t} \sin(x^2+y^2-t^2)\,dy$. Using the differentiation formula for integrals with parameters:

$$F'(t) = 2t \cdot g(t^2,t) + \int_0^{t^2} \frac{\partial}{\partial t} g(x,t)\,dx.$$

Note that $g(t^2,t) = \int_{t^2-t}^{t^2+t} \sin(t^4+y^2-t^2)\,dy$. For the partial derivative inside the integral, we apply the General Leibniz Rule to $g(x,t)$:

$$\begin{aligned}\frac{\partial}{\partial t}\int_{x-t}^{x+t} \sin(x^2+y^2-t^2)\,dy &= \sin[x^2+(x+t)^2-t^2]\cdot(1)\\ &\quad - \sin[x^2+(x-t)^2-t^2]\cdot(-1)\\ &\quad + \int_{x-t}^{x+t}(-2t)\cos(x^2+y^2-t^2)\,dy.\end{aligned}$$

Simplifying the boundary terms:

$$\sin(2x^2+2xt)+\sin(2x^2-2xt) = 2\sin(2x^2)\cos(2xt).$$

Thus:

$$F'(t) = 2t\int_{x-t}^{x+t} \sin(t^4+y^2-t^2)\,dy + 2\int_0^{t^2} \sin(2x^2)\cos(2xt)\,dx - 2t\int_0^{t^2} dx \int_{x-t}^{x+t} \cos(x^2+y^2-t^2)\,dy.$$

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Methods for Evaluating Integrals

If computing $\int_a^b f(x,t)\,dx$ directly is difficult, two common methods involve parameters:

Differentiation: Compute $\int_a^b f_t(x,t)\,dx$ first, then integrate the result with respect to t.

Integration: Express $f(x,t)$ as an integral, then interchange the order of integration.

Example 0.3. Differentiation under the Integral. Compute

$$I(x) = \int_0^{\pi/2} \ln(\sin^2\theta + x^2\cos^2\theta)\,d\theta$$

for $0 < x < +\infty$.

範例

Solution

Let $f(x,\theta) = \ln(\sin^2\theta + x^2\cos^2\theta)$. For any $x_0 \in (0,\infty)$, f and f_x are continuous on appropriate compact domains. Differentiation yields:

$$I'(x) = \int_0^{\pi/2} \frac{2x\cos^2\theta}{\sin^2\theta + x^2\cos^2\theta}\,d\theta = 2x\int_0^{\pi/2} \frac{d\theta}{x^2+\tan^2\theta}.$$

Let $t = \tan\theta$:

$$I'(x) = 2x\int_0^{+\infty} \frac{1}{x^2+t^2}\cdot\frac{1}{1+t^2}\,dt.$$

Using partial fractions for $x \neq 1$:

$$I'(x) = \frac{2x}{x^2-1}\int_0^{+\infty}\left(\frac{1}{1+t^2} - \frac{1}{x^2+t^2}\right)dt = \frac{2x}{x^2-1}\left(\frac{\pi}{2} - \frac{1}{x}\frac{\pi}{2}\right) = \frac{\pi}{1+x}.$$

Integrating gives $I(x) = \pi\ln(1+x) + C$. By continuity, this holds for $x = 1$. At $x = 1$, $I(1) = \int_0^{\pi/2} \ln(1)\,d\theta = 0$, so $C = -\pi\ln 2$.

$$I(x) = \pi\ln\frac{1+x}{2}.$$

■

Example 0.4. Interchanging Order of Integration. Find

$$I(\alpha) = \int_0^{\pi/2} \ln\frac{1+\alpha\cos x}{1-\alpha\cos x}\cdot\frac{1}{\cos x}\,dx$$

for $|\alpha| < 1$.

範例

Solution

Observe that the integrand can be written as an integral:

$$\frac{\ln(1+\alpha\cos x)}{\cos x} - \frac{\ln(1-\alpha\cos x)}{\cos x} = \int_{-\alpha}^{\alpha}\frac{dy}{1+y\cos x}.$$

Let $f(x,y) = \frac{1}{1+y\cos x}$. For $\alpha \in (-1,1)$, we can interchange the order:

$$I(\alpha) = \int_{-\alpha}^{\alpha} dy \int_0^{\pi/2}\frac{dx}{1+y\cos x}.$$

The inner integral evaluates to:

$$\int_0^{\pi/2}\frac{dx}{1+y\cos x} = \frac{2}{\sqrt{1-y^2}}\arctan\left(\sqrt{\frac{1-y}{1+y}}\right).$$

Thus:

$$I(\alpha) = \int_{-\alpha}^{\alpha}\frac{2}{\sqrt{1-y^2}}\arctan\sqrt{\frac{1-y}{1+y}}\,dy.$$

Using the symmetry of the arctan arguments (summing to $\pi/2$ roughly), or direct evaluation:

$$I(\alpha) = \pi \int_0^\alpha \frac{dy}{\sqrt{1-y^2}} = \pi \arcsin \alpha.$$

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Example 0.5. Proof of Constant Integral. Prove

$$\int_0^{2\pi} e^{t\cos\theta} \cos(t\sin\theta)\, d\theta = 2\pi.$$

範例

Proof

Let $f(t)$ be the integral. We show $f(t) \equiv f(0) = 2\pi$ by showing $f'(t) \equiv 0$.

$$f'(t) = \int_0^{2\pi} e^{t\cos\theta} \cos(t\sin\theta + \theta)\, d\theta.$$

By induction, $f^{(n)}(t) = \int_0^{2\pi} e^{t\cos\theta} \cos(t\sin\theta + n\theta)\, d\theta$. Thus $f^{(n)}(0) = \int_0^{2\pi} \cos(n\theta)\, d\theta = 0$ for $n \geq 1$. By Taylor's Theorem:

$$f(t) = f(0) + \sum_{k=1}^{n} \frac{t^k}{k!} f^{(k)}(0) + R_n = 2\pi + R_n.$$

Using the estimate $|f^{(n)}(\xi)| \leq 2\pi e^{|t|}$, the remainder term vanishes as $n \to \infty$. Thus $f(t) = 2\pi$.

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We can also prove it using Green's theorem

We will properly introduce this in the upcoming notes.

Proof

$$f'(t) = \oint_{x^2+y^2=1} e^{tx}[\cos(ty)\, dy + \sin(ty)\, dx].$$

Using Green's formula on the unit disk D:

$$f'(t) = \iint_D \left(\frac{\partial}{\partial x}[e^{tx}\cos(ty)] - \frac{\partial}{\partial y}[e^{tx}\sin(ty)] \right) dxdy.$$

Since $\frac{\partial}{\partial x}[e^{tx}\cos(ty)] = te^{tx}\cos(ty)$ and $\frac{\partial}{\partial y}[e^{tx}\sin(ty)] = te^{tx}\cos(ty)$, the integrand is 0. Thus $f'(t) = 0$.

■

Example 0.6. Embedding Method. Find

$$I = \int_0^1 \frac{\ln(1+x)}{1+x^2}\, dx.$$

範例

Solution

Introduce a parameter α and define $I(\alpha) = \int_0^1 \frac{\ln(1+\alpha x)}{1+x^2}\,dx$. Then $I(1) = I$ and $I(0) = 0$. Differentiating:

$$I'(\alpha) = \int_0^1 \frac{x}{(1+x^2)(1+\alpha x)}\,dx = \frac{1}{1+\alpha^2}\int_0^1 \left(\frac{\alpha + x}{1+x^2} - \frac{\alpha}{1+\alpha x}\right)dx.$$

Evaluating the integral:

$$I'(\alpha) = \frac{1}{1+\alpha^2}\left[\frac{\alpha\pi}{4} + \frac{1}{2}\ln 2 - \ln(1+\alpha)\right].$$

Integrating back from 0 to 1:

$$I(1) = \int_0^1 \frac{\frac{\alpha\pi}{4} + \frac{1}{2}\ln 2}{1+\alpha^2}\,d\alpha - \int_0^1 \frac{\ln(1+\alpha)}{1+\alpha^2}\,d\alpha.$$

The last term is exactly $I(1)$. Thus:

$$2I(1) = \frac{\pi}{4}\int_0^1 \frac{\alpha}{1+\alpha^2}\,d\alpha + \frac{1}{2}\ln 2\int_0^1 \frac{d\alpha}{1+\alpha^2} = \frac{\pi}{4}\ln 2,$$

which yields $I(1) = \frac{\pi}{8}\ln 2$.

■

0.2 *Improper Integrals with Parameters*

The transition from proper to improper integrals mirrors the transition from finite sums to infinite series. Just as a series of functions $\sum f_n(x)$ may be viewed as a discrete summation dependent on a parameter, an improper integral with a parameter $\int_a^\infty f(x,t)\,dx$ represents a continuous summation. Consequently, the analytic properties of these integrals are governed by a concept analogous to the uniform convergence of series.

Uniform Convergence

Let T be a subset of $\mathbb{R}$ (typically an interval) and let $f : [a, +\infty) \times T \to \mathbb{R}$. We assume that for every fixed $t \in T$, the improper integral $\int_a^\infty f(x,t)\,dx$ converges. We denote the value of the integral by $\varphi(t)$.

Definition 0.2. ***Uniform Convergence.***
The integral $\int_a^\infty f(x,t)\,dx$ is said to **converge uniformly** with respect to t on T if for every $\epsilon > 0$, there exists $A_0 = A_0(\epsilon) > a$ such that for all $A > A_0$ and for all $t \in T$:

$$\left|\int_a^\infty f(x,t)\,dx - \int_a^A f(x,t)\,dx\right| = \left|\int_A^\infty f(x,t)\,dx\right| < \epsilon.$$

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The crucial distinction between pointwise and uniform convergence is that A_0 depends only on ϵ and not on the parameter t.

