Multivariable Calculus & Implicit Functions

Gudfit

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Review: Analytic Geometry

Euclidean geometry, historically viewed as a system of axioms and constructions, was revolutionised by Descartes through the introduction of coordinates. By labelling points with numbers, geometric properties such as parallelism and intersection became amenable to algebraic verification. This fusion of algebra and geometry, known as *analytic geometry*, forms the foundation of modern calculus.
While elementary calculus often relies on trigonometry to analyse planar figures, the study of higher-dimensional spaces requires a more robust framework. Vector notation provides the most efficient language for three-dimensional (and n-dimensional) analysis, facilitating the description of lines, planes, and curves. This chapter is a review of the geometric and algebraic structure of Euclidean space, $\mathbb{R}^n$, which serves as the domain for multivariable functions.

0.1 Vectors in Euclidean Space

We begin by defining the space $\mathbb{R}^n$ and the fundamental objects within it: points and vectors.

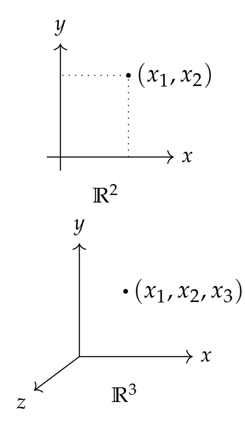

Figure 1: Points in $\mathbb{R}^2$ are ordered pairs; points in $\mathbb{R}^3$ are ordered triples.

The Structure of $\mathbb{R}^n$

The set of real numbers is denoted by $\mathbb{R}$. The Cartesian product of $\mathbb{R}$ with itself n times forms the n-dimensional Euclidean space.

> **Definition 0.1.** ***Euclidean Space.***
> For any positive integer n, the n-dimensional space $\mathbb{R}^n$ is the set of all ordered n-tuples of real numbers:
>
> $$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ copies}} = \{(x_1, x_2, \ldots, x_n) \mid x_j \in \mathbb{R} \text{ for } 1 \le j \le n\}.$$
>
> · For $n = 2$, $\mathbb{R}^2$ represents the plane.
> · For $n = 3$, $\mathbb{R}^3$ represents three-dimensional space.

定義

Two n-tuples are equal if and only if their corresponding entries are identical.

$$(v_1, \dots, v_n) = (w_1, \dots, w_n) \iff v_j = w_j \text{ for all } j.$$

Distance and Metric

The geometry of $\mathbb{R}^n$ is determined by the metric (distance function). The standard Euclidean distance is a generalisation of the Pythagorean theorem.

Definition 0.2. ***Euclidean Distance.***
Let $P = (x_1, \dots, x_n)$ and $Q = (y_1, \dots, y_n)$ be points in $\mathbb{R}^n$. The Euclidean distance between P and Q is defined as:

$$d(P, Q) = \sqrt{\sum_{j=1}^{n} (y_j - x_j)^2}.$$

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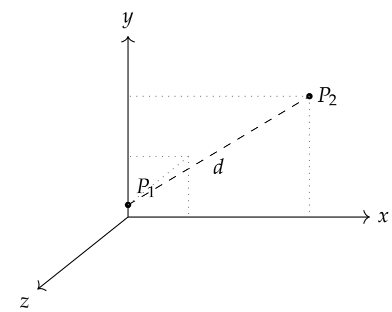

Figure 2: The distance between points in $\mathbb{R}^3$ corresponds to the length of the segment connecting them, derived via the Pythagorean theorem.

Vectors and Operations

While a point in $\mathbb{R}^n$ represents a location, a *vector* represents a magnitude and direction. We denote vectors with an arrow, e.g., $\boldsymbol{v}$. There is a natural isomorphism between points $P = (p_1, \dots, p_n)$ and position vectors $\boldsymbol{p} = \langle p_1, \dots, p_n \rangle$ originating from the origin O.
The vector from a point A to a point B is denoted by $\boldsymbol{AB}$ and is calculated by the difference of their coordinates:

$$\boldsymbol{AB} = B - A = \langle b_1 - a_1, \dots, b_n - a_n \rangle.$$

Definition 0.3. ***Vector Algebra.***
Let $\boldsymbol{x} = \langle x_1, \dots, x_n \rangle$ and $\boldsymbol{y} = \langle y_1, \dots, y_n \rangle$ be vectors in $\mathbb{R}^n$, and let $c \in \mathbb{R}$ be a scalar.