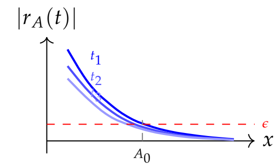

Figure 2: Uniform convergence: the tail remainder $r_A(t) = \int_A^\infty f(x,t)\,dx$ falls below ϵ for *all* t once $A > A_0$.

Tests for Uniform Convergence

We establish several criteria to determine uniform convergence, which are direct analogues of tests for series of functions.

Theorem 0.1. ***Cauchy's Criterion.***
The integral $\int_a^\infty f(x,t)\,dx$ converges uniformly on T if and only if for every $\epsilon > 0$, there exists $A_0 > a$ such that for all $A_1, A_2 > A_0$ and for all $t \in T$:

$$\left| \int_{A_1}^{A_2} f(x,t)\,dx \right| < \epsilon.$$

定理

Theorem 0.2. ***Weierstrass M-Test.***
Let $f(x,t)$ be defined on $[a, +\infty) \times T$. Suppose there exists a non-negative function $F(x)$ defined on $[a, +\infty)$ such that:

1. $|f(x,t)| \le F(x)$ for all $x \ge a$ and $t \in T$.
2. The improper integral $\int_a^\infty F(x)\,dx$ converges.

Then $\int_a^\infty f(x,t)\,dx$ converges uniformly (and absolutely) on T.

定理

For integrals that are conditionally convergent, the M-test fails. In such cases, we employ the tests of Abel and Dirichlet, which rely on the interplay between a monotonic term and an oscillatory or bounded term.

Theorem 0.3. ***Abel's Test.***
The integral $\int_a^\infty f(x,t)g(x,t)\,dx$ converges uniformly on T if:

1. $\int_a^\infty f(x,t)\,dx$ converges uniformly on T.
2. For each t, $g(x,t)$ is monotonic with respect to x, and $g(x,t)$ is uniformly bounded on $[a, +\infty) \times T$.

定理

Theorem 0.4. ***Dirichlet's Test.***
The integral $\int_a^\infty f(x,t)g(x,t)\,dx$ converges uniformly on T if:

1. The partial integrals $F(A,t) = \int_a^A f(x,t)\,dx$ are uniformly bounded; i.e., there exists $M > 0$ such that $|\int_a^A f(x,t)\,dx| \le M$ for all $A \ge a$ and $t \in T$.
2. For each t, $g(x,t)$ is monotonic with respect to x, and $g(x,t) \to 0$ as $x \to \infty$ uniformly with respect to t.

定理

Finally, for monotonic integrands, convergence to a continuous limit implies uniformity.

> **Theorem 0.5.** ***Dini's Theorem for Integrals.***
> Let $f(x,t)$ be continuous and non-negative on $[a,+\infty) \times [\alpha,\beta]$. If $\varphi(t) = \int_a^\infty f(x,t)\,dx$ is continuous on $[\alpha,\beta]$, then the integral converges uniformly on $[\alpha,\beta]$.
>
> 定理

> *Note*
>
> While the definitions above focus on infinite limits (Type I), the theory applies identically to improper integrals with finite singular points (Type II). For an integral $\int_a^b f(x,t)\,dx$ with a singularity at b, uniform convergence requires $|\int_{b-\delta}^b f(x,t)\,dx| < \epsilon$ for sufficiently small δ, independent of t.

Disproving Uniform Convergence

To show that an integral does **not** converge uniformly, one typically employs one of the following strategies:

Negation of Cauchy Criterion: Show that there exists $\epsilon_0 > 0$ such that for any A_0, one can find $A_1, A_2 > A_0$ and a specific parameter $t \in T$ where $|\int_{A_1}^{A_2} f(x,t)\,dx| \geq \epsilon_0$.

Limit Point Divergence: If t_0 is an accumulation point of T, and the integral converges for all $t \in T \setminus \{t_0\}$ but diverges at t_0, then convergence cannot be uniform on T.

Discontinuity: If f is continuous but the resulting integral function $\varphi(t)$ is discontinuous, convergence is not uniform (this is the contrapositive of the property that uniform convergence preserves continuity).

We illustrate these tests with examples where the integrand often changes sign.

> **Example 0.7.** Abel and Dirichlet Applications. Investigate the uniform convergence of
>
> $$I(y) = \int_0^\infty \frac{\sin x^2}{1+x^y}\,dx$$
>
> for $y \in [0,+\infty)$.
>
> 範例

> *Method 1 (Abel's Test).*
>
> Consider the factorisation $f(x,y) = \sin x^2$ and $g(x,y) = \frac{1}{1+x^y}$. First,

observe that $\int_0^\infty \sin x^2 \, dx$ converges (the Fresnel integral). Since this integral is independent of y, it converges uniformly with respect to y. For any fixed $y \geq 0$, the function $g(x,y) = \frac{1}{1+x^y}$ is monotonic decreasing in x (for $x > 0$). Furthermore, $|g(x,y)| \leq 1$ for all $x, y \geq 0$. Thus, by Abel's Test, the integral converges uniformly on $[0, +\infty)$.

■

Method 2 (Dirichlet's Test).

Rewrite the integrand as:

$$\frac{\sin x^2}{1+x^y} = \left(x \sin x^2\right) \cdot \left(\frac{1}{x(1+x^y)}\right).$$

Let $f(x) = x \sin x^2$. The partial integral is:

$$\left|\int_0^A x \sin x^2 \, dx\right| = \left|\left[-\frac{1}{2}\cos x^2\right]_0^A\right| = \frac{1}{2}|1 - \cos A^2| \leq 1.$$

Thus, the partial integrals are uniformly bounded. Let $h(x,y) = \frac{1}{x(1+x^y)}$. For fixed y, this is monotonic in x. Moreover:

$$\left|\frac{1}{x(1+x^y)}\right| \leq \frac{1}{x}.$$

Since $\frac{1}{x} \to 0$ as $x \to \infty$ independently of y, $h(x,y) \to 0$ uniformly. By Dirichlet's Test, the integral converges uniformly.

■

Example 0.8. Convergence on Domains. Discuss the uniform convergence of

$$I(\alpha) = \int_1^\infty \frac{\sin x}{x^\alpha} \, dx$$

on (1) $[\alpha_0, +\infty)$ with $\alpha_0 > 0$, and (2) $(0, +\infty)$.

On $[\alpha_0, +\infty)$. The partial integrals of $\sin x$ are uniformly bounded by 2. The function $g(x,\alpha) = \frac{1}{x^\alpha}$ is monotonic in x. Furthermore, for $\alpha \geq \alpha_0$:

$$\left|\frac{1}{x^\alpha}\right| \leq \frac{1}{x^{\alpha_0}}.$$

Since $x^{-\alpha_0} \to 0$ as $x \to \infty$, the convergence of g to 0 is uniform. By Dirichlet's Test, $I(\alpha)$ converges uniformly.

On $(0, +\infty)$. The integrand is continuous on $[1, \infty) \times (0, \infty)$. However, consider the limit as $\alpha \to 0^+$. The pointwise limit is $\int_1^\infty \sin x \, dx$, which diverges. Since $\alpha = 0$ is an accumulation point of the domain where the integral diverges, $I(\alpha)$ cannot converge uniformly on $(0, +\infty)$.

範例

Example 0.9. Improper Integral of Type II. Discuss the uniform convergence of $I(p) = \int_0^1 x^{p-1} \ln^2 x \, dx$ for (1) $p \geq p_0 > 0$ and (2) $p > 0$. The singularity is at $x = 0$.

On $[p_0, +\infty)$. For $x \in (0, 1)$ and $p \geq p_0$, we have $|x^{p-1} \ln^2 x| = x^{p-1} \ln^2 x \leq x^{p_0-1} \ln^2 x$. The integral $\int_0^1 x^{p_0-1} \ln^2 x \, dx$ converges (one may verify this by substitution $x = e^{-t}$). By the M-Test, convergence is uniform.

On $(0, +\infty)$. We suspect non-uniformity because $\int_0^1 x^{-1} \ln^2 x \, dx$ diverges. We verify this using the negation of the definition. Consider the integral on a small interval $[0, \xi]$:

$$\left| \int_0^\xi x^{p-1} \ln^2 x \, dx \right| = \int_0^\xi x^{p-1} \ln^2 x \, dx.$$

Using the substitution $u = x/\xi$ (so $x = \xi u$) is essentially scaling, but we can estimate directly. Since $\ln^2 x$ is decreasing near 0, for $x \in (0, \xi)$:

$$\int_0^\xi x^{p-1} \ln^2 x \, dx \geq \ln^2 \xi \int_0^\xi x^{p-1} \, dx = \ln^2 \xi \left[\frac{x^p}{p} \right]_0^\xi = \frac{\xi^p \ln^2 \xi}{p}.$$

We choose parameters to make this large. Let $p = \xi$ and consider $\xi \to 0^+$.

$$\lim_{\xi \to 0^+} \frac{\xi^\xi \ln^2 \xi}{\xi} = \lim_{\xi \to 0^+} \xi^\xi \cdot \frac{\ln^2 \xi}{\xi} = 1 \cdot (+\infty) = +\infty.$$

Thus, for any ξ_0, we can choose $\xi < \xi_0$ and $p = \xi$ such that the tail integral is arbitrarily large (specifically ≥ 1). This violates the definition of uniform convergence.

範例

Example 0.10. Proof by Contradiction. Prove that

$$I(t) = \int_1^\infty \frac{x \sin tx}{a^2 + x^2} \, dx$$

does **not** converge uniformly on $(0, +\infty)$.

範例

Proof

Suppose for the sake of contradiction that $I(t)$ converges uniformly on $(0, +\infty)$. We may write:

$$\frac{\sin tx}{x} = \frac{x \sin tx}{a^2 + x^2} \cdot \frac{a^2 + x^2}{x^2}.$$

Let

$$f(x,t) = \frac{x \sin tx}{a^2 + x^2}$$

and

$$g(x) = 1 + \frac{a^2}{x^2}.$$

The function $g(x)$ is monotonic decreasing for $x \geq 1$ and bounded by $1 + a^2$. If $\int f(x,t)\,dx$ converges uniformly, then by Abel's Test, the product integral $\int_1^\infty \frac{\sin tx}{x}\,dx$ must also converge uniformly on $(0, +\infty)$. However, it is a known result (demonstrable via the Cauchy criterion with $t = 1/A$) that $\int_1^\infty \frac{\sin tx}{x}\,dx$ does not converge uniformly near $t = 0$. This contradiction implies $I(t)$ is not uniformly convergent.

■

0.3 *Analytic Properties of Improper Integrals*

Similar to series of functions, improper integrals with parameters possess key analytic properties such as continuity, differentiability, and integrability. These properties hold provided the convergence is uniform.

We state the fundamental propositions for improper integrals of the form $\varphi(t) = \int_a^\infty f(x,t)\,dx$, where f is defined on $[a, +\infty) \times [\alpha, \beta]$.

Proposition 0.6. ***Continuity of Improper Integrals.***
Let $f(x,t)$ be continuous on $[a, +\infty) \times [\alpha, \beta]$. If the integral $\int_a^\infty f(x,t)\,dx$ converges uniformly to $\varphi(t)$ on $[\alpha, \beta]$, then $\varphi(t)$ is continuous on $[\alpha, \beta]$. That is, for any $t_0 \in [\alpha, \beta]$:

$$\lim_{t \to t_0} \int_a^\infty f(x,t)\,dx = \int_a^\infty \lim_{t \to t_0} f(x,t)\,dx.$$

命题

Proposition 0.7. ***Differentiability Under the Integral Sign.***
Let $f(x,t)$ and its partial derivative $f_t(x,t)$ be continuous on $[a, +\infty) \times [\alpha, \beta]$. Suppose:

1. The integral $\int_a^\infty f(x,t)\,dx$ converges to $\varphi(t)$ for $t \in [\alpha, \beta]$.
2. The integral of the derivative $\int_a^\infty f_t(x,t)\,dx$ converges uniformly on $[\alpha, \beta]$.

Then $\varphi(t)$ is differentiable on $[\alpha, \beta]$ and:

$$\varphi'(t) = \frac{d}{dt} \int_a^\infty f(x,t)\,dx = \int_a^\infty \frac{\partial f}{\partial t}(x,t)\,dx.$$

命题

Proposition 0.8. ***Interchanging Order of Integration.***
There are two primary cases for interchanging improper integrals:

Case 1 (Finite Interval): Under the conditions of the Continuity Proposition, $\varphi(t)$ is integrable on $[\alpha, \beta]$, and:

$$\int_{\alpha}^{\beta} dt \int_{a}^{\infty} f(x,t)\, dx = \int_{a}^{\infty} dx \int_{\alpha}^{\beta} f(x,t)\, dt.$$

Case 2 (Infinite Interval): Let $f(x,t)$ be continuous for $x \geq a, t \geq c$. Suppose:

1. $\int_a^\infty f(x,t)\, dx$ converges uniformly with respect to t on any finite interval.
2. $\int_c^\infty f(x,t)\, dt$ converges uniformly with respect to x on any finite interval.
3. At least one of the iterated integrals of the absolute value exists:

$$\int_{c}^{\infty} dt \int_{a}^{\infty} |f(x,t)|\, dx \quad \text{or} \quad \int_{a}^{\infty} dx \int_{c}^{\infty} |f(x,t)|\, dt.$$

Then both iterated integrals of $f(x,t)$ exist and are equal:

$$\int_{c}^{\infty} dt \int_{a}^{\infty} f(x,t)\, dx = \int_{a}^{\infty} dx \int_{c}^{\infty} f(x,t)\, dt.$$

命题

Note

If $f(x,t) \geq 0$, Dini's Theorem ensures that the existence of one iterated integral implies the existence of the other, without requiring the explicit absolute convergence check.

We apply these properties to evaluate improper integrals, often using standard results like the Dirichlet integral $\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$ or the Gaussian integral $\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$.

Example 0.11. Integration by Parts with Parameters. Find

$$I = \int_0^{+\infty} \frac{1}{x^2}(e^{-\alpha x^2} - 1)\, dx$$

for $\alpha > 0$.

範例

Solution

The integrand has a removable singularity at $x = 0$ since the limit is $-\alpha$. We use integration by parts to reduce the power of $1/x^2$. Let $u = e^{-\alpha x^2} - 1$ and $dv = x^{-2}dx$. Then $v = -1/x$ and

$du = -2\alpha x e^{-\alpha x^2} dx$.

$$I = \left[-\frac{1}{x}(e^{-\alpha x^2} - 1) \right]_0^{+\infty} + \int_0^{+\infty} \frac{1}{x}(-2\alpha x e^{-\alpha x^2})\, dx.$$

The boundary term vanishes at $+\infty$. At 0, using L'Hôpital's rule, the limit is 0.

$$I = -2\alpha \int_0^{+\infty} e^{-\alpha x^2}\, dx.$$

Using the substitution $u = \sqrt{\alpha}x$, we have $\int_0^\infty e^{-\alpha x^2}\, dx = \frac{1}{\sqrt{\alpha}} \frac{\sqrt{\pi}}{2}$.

$$I = -2\alpha \left(\frac{\sqrt{\pi}}{2\sqrt{\alpha}} \right) = -\sqrt{\pi\alpha}.$$

■

Example 0.12. Continuity via Uniform Convergence. Prove that

$$F(\alpha) = \int_0^{+\infty} \frac{x}{2 + x^\alpha}\, dx$$

is continuous on $(2, +\infty)$.

範例

Proof

We establish continuity on any ray $[2 + \epsilon, +\infty)$ where $\epsilon > 0$. For $\alpha \geq 2 + \epsilon$ and $x \geq 1$:

$$\left| \frac{x}{2 + x^\alpha} \right| \leq \frac{x}{x^\alpha} = \frac{1}{x^{\alpha - 1}} \leq \frac{1}{x^{1+\epsilon}}.$$

Since $\int_1^\infty x^{-(1+\epsilon)}\, dx$ converges, the M-Test implies that $\int_1^{+\infty} \frac{x}{2+x^\alpha}\, dx$ converges uniformly for $\alpha \in [2 + \epsilon, +\infty)$. The integral on $[0, 1]$ is proper and thus continuous. Since uniform convergence preserves continuity, $F(\alpha)$ is continuous on $[2 + \epsilon, +\infty)$. Since ϵ is arbitrary, F is continuous on $(2, +\infty)$.

■

Example 0.13. Continuity and Differentiability. Let $b \neq 0$. Prove that $F(a) = \int_0^{+\infty} \frac{1}{t}(1 - e^{-at}) \cos bt\, dt$ is continuous on $[0, +\infty)$ and differentiable on $(0, +\infty)$.

範例

Proof of Continuity.

Define $f(t, a) = \frac{1}{t}(1 - e^{-at}) \cos bt$ for $t > 0$ and $f(0, a) = a$. This function is continuous. We split the integral at $t = 1$. The part $\int_0^1$ is a proper parameter integral and is thus continuous.
For $\int_1^{+\infty}$, we apply Abel's Test. The integral $\int_1^\infty \frac{\cos bt}{t}\, dt$ converges

uniformly with respect to a (it is independent of a and convergent). The factor $(1 - e^{-at})$ is monotonic in t for fixed a and bounded by 2. Thus, the improper integral converges uniformly for $a \in [0, +\infty)$, implying continuity.

証明終

Differentiability.

The partial derivative is $f_a(t, a) = e^{-at} \cos bt$. For any $\epsilon > 0$, if $a \geq \epsilon$, then $|e^{-at} \cos bt| \leq e^{-\epsilon t}$. Since $\int_0^\infty e^{-\epsilon t}\, dt$ converges, the integral of the derivative converges uniformly on $[\epsilon, +\infty)$ by the M-Test. Thus $F(a)$ is differentiable on $[\epsilon, +\infty)$, and by extension on $(0, +\infty)$, with:

$$F'(a) = \int_0^{+\infty} e^{-at} \cos bt\, dt.$$

証明終

Example 0.14. Evaluating the Integral. Find the value of

$$F(a) = \int_0^{+\infty} \frac{1}{t}(1 - e^{-at}) \cos bt\, dt.$$

範例

Solution

We can evaluate this in two ways.