1. **Addition:** $\boldsymbol{x} + \boldsymbol{y} = \langle x_1 + y_1, \dots, x_n + y_n \rangle$.
2. **Scalar Multiplication:** $c\boldsymbol{x} = \langle cx_1, \dots, cx_n \rangle$.

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Geometrically, vector addition follows the "tip-to-tail" rule. Scalar multiplication corresponds to scaling the length of the vector (and reversing direction if $c < 0$).

Standard Basis and Components

It is often convenient to decompose vectors into components along coordinate axes. We define the standard basis vectors for $\mathbb{R}^n$ as:

$$\hat{x}_j = \langle 0, \dots, 1, \dots, 0 \rangle,$$

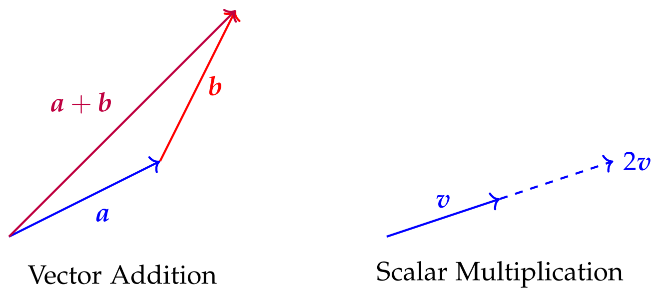

Figure 3: Geometric interpretation of vector operations.

where the 1 is in the j-th position. In $\mathbb{R}^2$ and $\mathbb{R}^3$, these are often explicitly denoted:

- In $\mathbb{R}^2$: $\hat{x} = \langle 1, 0 \rangle$, $\hat{y} = \langle 0, 1 \rangle$.
- In $\mathbb{R}^3$: $\hat{x} = \langle 1, 0, 0 \rangle$, $\hat{y} = \langle 0, 1, 0 \rangle$, $\hat{z} = \langle 0, 0, 1 \rangle$.

Any vector $\boldsymbol{v} = \langle v_1, \ldots, v_n \rangle$ can be uniquely expressed as a linear combination of basis vectors:

$$\boldsymbol{v} = \sum_{j=1}^{n} v_j \hat{x}_j.$$

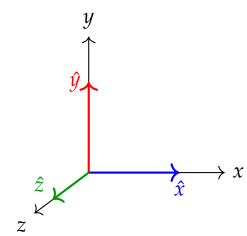

Figure 4: The standard basis vectors in $\mathbb{R}^3$ are mutually orthogonal unit vectors aligned with the coordinate axes.

We distinguish between the **scalar component** v_j (a real number) and the **vector component** $v_j \hat{x}_j$ (a vector).

> **Example 0.1.** Decomposition in $\mathbb{R}^3$. The vector $\boldsymbol{v} = \langle 3, -2, 5 \rangle$ can be written as:
>
> $$\boldsymbol{v} = 3\hat{x} - 2\hat{y} + 5\hat{z}.$$
>
> Here, the scalar component in the z-direction is 5, while the vector component is $5\hat{z} = \langle 0, 0, 5 \rangle$.
>
> 範例

0.2 *The Dot Product and Norm*

The Euclidean metric is induced by an inner product known as the dot product. This operation takes two vectors and returns a scalar, encoding information about length and angle.

> **Definition 0.4.** ***Dot Product.***
> Let $\boldsymbol{v} = \langle v_1, \ldots, v_n \rangle$ and $\boldsymbol{w} = \langle w_1, \ldots, w_n \rangle$. The dot product $\boldsymbol{v} \bullet \boldsymbol{w}$ is defined as:
>
> $$\boldsymbol{v} \bullet \boldsymbol{w} = \sum_{j=1}^{n} v_j w_j.$$
>
> 定義

Proposition 0.1. ***Properties of the Dot Product.***
For any vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$ and scalar $c \in \mathbb{R}$:

1. **Commutativity:** $\boldsymbol{u} \bullet \boldsymbol{v} = \boldsymbol{v} \bullet \boldsymbol{u}$.
2. **Distributivity:** $\boldsymbol{u} \bullet (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \bullet \boldsymbol{v} + \boldsymbol{u} \bullet \boldsymbol{w}$.
3. **Scalar Associativity:** $(c\boldsymbol{u}) \bullet \boldsymbol{v} = c(\boldsymbol{u} \bullet \boldsymbol{v})$.
4. **Positive Definiteness:** $\boldsymbol{u} \bullet \boldsymbol{u} \geq 0$, and $\boldsymbol{u} \bullet \boldsymbol{u} = 0$ if and only if $\boldsymbol{u} = \boldsymbol{0}$.