Integration of Derivative From the previous example, for $a > 0$:

$$F'(a) = \int_0^{+\infty} e^{-at} \cos bt\, dt = \frac{a}{a^2 + b^2}.$$

Integrating with respect to a:

$$F(a) = \frac{1}{2} \ln(a^2 + b^2) + C.$$

Since $F(a)$ is continuous at $a = 0$ and $F(0) = 0$:

$$0 = \frac{1}{2} \ln(b^2) + C \implies C = -\frac{1}{2} \ln(b^2).$$

$$F(a) = \frac{1}{2} \ln\left(\frac{a^2 + b^2}{b^2}\right).$$

Embedding / Fubini's Theorem We rewrite the term $\frac{1}{t}(1 - e^{-at})$ as an integral:

$$\frac{1 - e^{-at}}{t} = \int_0^a e^{-yt}\, dy.$$

Thus $F(a) = \int_0^{+\infty} \left(\int_0^a e^{-yt}\, dy\right) \cos bt\, dt$. We interchange the order of integration. This is justified for $a > 0$ because

$\int_0^\infty |e^{-yt}\cos bt|\,dt$ converges for $y > 0$, and the convergence is uniform on intervals bounded away from $y = 0$.

$$F(a) = \int_0^a \left(\int_0^{+\infty} e^{-yt} \cos bt\, dt \right) dy = \int_0^a \frac{y}{y^2 + b^2}\, dy.$$

$$F(a) = \left[\frac{1}{2} \ln(y^2 + b^2) \right]_0^a = \frac{1}{2} \ln(a^2 + b^2) - \frac{1}{2} \ln(b^2).$$

■

0.4 Exercises

1. **Evaluate the Parameter Integral.** Compute the function $F(\theta) = \int_0^\pi \ln(1 + \theta \cos x)\, dx$ for $|\theta| < 1$.

   > *Remark.*
   >
   > Differentiate with respect to θ and evaluate the resulting integral using the substitution $t = \tan(x/2)$ or standard residues.

2. **Differentiation with Variable Limits.** Let $f(s,t)$ be a differentiable function. Find the derivative $F'(x)$ of the function:

   $$F(x) = \int_0^x dt \int_{t^2}^{x^2} f(t,s)\, ds.$$

   > *Remark.*
   >
   > Apply the General Leibniz Rule carefully, noting that x appears in the outer limit and the inner limit.

3. **Continuity of Parameter Integrals.** Let $f(x,y)$ be continuous and bounded on the rectangle $(a,b) \times (c,d)$. Prove that the function

   $$I(x) = \int_c^d f(x,y)\, dy$$

   is continuous on the interval (a,b).

4. **Derivative of a Logarithmic Potential.** Let $R > 0$. Consider the integral:

   $$I(a) = \int_0^{2\pi} \ln(R^2 + a^2 - 2aR\cos\theta)\, d\theta \quad \text{for } |a| < R.$$

   Prove that $I'(a) = 0$. What is the value of $I(a)$?

5. **Interchanging Integration Order.** For $a, b > 0$, evaluate the integral:

   $$\int_0^1 \frac{x^b - x^a}{\ln x} \sin\left(\ln \frac{1}{x} \right) dx.$$

> *Remark.*
>
> Express the fraction as an integral with respect to a parameter, interchange the order, and evaluate.

6. **Second Derivative Identity.** Let f be a continuous function. Define
$$F(t) = \int_0^a dx \int_0^x f(x+y+t)\,dy.$$
Prove that the second derivative satisfies:
$$F''(t) = f(t+2a) - 2f(t+a) + f(t).$$

7. **The Gaussian Integral via Parameters.** Define the functions:
$$f(t) = \left(\int_0^t e^{-x^2}\,dx\right)^2, \quad g(t) = \int_0^1 \frac{e^{-t^2(1+x^2)}}{1+x^2}\,dx.$$
   (a) Prove that $f'(t) + g'(t) = 0$ for all t.
   (b) Deduce that $f(t) + g(t) = \frac{\pi}{4}$.
   (c) By taking the limit as $t \to \infty$, compute the value of the Gaussian integral $\int_0^{+\infty} e^{-x^2}\,dx$.

8. **Testing Uniform Convergence.** Investigate the uniform convergence of the following integrals on the specified domains. Justify your answers using the M-test, Abel's test, Dirichlet's test, or by disproving uniform convergence.

   (1) $\displaystyle\int_0^{+\infty} e^{-(1+a^2)t} \sin t\,dt, \quad a \in (-\infty, +\infty).$

   (2) $\displaystyle\int_0^{+\infty} \frac{\cos xy}{\sqrt{x+y}}\,dx, \quad y \in [y_0, +\infty),$ where $y_0 > 0$.

   (3) $\displaystyle\int_0^{+\infty} e^{-tx^2}\,dx, \quad t \in (0, +\infty).$

   (4) $\displaystyle\int_1^{+\infty} e^{-\alpha x} \frac{\cos x}{\sqrt{x}}\,dx, \quad \alpha \in [0, +\infty).$

   (5) $\displaystyle\int_0^{+\infty} e^{-(x-y)^2}\,dx, \quad y \in (-\infty, +\infty).$

   (6) $\displaystyle\int_0^{+\infty} x \ln x e^{-t\sqrt{x}}\,dx, \quad$ (i) $t \in [t_0, +\infty)$, $t_0 > 0$; $\quad$ (ii) $t \in (0, +\infty)$.

   (7) $\displaystyle\int_1^{+\infty} \frac{1 - e^{-ut}}{t} \cos t\,dt, \quad u \in [0, 1].$

   (8) $\displaystyle\int_0^{+\infty} \frac{\alpha t}{1+\alpha^2 t^2} e^{-\alpha^2 t^2} \cos(\alpha^2 t^2)\,dt, \quad \alpha \in (0, +\infty).$

   (9) $\displaystyle\int_0^{+\infty} e^{-x^2(1+y^2)} \sin y\,dy, \quad x \in (0, +\infty).$

   (10) $\displaystyle\int_0^{+\infty} \frac{\alpha\,dx}{1+\alpha^2 x^2}, \quad \alpha \in (0, 1).$

(11) $\displaystyle\int_0^2 \frac{x^t}{\sqrt{(x-1)(x-2)}}\,dx, \quad |t| < \frac{1}{2}.$

(12) $\displaystyle\int_0^1 (1-x)^{u-1}\,dx,$ (i) $u \in [a, +\infty)$, $a > 0$; (ii) $u \in (0, +\infty)$.

9. **Convexity of Convergence Domain.** Suppose the integral $\int_0^{+\infty} x^\lambda f(x)\,dx$ converges for $\lambda = a$ and $\lambda = b$ with $a < b$. Prove that the integral converges uniformly for $\lambda \in [a, b]$.

10. **Non-Uniform Convergence.** Prove that the integral $\int_0^{+\infty} xe^{-xy}\,dy$ does **not** converge uniformly with respect to x on the interval $(0, +\infty)$. Note that the variable of integration is y.

11. **Leibniz Rule for Improper Integrals.** Let $\int_a^\infty f(x,y)\,dy$ converge for x in a neighborhood $U(x_0)$. Suppose the partial derivative $f_x(x,y)$ exists and converges to $f_x(x_0,y)$ uniformly in y on any finite interval as $x \to x_0$. If $\int_a^\infty f_x(x,y)\,dy$ converges uniformly on $U(x_0)$, prove that:

$$\frac{d}{dx}\left(\int_a^\infty f(x,y)\,dy\right)\bigg|_{x=x_0} = \int_a^\infty f_x(x_0,y)\,dy.$$

12. **Continuity Analysis.** Let $F(\alpha) = \int_0^\infty \frac{\sin((1-\alpha^2)x)}{x}\,dx$. Determine the domain of continuity of $F(\alpha)$ and identifying any points of discontinuity. Specifically, examine $\alpha = \pm 1$.

13. **Continuity without Uniform Convergence.** Prove that the Fresnel-type integral $F(\alpha) = \int_0^\infty \frac{\sin(\alpha x^2)}{x}\,dx$ is continuous on $(0, +\infty)$, even though the convergence is not uniform on this domain.

14. **Differentiability Analysis.** Consider the function $F(x) = \int_1^\infty \frac{xe^{-yx}}{y}\,dy$.

    (a) Prove that $F(x)$ is continuous on $[0, +\infty)$.

    (b) Prove that $F(x)$ is differentiable on $(0, +\infty)$ and justify the formula:

$$F'(x) = \int_1^\infty \frac{\partial}{\partial x}\left(\frac{xe^{-yx}}{y}\right)dy.$$

15. **Parameter in the Denominator.** Let $F(\alpha) = \int_0^\infty \frac{\sin x}{x(\pi-x)^2-\alpha}\,dx$. Prove that $F(\alpha)$ is continuous for $\alpha \in (0, 2)$.

16. **Limit Calculation.** Let

$$F(y) = \int_0^\infty ye^{-x^2y^2}\cos(x(1-y))\,dx.$$

Compute the limit $\lim\limits_{y\to 1^-} F(y)$.

17. **Standard Integral Evaluations.** Use known results (Dirichlet, Gaussian) to evaluate:

    (a) $\displaystyle\int_{-\infty}^\infty \left(\frac{\sin x}{x}\right)^2 dx.$

(b) $\displaystyle\int_0^\infty \frac{\sin^3 x}{x}\, dx$.

(c) $\displaystyle\int_0^\infty x^2 e^{-\alpha x^2}\, dx$ for $\alpha > 0$.

(d) $\displaystyle\int_{-\infty}^\infty e^{-(ax^2+bx+c)}\, dx$ for $a > 0$.

18. Evaluation via Differentiation. Use differentiation under the integral sign to evaluate:

(a) $I(a) = \displaystyle\int_0^\infty e^{-ax} \frac{\sin x}{x}\, dx$ for $a \geq 0$.

(b) $I(\alpha) = \displaystyle\int_0^\infty \frac{\arctan(\alpha x)}{x(1+x^2)}\, dx$ for $\alpha \geq 0$.

(c) $f(x) = \displaystyle\int_1^\infty \frac{1}{y} x e^{-xy}\, dy$ for $x \geq 0$.

19. Evaluation via Integration. Use integration with respect to a parameter to evaluate:

(a) $I(\alpha) = \displaystyle\int_0^{+\infty} \frac{\arctan(\alpha x)}{x(1+x^2)}\, dx$ for $\alpha > 0$.

> *Remark.*
> Use the identity $\displaystyle\frac{\arctan \alpha x}{x} = \int_0^1 \frac{dy}{1+y^2x^2}$.

(b) $\displaystyle\int_0^{+\infty} \frac{\cos \beta x}{x^2+\alpha^2}\, dx$.

> *Remark.*
> Use the identity $\displaystyle\frac{1}{x^2+\alpha^2} = \lim_{\varepsilon\to 0^+} \int_\varepsilon^{+\infty} e^{-t(x^2+\alpha^2)}\, dt$.

20. Evaluation of Parameter Integrals. Compute:

$$I(\alpha) = \int_0^1 \frac{\ln(1-\alpha^2 x^2)}{\sqrt{1-x^2}}\, dx \quad \text{for } |\alpha| \leq 1.$$

21. Fourier Cosine Transform. Compute the integral:

$$I(y) = \int_0^{+\infty} e^{-x^2} \cos(2y\alpha x)\, dx \quad \text{for } -\infty < y < +\infty.$$

1

The Beta and Gamma Functions

In the analysis of integrals and differential equations, one frequently encounters solutions that cannot be expressed in terms of elementary functions. To address this, we introduce **special functions**, often defined via improper integrals dependent on parameters. This chapter focuses on two of the most significant special functions: the Beta function (B-function) and the Gamma function (Γ-function). These functions allow for the evaluation of a wide class of definite integrals and serve as the continuous analogues of binomial coefficients and factorials.

1.1 The Beta Function

The Beta function, also known as the Euler integral of the first kind, is a function of two variables defined by a definite integral.

> **Definition 1.1.** ***Beta Function.***
> For $p > 0$ and $q > 0$, the **Beta function** is defined by the integral:
>
> $$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx.$$

定義

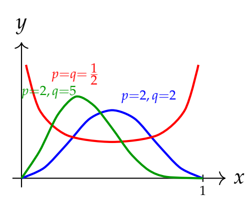

Figure 1.1: The Beta integrand $x^{p-1}(1-x)^{q-1}$ for various (p,q) values on $[0,1]$.

While defined on the unit interval, the Beta function admits several equivalent integral representations that are often more suitable for specific calculations.

> **Proposition 1.1.** ***Integral Representations.***
> For $p, q > 0$, the following representations hold:
> 1. **Trigonometric Form:**
>
> $$B(p,q) = 2\int_0^{\pi/2} \sin^{2p-1}\theta\cos^{2q-1}\theta\,d\theta.$$

2. **Infinite Interval Form:**

$$B(p,q) = \int_0^\infty \frac{y^{p-1}}{(1+y)^{p+q}}\,dy = \int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}}\,dy.$$

命題

Trigonometric Form.

Let $x = \sin^2\theta$. Then $dx = 2\sin\theta\cos\theta\,d\theta$. As x ranges from 0 to 1, θ ranges from 0 to $\pi/2$. Substituting into the definition:

$$\begin{aligned} B(p,q) &= \int_0^{\pi/2} (\sin^2\theta)^{p-1}(1-\sin^2\theta)^{q-1}\cdot 2\sin\theta\cos\theta\,d\theta \\ &= 2\int_0^{\pi/2} \sin^{2p-2}\theta\cos^{2q-2}\theta\sin\theta\cos\theta\,d\theta \\ &= 2\int_0^{\pi/2} \sin^{2p-1}\theta\cos^{2q-1}\theta\,d\theta. \end{aligned}$$

証明終

Infinite Interval Form.

Let $x = \frac{y}{1+y}$. Then $1-x = \frac{1}{1+y}$ and $dx = \frac{1}{(1+y)^2}\,dy$. The limits $0 \to 1$ transform to $0 \to \infty$.

$$\begin{aligned} B(p,q) &= \int_0^\infty \left(\frac{y}{1+y}\right)^{p-1}\left(\frac{1}{1+y}\right)^{q-1}\frac{1}{(1+y)^2}\,dy \\ &= \int_0^\infty \frac{y^{p-1}}{(1+y)^{p-1+q-1+2}}\,dy \\ &= \int_0^\infty \frac{y^{p-1}}{(1+y)^{p+q}}\,dy. \end{aligned}$$

The second equality follows from the symmetry property $B(p,q) = B(q,p)$, or by the substitution $z = 1/y$.

証明終

We now establish the fundamental algebraic and analytic properties of the Beta function.

Proposition 1.2. ***Properties of the Beta Function.***

1. **Symmetry:** $B(p,q) = B(q,p)$.
2. **Regularity:** $B(p,q)$ is continuous and possesses continuous partial derivatives of all orders on its domain.
3. **Recurrence Relations:**

$$B(p,q+1) = \frac{q}{p+q}B(p,q), \quad B(p+1,q) = \frac{p}{p+q}B(p,q).$$

4. **Integer Values:** If $m, n \in \mathbb{Z}^+$, then

$$B(m,n) = \frac{(m-1)!(n-1)!}{(m+n-1)!}.$$

命題

Symmetry.

Let $u = 1 - x$ in the definition. Then $du = -dx$, and the limits swap:

$$B(p,q) = \int_1^0 (1-u)^{p-1} u^{q-1} (-du) = \int_0^1 u^{q-1} (1-u)^{p-1}\, du = B(q,p).$$

証明終

Recurrence.

We apply integration by parts to $B(p,q+1) = \int_0^1 x^{p-1}(1-x)^q\, dx$. Let $u = (1-x)^q$ and $dv = x^{p-1}dx$. Then $du = -q(1-x)^{q-1}dx$ and $v = \frac{x^p}{p}$.

$$\begin{aligned} B(p,q+1) &= \left[\frac{x^p}{p}(1-x)^q\right]_0^1 + \frac{q}{p}\int_0^1 x^p(1-x)^{q-1}\, dx \\ &= 0 + \frac{q}{p} B(p+1,q). \end{aligned}$$

Using the identity

$$B(p,q+1) + B(p+1,q) = \int_0^1 x^{p-1}(1-x)^{q-1}[(1-x)+x]\, dx = B(p,q),$$

we solve the system:

$$B(p+1,q) = \frac{p}{q} B(p,q+1) \implies B(p,q+1) + \frac{p}{q} B(p,q+1) = B(p,q).$$

Thus $B(p,q+1)\left(1 + \frac{p}{q}\right) = B(p,q)$, which implies

$$B(p,q+1) = \frac{q}{p+q} B(p,q).$$

証明終

1.2 *The Gamma Function*

The Gamma function, or Euler integral of the second kind, extends the factorial function to real and complex arguments.

Definition 1.2. *Gamma Function.*
For $x > 0$, the **Gamma function** is defined by the improper integral:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt.$$

定義

An equivalent definition, derived from the infinite product expansion, is the Euler-Gauss formula.

Theorem 1.1. ***Euler-Gauss Formula.***
For $x > 0$:

$$\Gamma(x) = \lim_{n\to\infty} \frac{n! n^x}{x(x+1)\cdots(x+n)}.$$

定理

The Gamma function satisfies several critical analytic identities.

Proposition 1.3. ***Properties of the Gamma Function.***

Relation to Beta Function:

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad p, q > 0.$$

Recurrence Formula:

$$\Gamma(x+1) = x\Gamma(x), \quad x > 0.$$

Since $\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1$, this implies $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$.

Differentiability and Convexity: $\Gamma(x)$ is infinitely differentiable on $(0,\infty)$ with

$$\Gamma^{(n)}(x) = \int_0^\infty t^{x-1} (\ln t)^n e^{-t}\, dt.$$

The function $\Gamma(x)$ is positive and log-convex, and $\ln \Gamma(x)$ is strictly convex on $(0,\infty)$.

Legendre Duplication Formula:

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}} \Gamma(x)\Gamma\left(x + \frac{1}{2}\right).$$

Reflection Formula: For $0 < x < 1$:

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}.$$

命題

Recurrence Formula.

We apply integration by parts to $\Gamma(x+1) = \int_0^\infty t^x e^{-t}\, dt$. Let $u = t^x$ and $dv = e^{-t}dt$. Then $du = xt^{x-1}dt$ and $v = -e^{-t}$.

$$\Gamma(x+1) = \left[-t^x e^{-t}\right]_0^\infty + x \int_0^\infty t^{x-1} e^{-t}\, dt.$$

The boundary term vanishes at both limits for $x > 0$. Thus $\Gamma(x+1) = x\Gamma(x)$.

証明終

Convexity of $\ln\Gamma(x)$.