命题

Proof

These properties follow directly from the properties of real number arithmetic. For example, commutativity holds because $u_j v_j = v_j u_j$ for all j. Positive definiteness holds because $\boldsymbol{u} \bullet \boldsymbol{u} = \sum u_j^2$, which is a sum of squares.

■

Norm and Distance

The length (or norm) of a vector is defined via the dot product.

Definition 0.5. ***Norm.***
The norm of a vector $\boldsymbol{v} \in \mathbb{R}^n$, denoted $\|\boldsymbol{v}\|$, is given by:

$$\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v} \bullet \boldsymbol{v}} = \sqrt{\sum_{j=1}^{n} v_j^2}.$$

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By *definition* 0.2, $d(P, Q) = \|\boldsymbol{Q} - \boldsymbol{P}\|$.

Theorem 0.1. ***Cauchy-Schwarz Inequality.***
For any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$:

$$|\boldsymbol{x} \bullet \boldsymbol{y}| \leq \|\boldsymbol{x}\| \|\boldsymbol{y}\|.$$

Equality holds if and only if one vector is a scalar multiple of the other.

定理

Proof

If $\boldsymbol{x} = \boldsymbol{0}$ or $\boldsymbol{y} = \boldsymbol{0}$, the inequality holds trivially ($0 \leq 0$). Assume $\boldsymbol{x}, \boldsymbol{y} \neq \boldsymbol{0}$. Consider the square of the norm of a linear combination of unit vectors. Let $\hat{x} = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$ and $\hat{y} = \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|}$. Then $\|\hat{x}\| = \|\hat{y}\| = 1$. Observe that:

$$\begin{aligned} 0 \le \|\hat{x} \pm \hat{y}\|^2 &= (\hat{x} \pm \hat{y}) \bullet (\hat{x} \pm \hat{y}) \\ &= \hat{x} \bullet \hat{x} \pm 2(\hat{x} \bullet \hat{y}) + \hat{y} \bullet \hat{y} \\ &= 1 \pm 2(\hat{x} \bullet \hat{y}) + 1 \\ &= 2 \pm 2(\hat{x} \bullet \hat{y}). \end{aligned}$$

This implies $\mp 2(\hat{x} \bullet \hat{y}) \le 2$, or $|\hat{x} \bullet \hat{y}| \le 1$. Substituting the definitions of $\hat{x}$ and $\hat{y}$:

$$\left| \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} \bullet \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \right| \le 1 \implies |\boldsymbol{x} \bullet \boldsymbol{y}| \le \|\boldsymbol{x}\| \|\boldsymbol{y}\|.$$

■

The Cauchy-Schwarz inequality allows us to prove the Triangle Inequality, which asserts that the straight line is the shortest path between two points.

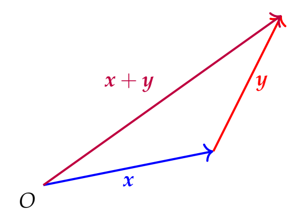

Figure 5: The triangle inequality: the length of $\boldsymbol{x} + \boldsymbol{y}$ (direct path) is at most the sum of the lengths of $\boldsymbol{x}$ and $\boldsymbol{y}$.

Theorem 0.2. ***Triangle Inequality.***
For any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$:

$$\|\boldsymbol{x} + \boldsymbol{y}\| \le \|\boldsymbol{x}\| + \|\boldsymbol{y}\|.$$

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Proof

We square the left side:

$$\begin{aligned} \|\boldsymbol{x} + \boldsymbol{y}\|^2 &= (\boldsymbol{x} + \boldsymbol{y}) \bullet (\boldsymbol{x} + \boldsymbol{y}) \\ &= \|\boldsymbol{x}\|^2 + 2(\boldsymbol{x} \bullet \boldsymbol{y}) + \|\boldsymbol{y}\|^2. \end{aligned}$$

By *theorem* 0.1, $\boldsymbol{x} \bullet \boldsymbol{y} \le |\boldsymbol{x} \bullet \boldsymbol{y}| \le \|\boldsymbol{x}\| \|\boldsymbol{y}\|$. Thus:

$$\|\boldsymbol{x} + \boldsymbol{y}\|^2 \le \|\boldsymbol{x}\|^2 + 2\|\boldsymbol{x}\| \|\boldsymbol{y}\| + \|\boldsymbol{y}\|^2 = (\|\boldsymbol{x}\| + \|\boldsymbol{y}\|)^2.$$

Taking the square root (since norms are non-negative) yields the result.