We must show that $\frac{d^2}{dx^2}\ln\Gamma(x) > 0$. Computing the derivatives:

$$(\ln\Gamma(x))' = \frac{\Gamma'(x)}{\Gamma(x)}, \quad (\ln\Gamma(x))'' = \frac{\Gamma(x)\Gamma''(x) - (\Gamma'(x))^2}{\Gamma(x)^2}.$$

We require $\Gamma(x)\Gamma''(x) - (\Gamma'(x))^2 > 0$. From the integral definitions:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt, \quad \Gamma'(x) = \int_0^\infty t^{x-1}(\ln t)e^{-t}\,dt, \quad \Gamma''(x) = \int_0^\infty t^{x-1}(\ln t)^2 e^{-t}\,dt.$$

Let $f(t) = t^{(x-1)/2}e^{-t/2}$ and $g(t) = t^{(x-1)/2}e^{-t/2}\ln t$. Then:

$$\Gamma(x) = \int f^2, \quad \Gamma''(x) = \int g^2, \quad \Gamma'(x) = \int fg.$$

By the Cauchy-Schwarz inequality, $(\int fg)^2 \leq (\int f^2)(\int g^2)$. Equality holds only if $g(t) = cf(t)$, i.e., $\ln t = c$, which is impossible on $(0,\infty)$. Thus the inequality is strict, and $\ln\Gamma(x)$ is strictly convex.

証明終

Remark (Analytic Continuation).

The domain of $\Gamma(x)$ may be extended beyond $x > 0$ using the recurrence relation rewritten as:

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}.$$

For $x \in (-1,0)$, the right-hand side is well-defined. Iterating this process allows $\Gamma(x)$ to be defined for all $x \in \mathbb{R} \setminus \{0,-1,-2,\ldots\}$. The points $0,-1,-2,\ldots$ are simple poles.

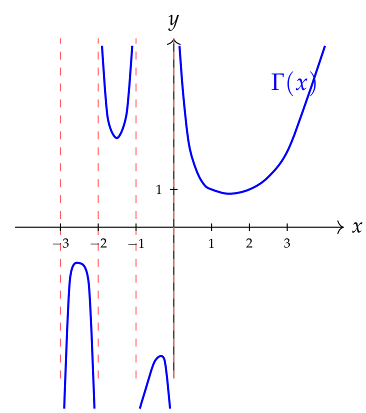

Figure 1.2: The analytically continued Gamma function with poles at $0,-1,-2,\ldots$

1.3 Applications and Further Properties

We apply the theory of Beta and Gamma functions to evaluate complex definite integrals and establish the remaining properties stated in the previous chapter.

Example 1.1. Dirichlet Integral on a Tetrahedron. Let V be the region in the first octant bounded by the coordinate planes and the plane $x+y+z=1$. Evaluate

$$I = \iiint_V x^{a-1}y^{b-1}z^{c-1}\,dx\,dy\,dz, \quad (a,b,c>0).$$

範例

Solution

We evaluate the integral iteratively. The region V is defined by $0 \leq x \leq 1$, $0 \leq y \leq 1-x$, and $0 \leq z \leq 1-x-y$.

$$I = \int_0^1 x^{a-1}\,dx \int_0^{1-x} y^{b-1}\,dy \int_0^{1-x-y} z^{c-1}\,dz.$$

The innermost integral is $\frac{1}{c}(1-x-y)^c$. Substituting this back:

$$I = \frac{1}{c}\int_0^1 x^{a-1}\,dx \int_0^{1-x} y^{b-1}(1-x-y)^c\,dy.$$

Let $y = (1-x)t$. Then $dy = (1-x)dt$, and the limits for t are 0 to 1.

$$\begin{aligned}\int_0^{1-x} y^{b-1}(1-x-y)^c\,dy &= \int_0^1 (1-x)^{b-1}t^{b-1}(1-x)^c(1-t)^c(1-x)\,dt \\ &= (1-x)^{b+c}\int_0^1 t^{b-1}(1-t)^c\,dt \\ &= (1-x)^{b+c}B(b,c+1).\end{aligned}$$

Thus:

$$I = \frac{1}{c}B(b,c+1)\int_0^1 x^{a-1}(1-x)^{b+c}\,dx = \frac{1}{c}B(b,c+1)B(a,b+c+1).$$

Expressing $B(p,q)$ in terms of Gamma functions:

$$I = \frac{1}{c}\frac{\Gamma(b)\Gamma(c+1)}{\Gamma(b+c+1)}\frac{\Gamma(a)\Gamma(b+c+1)}{\Gamma(a+b+c+1)}.$$

Using $\Gamma(z+1) = z\Gamma(z)$, we obtain:

$$I = \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b+c+1)}.$$

■

Example 1.2. Generalised Spherical Integral. Determine the values of α, β, γ for which the integral

$$I = \iiint_D \frac{dx\,dy\,dz}{1 + x^\alpha + y^\beta + z^\gamma}$$

converges, and evaluate it. Here $D = \{(x,y,z) \mid x \geq 0, y \geq 0, z \geq 0\}$.

範例

Solution

We require $\alpha, \beta, \gamma > 0$. Let $x = u^{2/\alpha}$, $y = v^{2/\beta}$, and $z = w^{2/\gamma}$. The integral transforms to:

$$I = \frac{8}{\alpha\beta\gamma} \iiint_{\Omega} \frac{u^{\frac{2}{\alpha}-1} v^{\frac{2}{\beta}-1} w^{\frac{2}{\gamma}-1}}{1+u^2+v^2+w^2}\, du\, dv\, dw,$$

where Ω is the first octant. Switching to spherical coordinates (ρ, θ, ϕ):

$$I = \frac{8}{\alpha\beta\gamma} \int_0^{\pi/2} \cos^{\frac{2}{\alpha}-1}\theta \sin^{\frac{2}{\beta}-1}\theta\, d\theta \int_0^{\pi/2} \sin^{2\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)-1}\phi \cos^{\frac{2}{\gamma}-1}\phi\, d\phi \int_0^{\infty} \frac{\rho^{2\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)-1}}{1+\rho^2}\, d\rho.$$

The angular integrals reduce to Beta functions:

$$\int_0^{\pi/2} \cos^{\frac{2}{\alpha}-1}\theta \sin^{\frac{2}{\beta}-1}\theta\, d\theta = \frac{1}{2} B\left(\frac{1}{\alpha}, \frac{1}{\beta}\right).$$

$$\int_0^{\pi/2} \sin^{2(\frac{1}{\alpha}+\frac{1}{\beta})-1}\phi \cos^{\frac{2}{\gamma}-1}\phi\, d\phi = \frac{1}{2} B\left(\frac{1}{\alpha}+\frac{1}{\beta}, \frac{1}{\gamma}\right).$$

The radial integral converges if and only if $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} < 1$. Its value is $\frac{1}{2}B(\Sigma, 1-\Sigma)$ where $\Sigma = \frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$. Combining these:

$$I = \frac{1}{\alpha\beta\gamma} \Gamma\left(\frac{1}{\alpha}\right) \Gamma\left(\frac{1}{\beta}\right) \Gamma\left(\frac{1}{\gamma}\right) \Gamma\left(1 - \left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)\right).$$

■

Example 1.3. Gaussian Integral with Parameter. Evaluate $I(t) = \int_0^\infty e^{-(x^2+t^2/x^2)}\, dx$ for $t > 0$.

範例

Solution

We solve this two ways.

Differentiation. Fix $t_0 > 0$ and consider t in a compact neighborhood of t_0. Split the domain at $x = 1$.

On $[1, \infty)$,

$$e^{-(x^2+t^2/x^2)} \le e^{-x^2/2}.$$

On $(0, 1]$,

$$e^{-(x^2+t^2/x^2)} \le e^{-c/x^2}$$

with $c = \frac{1}{2}t_0^2$, and

$$\frac{t}{x^2} e^{-(x^2+t^2/x^2)} \le \frac{2t_0}{x^2} e^{-c/x^2}$$

.

Both $e^{-x^2/2}$ and $\frac{1}{x^2}e^{-c/x^2}$ are integrable, giving an L^1 dominator independent of t in that neighborhood. Dominated convergence then permits differentiation under the integral:

$$I'(t) = \int_0^\infty e^{-(x^2+t^2/x^2)} \left(-\frac{2t}{x^2}\right) dx.$$

Substitute $x = t/y$, so $dx = -t/y^2\, dy$.

$$I'(t) = \int_\infty^0 e^{-(t^2/y^2+y^2)} \left(-\frac{2t}{(t/y)^2}\right) \left(-\frac{t}{y^2}\right) dy = -2\int_0^\infty e^{-(y^2+t^2/y^2)}\, dy = -2I(t).$$

Thus $I(t) = Ce^{-2t}$. Since $I(0) = \frac{\sqrt{\pi}}{2}$, we have $I(t) = \frac{\sqrt{\pi}}{2}e^{-2t}$.

Substitution. Let $y = t/x$. Then

$$I(t) = \int_0^\infty e^{-(t^2/y^2+y^2)} \frac{t}{y^2}\, dy.$$

Summing the two expressions:

$$2I(t) = \int_0^\infty e^{-(x^2+t^2/x^2)} \left(1+\frac{t}{x^2}\right) dx = e^{-2t}\int_0^\infty e^{-(x-t/x)^2}\, d\left(x-\frac{t}{x}\right).$$

The integral becomes $\int_{-\infty}^\infty e^{-u^2}\, du = \sqrt{\pi}$, yielding the same result.