■

Orthogonality and Components

The dot product provides a precise algebraic definition for perpendicularity.

Definition 0.6. ***Orthogonality.***
Two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are **orthogonal** if and only if $\boldsymbol{v} \bullet \boldsymbol{w} = 0$. A set of vectors is **orthonormal** if every pair is orthogonal and every vector has unit length (norm 1).

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The standard basis $\{\hat{x}_1, \ldots, \hat{x}_n\}$ forms an orthonormal set, captured by the Kronecker delta relation:

$$\hat{x}_i \bullet \hat{x}_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

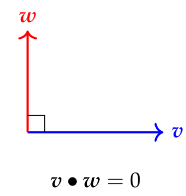

Figure 6: Orthogonal vectors meet at a right angle. The dot product of perpendicular vectors is zero.

Using orthogonality, we can extract the components of a vector using the dot product. If $v = \sum v_j \hat{x}_j$, then:

$$v \bullet \hat{x}_k = \left(\sum_{j=1}^{n} v_j \hat{x}_j \right) \bullet \hat{x}_k = \sum_{j=1}^{n} v_j (\hat{x}_j \bullet \hat{x}_k) = v_k.$$

Thus, any vector can be decomposed as:

$$v = \sum_{j=1}^{n} (v \bullet \hat{x}_j) \hat{x}_j.$$

This is known as an *orthogonal decomposition*.

Example 0.2. Components via Dot Product. Let $v = \langle 2, 5 \rangle$ in $\mathbb{R}^2$. The component in the direction of $\hat{x} = \langle 1, 0 \rangle$ is:

$$v \bullet \hat{x} = (2)(1) + (5)(0) = 2.$$

The component in the direction of $\hat{y} = \langle 0, 1 \rangle$ is:

$$v \bullet \hat{y} = (2)(0) + (5)(1) = 5.$$

This confirms $v = 2\hat{x} + 5\hat{y}$.

範例

Angles and Direction

The Cauchy-Schwarz inequality implies that for non-zero vectors x, y, the ratio $\frac{x \bullet y}{\|x\| \|y\|}$ lies in the interval $[-1, 1]$. This allows us to define the angle θ between vectors.

Definition 0.7. ***Angle Between Vectors.***
The angle $\theta \in [0, \pi]$ between two non-zero vectors x and y is defined by:

$$\cos\theta = \frac{x \bullet y}{\|x\| \|y\|}.$$

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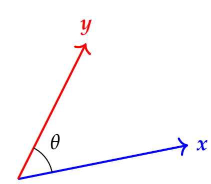

Figure 7: The angle θ between two vectors is determined by their dot product: $\cos\theta = \frac{x \bullet y}{\|x\| \|y\|}$.

This leads to the geometric form of the dot product: $x \bullet y = \|x\| \|y\| \cos\theta$.

- If $x \bullet y > 0$, the angle is acute.
- If $x \bullet y < 0$, the angle is obtuse.
- If $x \bullet y = 0$, the vectors are orthogonal ($\theta = \pi/2$).

Example 0.3. Calculating Angles. Calculate the angle between $\boldsymbol{u} = \langle 1, 1\rangle$ and $\boldsymbol{v} = \langle 0, 1\rangle$.

$$\boldsymbol{u} \bullet \boldsymbol{v} = 1(0) + 1(1) = 1.$$
$$\|\boldsymbol{u}\| = \sqrt{1^2 + 1^2} = \sqrt{2}.$$
$$\|\boldsymbol{v}\| = \sqrt{0^2 + 1^2} = 1.$$

Thus, $\cos\theta = \frac{1}{\sqrt{2}\cdot 1} = \frac{1}{\sqrt{2}}$, which implies $\theta = \frac{\pi}{4}$.