■

Example 1.4. Integral Representation of Riemann Zeta Function. Prove that for $s > 1$, $\zeta(s) = \sum_{n=1}^\infty n^{-s}$ satisfies

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x-1}\, dx.$$

範例

Proof

We expand the term $(e^x - 1)^{-1}$ as a geometric series $e^{-x}(1 - e^{-x})^{-1} = \sum_{n=1}^\infty e^{-nx}$ for $x > 0$.

$$\int_0^\infty \frac{x^{s-1}}{e^x-1}\, dx = \int_0^\infty x^{s-1}\sum_{n=1}^\infty e^{-nx}\, dx.$$

Since the series of functions $f_n(x) = x^{s-1}e^{-nx}$ is positive, we may interchange summation and integration (by the Monotone Convergence Theorem):

$$\sum_{n=1}^\infty \int_0^\infty x^{s-1}e^{-nx}\, dx.$$

Let $u = nx$, so $dx = du/n$. Then

$$\int_0^\infty x^{s-1}e^{-nx}\, dx = \frac{1}{n^s}\int_0^\infty u^{s-1}e^{-u}\, du = \frac{\Gamma(s)}{n^s}.$$

Summing over n yields $\Gamma(s)\zeta(s)$.

■

Example 1.5. A Logarithmic Integral. Evaluate

$$\int_0^1 \frac{\ln x}{1-x}\,dx.$$

範例

Solution

Let $x = e^{-t}$. Then $dx = -e^{-t}dt$ and the range $[0,1]$ maps to $[\infty, 0]$.

$$\int_0^1 \frac{\ln x}{1-x}\,dx = \int_\infty^0 \frac{-t}{1-e^{-t}}(-e^{-t})\,dt = -\int_0^\infty \frac{t}{e^t-1}\,dt.$$

Using the representation of $\zeta(s)$ from the previous example with $s = 2$:

$$\int_0^\infty \frac{t^{2-1}}{e^t-1}\,dt = \Gamma(2)\zeta(2) = 1! \cdot \frac{\pi^2}{6}.$$

Thus the integral is $-\frac{\pi^2}{6}$.

■

Example 1.6. Volume of the Generalized Viviani Body. Let Ω be the region bounded by the sphere $x^2 + y^2 + z^2 \le a^2$ and the cylinder $x^2 + y^2 \le ay$ (where $a > 0$). Evaluate

$$I = \iiint_\Omega (\sqrt{a^2 - x^2 - y^2})^p\,dx\,dy\,dz, \quad p \ge 0.$$

範例

Solution

We employ cylindrical coordinates (r, θ, z). The cylinder equation $r^2 \le ar\sin\theta$ implies $r \le a\sin\theta$. Since $r \ge 0$, we require $\sin\theta \ge 0$, so $\theta \in [0, \pi]$. The bounds are $0 \le \theta \le \pi$, $0 \le r \le a\sin\theta$, and $|z| \le \sqrt{a^2 - r^2}$.

$$\begin{aligned} I &= \int_0^\pi d\theta \int_0^{a\sin\theta} r\,dr \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} (a^2 - r^2)^{p/2}\,dz \\ &= \int_0^\pi d\theta \int_0^{a\sin\theta} 2r(a^2 - r^2)^{\frac{p+1}{2}}\,dr. \end{aligned}$$

Let $u = a^2 - r^2$. Then $du = -2rdr$. When $r = 0$, $u = a^2$; when $r = a\sin\theta$, $u = a^2\cos^2\theta$.

$$\int_0^{a\sin\theta} 2r(a^2 - r^2)^{\frac{p+1}{2}}\,dr = \int_{a^2\cos^2\theta}^{a^2} u^{\frac{p+1}{2}}\,du = \frac{2}{p+3}a^{p+3}(1 - |\cos\theta|^{p+3}).$$

Integrating over $\theta \in [0, \pi]$:

$$I = \frac{2a^{p+3}}{p+3}\int_0^\pi (1 - |\cos\theta|^{p+3})\,d\theta = \frac{4a^{p+3}}{p+3}\left(\frac{\pi}{2} - \int_0^{\pi/2} \cos^{p+3}\theta\,d\theta\right).$$

The remaining integral is $\frac{1}{2}B(\frac{p+4}{2}, \frac{1}{2})$.

■

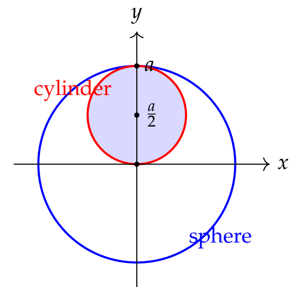

Figure 1.3: Cross-section of Viviani's body at $z = 0$. The red disk is the cylinder $x^2 + y^2 \leq ay$ (centered at $y = a/2$), internally tangent to the blue sphere at the origin.

Characterization of the Gamma Function

We now state and prove the theorem that uniquely characterizes the Gamma function based on the properties listed in the previous chapter.

Theorem 1.2. ***Bohr-Mollerup Theorem.***
If $f : (0, \infty) \to (0, \infty)$ satisfies:

1. $f(1) = 1$,
2. $f(x+1) = xf(x)$,
3. $\ln f(x)$ is convex,

then $f(x) \equiv \Gamma(x)$.

定理

Proof

Let $\phi(x) = \ln f(x)$. Then $\phi(x+1) = \phi(x) + \ln x$. Convexity implies that for $x \in (0,1)$ and $n \in \mathbb{N}$:

$$\phi(n+1) - \phi(n) \leq \frac{\phi(n+1+x) - \phi(n+1)}{x} \leq \phi(n+2) - \phi(n+1).$$

Since $\phi(n) = \ln((n-1)!)$, this reduces to:

$$\ln n \leq \frac{\phi(n+1+x) - \ln n!}{x} \leq \ln(n+1).$$

Using the recurrence, $\phi(n+1+x) = \phi(x) + \ln(x(x+1)\cdots(x+n))$. Substituting this:

$$\ln n \leq \frac{\phi(x) + \ln \prod_{k=0}^{n}(x+k) - \ln n!}{x} \leq \ln(n+1).$$

Rearranging for $\phi(x)$:

$$\ln n! - \ln \prod_{k=0}^{n}(x+k) + x \ln n \leq \phi(x) \leq \ln n! - \ln \prod_{k=0}^{n}(x+k) + x \ln(n+1).$$

Exponentiating, we find that $f(x)$ is squeezed between two sequences that both converge to the Euler-Gauss limit of $\Gamma(x)$. Thus $f(x) = \Gamma(x)$.

■

Fundamental Gamma Identities

The Bohr-Mollerup theorem provides a powerful method for proving identities: one simply verifies the three conditions for a candidate function.

Proposition 1.4. ***Legendre Duplication Formula.***
For $x > 0$:

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}}\Gamma(x)\Gamma\left(x + \frac{1}{2}\right).$$

命题

Proof of the Legendre Duplication Formula

Define

$$g(x) = \frac{2^{x-1}}{\sqrt{\pi}}\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x+1}{2}\right).$$

We verify the Bohr-Mollerup conditions for g.

1. $g(1) = \frac{1}{\sqrt{\pi}}\Gamma(1/2)\Gamma(1) = 1.$
2. $g(x+1) = \frac{2^x}{\sqrt{\pi}}\Gamma(\frac{x+1}{2})\Gamma(\frac{x}{2}+1) = \frac{2^x}{\sqrt{\pi}}\Gamma(\frac{x+1}{2})\frac{x}{2}\Gamma(\frac{x}{2}) = xg(x).$
3. Since $\ln\Gamma(y)$ is convex, $\ln g(x) = (x-1)\ln 2 - \frac{1}{2}\ln\pi + \ln\Gamma(x/2) + \ln\Gamma((x+1)/2)$ is a sum of convex functions, hence convex.

Thus $g(x) = \Gamma(x)$. Replacing x with $2x$ and rearranging yields the formula.

■

Proposition 1.5. ***Euler's Reflection Formula.***
For $0 < x < 1$:

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin\pi x}.$$

命题

Proof of the Reflection Formula

Using the relation between Beta and Gamma functions:

$$\Gamma(x)\Gamma(1-x) = B(x, 1-x) = \int_0^1 y^{x-1}(1-y)^{-x}\,dy.$$

Substituting $u = \frac{y}{1-y}$ (so $y = \frac{u}{1+u}$), the integral becomes:

$$\int_0^\infty \frac{u^{x-1}}{1+u}\,du.$$

Let $u = \tan^2\theta$, $\theta \in (0, \pi/2)$. Then $du = 2\tan\theta\sec^2\theta\,d\theta$ and $1 + u = \sec^2\theta$, so

$$\int_0^\infty \frac{u^{x-1}}{1+u}\,du = 2\int_0^{\pi/2}\tan^{2x-1}\theta\,d\theta = 2\int_0^{\pi/2}\sin^{2x-1}\theta\cos^{2(1-x)-1}\theta\,d\theta.$$

The last integral is the classical Wallis sine integral, which evaluates to $\frac{\pi}{2\sin\pi x}$; hence the original integral equals $\frac{\pi}{\sin\pi x}$, proving the reflection formula.

■

Theorem 1.3. ***Stirling's Approximation.***
As $x \to +\infty$:

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x.$$

定理

Proof
Substitute $t = x(1+u)$ in $\Gamma(x+1) = \int_0^\infty t^x e^{-t}\,dt$:

$$\Gamma(x+1) = x^{x+1}e^{-x}\int_{-1}^{\infty}((1+u)e^{-u})^x\,du.$$

Let $h(u) = u - \ln(1+u)$. Then $(1+u)e^{-u} = e^{-h(u)}$. Substitute $u = s\sqrt{2/x}$. The integral becomes:

$$\sqrt{\frac{2}{x}}\int_{-\infty}^{\infty} e^{-xh(s\sqrt{2/x})}\,ds.$$

For large x, $xh(s\sqrt{2/x}) \approx x\frac{1}{2}(s\sqrt{2/x})^2 = s^2$. The integral converges to $\sqrt{\pi}$. Thus $\Gamma(x+1) \sim x^x e^{-x}\sqrt{2x}\sqrt{\pi}$.