範例

0.3 *Projections and Decompositions*

The dot product allows us to decompose a vector into components relative to another vector, generalizing the concept of Cartesian components. This process is fundamental in approximation theory and physics, where one often needs the "shadow" of a force or displacement along a specific axis.

Definition 0.8. ***Vector Projection.***
Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be vectors in $\mathbb{R}^n$ with $\boldsymbol{u} \neq \boldsymbol{0}$. The **vector projection** of $\boldsymbol{v}$ onto $\boldsymbol{u}$, denoted $\text{proj}_{\boldsymbol{u}}(\boldsymbol{v})$, is the vector parallel to $\boldsymbol{u}$ that best approximates $\boldsymbol{v}$. It is defined by:

$$\text{proj}_{\boldsymbol{u}}(\boldsymbol{v}) = \left(\frac{\boldsymbol{v} \bullet \boldsymbol{u}}{\|\boldsymbol{u}\|^2}\right)\boldsymbol{u} = (\boldsymbol{v} \bullet \hat{u})\,\hat{u},$$

where $\hat{u} = \boldsymbol{u}/\|\boldsymbol{u}\|$ is the unit vector in the direction of $\boldsymbol{u}$. The scalar $\boldsymbol{v} \bullet \hat{u}$ is called the **scalar component** of $\boldsymbol{v}$ along $\boldsymbol{u}$, denoted $\text{comp}_{\boldsymbol{u}}(\boldsymbol{v})$.

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We can essentially split $\boldsymbol{v}$ into two parts: one parallel to $\boldsymbol{u}$ and one orthogonal to $\boldsymbol{u}$.

Proposition 0.2. ***Orthogonal Decomposition.***
Any vector $\boldsymbol{v}$ can be uniquely written as the sum of a vector parallel to $\boldsymbol{u}$ and a vector orthogonal to $\boldsymbol{u}$:

$$\boldsymbol{v} = \text{proj}_{\boldsymbol{u}}(\boldsymbol{v}) + \text{orth}_{\boldsymbol{u}}(\boldsymbol{v}),$$

where $\text{orth}_{\boldsymbol{u}}(\boldsymbol{v}) = \boldsymbol{v} - \text{proj}_{\boldsymbol{u}}(\boldsymbol{v})$ is orthogonal to $\boldsymbol{u}$.

命題

Proof

We simply check the orthogonality condition. Let $\boldsymbol{w} = \boldsymbol{v} - c\boldsymbol{u}$ where

$c = \frac{v \bullet u}{u \bullet u}$. Then:

$$w \bullet u = (v - cu) \bullet u = v \bullet u - c(u \bullet u) = v \bullet u - \left(\frac{v \bullet u}{u \bullet u}\right)(u \bullet u) = 0.$$

Thus, the remainder is orthogonal to u.

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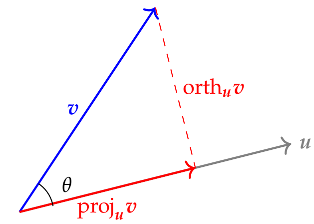

Figure 8: The projection of v onto u. The vector $\text{orth}_u v$ represents the "error" or the distance from the line spanned by u.

Example 0.4. Calculating Projections. Let $a = \langle 3, 4\rangle$ and $b = \langle 5, 12\rangle$. To find the projection of b onto a:

1. Compute the dot product: $a \bullet b = 3(5) + 4(12) = 15 + 48 = 63$.
2. Compute the squared norm of a: $\|a\|^2 = 3^2 + 4^2 = 25$.
3. Apply the formula:

$$\text{proj}_a(b) = \frac{63}{25}\langle 3, 4\rangle = \left\langle \frac{189}{25}, \frac{252}{25} \right\rangle.$$

The scalar component is $\text{comp}_a(b) = \frac{63}{\|a\|} = \frac{63}{5} = 12.6$.

範例

Example 0.5. Cube Diagonal. Consider a cube of side length α. We wish to find the angle ϕ between the main diagonal of the cube and one of its edges.