■

1.4 Exercises

1. **Evaluate via Gamma and Beta Functions.** Compute the values of the following integrals:
   (a) $\displaystyle\int_0^1 \frac{dx}{\sqrt{x\ln\frac{1}{x}}}$.
   (b) $\displaystyle\int_0^{+\infty} \frac{\sqrt{x}}{(1+x)^2}\,dx$.
2. **Express in Terms of Special Functions.** Represent the following integrals using the Γ or B functions:
   (a) $\displaystyle\int_0^{\pi/2} \tan^\alpha x\,dx$ for $|\alpha| < 1$.
   (b) $\displaystyle\int_{-1}^{1}(1+x)^a(1-x)^b\,dx$ for $a, b > -1$.
3. **Integer Beta Identity.** Let n be a positive integer and $p > 0$. Prove that:
$$B(p,n) = \frac{(n-1)!}{p(p+1)\cdots(p+n-1)}.$$

4. **Convexity of Log-Gamma.** Provide a detailed proof that $\ln\Gamma(x)$ is a convex function on $(0,\infty)$ by computing its second derivative and applying the Cauchy-Schwarz inequality to the integral representation.

5. **Proof of the Beta-Gamma Relation.** Prove the identity

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

by following these steps:

(a) Show that

$$\Gamma(p) = 2\int_0^{+\infty} u^{2p-1}e^{-u^2}\,du.$$

(b) Write the product $\Gamma(p)\Gamma(q)$ as a double integral over the first quadrant:

$$\Gamma(p)\Gamma(q) = 4\int_0^{\infty}\int_0^{\infty} u^{2p-1}v^{2q-1}e^{-(u^2+v^2)}\,du\,dv.$$

(c) Let $D(R) = \{(r,\theta) \mid 0 \le r \le R,\ 0 \le \theta \le \frac{\pi}{2}\}$ and $G(A) = [0,A]\times[0,A]$. Show that:

$$\iint_{D(R)} f(u,v)\,du\,dv \le \iint_{G(R)} f(u,v)\,du\,dv \le \iint_{D(R\sqrt{2})} f(u,v)\,du\,dv.$$

(d) Transform the integral over $D(R)$ to polar coordinates and show that its limit as $R\to\infty$ equals $\Gamma(p+q)B(p,q)$.

2

Further Exercises

1. **Duhamel's Principle.** Let $f(x,t)$ and $f_x(x,t)$ be continuous functions. Define

$$u(x,t) = \frac{1}{2a}\int_0^t d\tau \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)\,d\xi,$$

where $a > 0$ is the wave speed. Prove that $u(x,t)$ is the solution to the inhomogeneous wave equation:

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} + f(x,t), \quad \text{with } u(x,0) = 0,\, u_t(x,0) = 0.$$

2. **Bessel's Equation.** Let n be a positive integer. Consider the Bessel function of the first kind:

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\phi - x\sin\phi)\,d\phi.$$

Show by direct differentiation that $J_n(x)$ satisfies the differential equation:

$$x^2y'' + xy' + (x^2 - n^2)y = 0.$$

3. **Abel's Integral Equation.** Let $f(x)$ be continuously differentiable on $[0,1]$ with $f(0) = 0$. Define the fractional integral:

$$\varphi(x) = \int_0^x \frac{f(t)}{\sqrt{x-t}}\,dt, \quad x \in (0,1].$$

   (a) Prove that φ is continuously differentiable on $[0,1]$ and justify the formula:

$$\varphi'(x) = \int_0^x \frac{f'(t)}{\sqrt{x-t}}\,dt.$$

> *Remark.*
>
> The assumption $f(0) = 0$ removes the boundary term $\frac{f(x)}{\sqrt{x-t}}\big|_{t=x}$ when differentiating under the integral sign.

(b) Prove the inversion formula:

$$f(x) = \frac{1}{\pi} \int_0^x \frac{\varphi'(t)}{\sqrt{x-t}}\, dt.$$

4. **Localisation of the Dirichlet Integral.** Let $f(x)$ be monotonic on $[0, A]$ for some $A > 0$, of bounded variation on $[0, A]$, and assume $f \in L^1([A, \infty))$ (for example, $f(x) = O(1/x)$ as $x \to \infty$). Prove that:
$$\lim_{\alpha \to +\infty} \int_0^\alpha f(x) \frac{\sin \alpha x}{x}\, dx = \frac{\pi}{2} f(0^+).$$

5. **Tauberian-Type Limit.** Let $F(t) = t \int_0^{+\infty} e^{-tx} f(x)\, dx$, where $f(x)$ is bounded and integrable on every finite interval. If $\lim_{x \to +\infty} f(x) = L$, prove that:
$$\lim_{t \to 0^+} F(t) = L.$$

6. **Autocorrelation and Smoothing.** Let $f \in L^2(\mathbb{R})$ be a continuous function.

   (a) Prove that the autocorrelation $g(t) = \int_{-\infty}^{+\infty} f(t+u) f(u)\, du$ is continuous, bounded, and in fact uniformly continuous (translations are continuous in L^2 and Cauchy–Schwarz gives the bounds).

   (b) Show that for $\alpha > 0$,
   $$\sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{+\infty} e^{-\alpha t^2} g(t)\, dt = \int_{-\infty}^{+\infty} |\widehat{f}(\xi)|^2 e^{-\xi^2/(4\alpha)}\, d\xi,$$
   and deduce that
   $$\lim_{\alpha \to +\infty} \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{+\infty} e^{-\alpha t^2} g(t)\, dt = \int_{-\infty}^{+\infty} f^2(x)\, dx.$$

   > *Remark.*
   >
   > This is the Plancherel/heat-kernel identity; the limit removes the Gaussian factor in the Fourier side.

7. **Continuity of Singular Parameter Integrals.** Investigate the continuity of the following functions on the interval $(0, 1)$:

   (a) $f(\alpha) = \int_0^{+\infty} \frac{e^{-t}}{|\sin t|^\alpha}\, dt$ (the integral converges only for $0 < \alpha < 1$, since near $k\pi$ the integrand behaves like $|t - k\pi|^{-\alpha}$).

   (b) $g(\alpha) = \int_0^1 \frac{f(t)}{\sqrt{|t-\alpha|}}\, dt$, where f is bounded and integrable.

8. **Volume of a Generalized Body.** Find the volume of the solid region defined by $(x^2 + y^2)^2 + z^4 \leq y$.

> *Remark.*
>
> The body is rotationally symmetric about the z-axis; in cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$ the constraint is $r^4 + z^4 \le r\sin\theta$ with $\theta \in [0, \pi]$.

9. **Centroid of a Lamé Solid.** Find the x-coordinate of the centroid of the solid defined by:

$$\left(\frac{x}{a}\right)^{\frac{1}{n}} + \left(\frac{y}{b}\right)^{\frac{1}{n}} + \left(\frac{z}{c}\right)^{\frac{1}{n}} \le 1, \quad x, y, z \ge 0.$$

Express the result using Gamma functions.

10. **Moment of Inertia.** Compute the moment of inertia about the x-axis for the area enclosed by the astroid $x^{2/3} + y^{2/3} = R^{2/3}$.

11. **Gaussian Concentration.** Let f be continuous on $[0, 1]$. Prove:

$$\lim_{t\to+\infty} \int_0^1 te^{-t^2x^2} f(x)\,dx = \frac{\sqrt{\pi}}{2} f(0).$$

12. **Abel Continuity for Integrals.** Let $\int_0^{+\infty} f(x)\,dx$ be convergent. Prove that:

$$\lim_{y\to 0^+} \int_0^{+\infty} e^{-xy} f(x)\,dx = \int_0^{+\infty} f(x)\,dx.$$

13. **Fresnel Generalisation.** Evaluate the integral $\int_0^{+\infty} \cos(x^p)\,dx$ for $p > 1$.

> *Remark.*
>
> Use the substitution $u = x^p$ and express the result in terms of the Gamma function.

14. **Raabe's Integral.** Evaluate $\int_0^1 \ln\Gamma(x)\,dx$.

> *Remark.*
>
> Use the reflection formula $\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x)$ and symmetry.

15. **Laplace Integrals.** Evaluate the following integrals for $a, b > 0$ and $k \in \mathbb{Z}^+$:

$$I_k = \int_0^{+\infty} \frac{\cos bx}{(a^2+x^2)^k}\,dx, \quad J_k = \int_0^{+\infty} \frac{x\sin bx}{(a^2+x^2)^k}\,dx.$$

> *Remark.*
>
> Start with $k = 1$ where $I_1 = \frac{\pi}{2a}e^{-ab}$; when differentiating with respect to a, boundary terms vanish because the e^{-ab} factor (or the algebraic decay of the integrand) kills the endpoints.

16. Euler-Mascheroni Constant. Using the infinite product expansion for the Gamma function, prove that:

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right) = -\Gamma'(1).$$

17. Euler's Reflection Formula Proof.

(a) Using the roots of unity, prove the polynomial identity:

$$\sum_{k=0}^{n-1} x^k = \prod_{k=1}^{n-1}\left(x - e^{2\pi i k/n}\right).$$

(b) Deduce the trigonometric identity: $\prod_{k=1}^{n-1}\sin\frac{k\pi}{n} = \frac{n}{2^{n-1}}$.

(c) Prove the multiplication formula for the Gamma function:

$$\prod_{k=1}^{n-1}\Gamma\left(\frac{k}{n}\right) = \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}}.$$