Place the cube at the origin in $\mathbb{R}^3$. The edge lies along the vector $e = \langle \alpha, 0, 0\rangle$. The main diagonal connects $(0, 0, 0)$ to (α, α, α), given by $d = \langle \alpha, \alpha, \alpha\rangle$.

$$e \bullet d = \alpha^2 + 0 + 0 = \alpha^2.$$
$$\|e\| = \alpha.$$
$$\|d\| = \sqrt{\alpha^2 + \alpha^2 + \alpha^2} = \alpha\sqrt{3}.$$

Using the angle formula:

$$\cos\phi = \frac{\alpha^2}{(\alpha)(\alpha\sqrt{3})} = \frac{1}{\sqrt{3}}.$$

Thus, $\phi = \arccos(1/\sqrt{3}) \approx 54.74°$.

範例

Physical Application: Work

In physics, work is defined as the product of the force component in the direction of displacement and the magnitude of the displacement. This is elegantly captured by the dot product.

Definition 0.9. ***Work.***
If a constant force $\boldsymbol{F}$ is applied to an object moving along a displacement vector $\boldsymbol{d}$, the work done W is:

$$W = \boldsymbol{F} \bullet \boldsymbol{d} = \|\boldsymbol{F}\|\|\boldsymbol{d}\| \cos\theta,$$

where θ is the angle between the force and the displacement.

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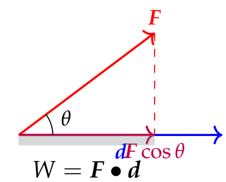

Figure 9: Work equals the component of force along the displacement times the distance: $W = \|\boldsymbol{F}\| \cos\theta \cdot \|\boldsymbol{d}\|$.

0.4 *The Cross Product in* $\mathbb{R}^3$

In $\mathbb{R}^2$, given a non-zero vector, there is a unique direction (up to sign) perpendicular to it. In $\mathbb{R}^3$, the orthogonal complement of a single vector is a plane. However, given *two* non-collinear vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, there exists a unique line perpendicular to both. The **cross product** provides a method to construct a specific vector along this line.

Definition 0.10. ***Cross Product.***
Let $\boldsymbol{u} = \langle u_1, u_2, u_3 \rangle$ and $\boldsymbol{v} = \langle v_1, v_2, v_3 \rangle$ be vectors in $\mathbb{R}^3$. The cross product $\boldsymbol{u} \times \boldsymbol{v}$ is the vector defined by:

$$\boldsymbol{u} \times \boldsymbol{v} = \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 \rangle.$$

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This formula is often memorised using the determinant of a formal matrix containing the standard basis vectors:

$$\boldsymbol{u} \times \boldsymbol{v} = \det \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} = (u_2v_3 - u_3v_2)\hat{x} - (u_1v_3 - u_3v_1)\hat{y} + (u_1v_2 - u_2v_1)\hat{z}.$$

Note

Unlike the dot product, the cross product is defined *only* in $\mathbb{R}^3$ (and technically $\mathbb{R}^7$, though that is an algebraic curiosity linked to octonions). It yields a **vector**, not a scalar.

Proposition 0.3. ***Properties of the Cross Product.***
For vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^3$ and scalar $c \in \mathbb{R}$:

1. **Anticommutativity:** $\boldsymbol{u} \times \boldsymbol{v} = -(\boldsymbol{v} \times \boldsymbol{u})$.
2. **Distributivity:** $\boldsymbol{u} \times (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} \times \boldsymbol{v}) + (\boldsymbol{u} \times \boldsymbol{w})$.
3. **Scalar Associativity:** $(c\boldsymbol{u}) \times \boldsymbol{v} = c(\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{u} \times (c\boldsymbol{v})$.
4. **Orthogonality:** $(\boldsymbol{u} \times \boldsymbol{v}) \bullet \boldsymbol{u} = 0$ and $(\boldsymbol{u} \times \boldsymbol{v}) \bullet \boldsymbol{v} = 0$.
5. **Parallelism:** $\boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{0}$ if and only if $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel (linearly

dependent).

命题

Proof

Anticommutativity follows from swapping rows in the determinant definition (which reverses the sign). Orthogonality can be verified by direct substitution. For example, the dot product $\boldsymbol{u} \bullet (\boldsymbol{u} \times \boldsymbol{v})$ corresponds to a determinant with two identical rows ($\boldsymbol{u}$ and $\boldsymbol{u}$), which is identically zero.

■

Geometry of the Cross Product

Just as the dot product relates to the cosine of the angle, the cross product relates to the sine.

Theorem 0.3. ***Lagrange's Identity.***
For any $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$:

$$\|\boldsymbol{u} \times \boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 \|\boldsymbol{v}\|^2 - (\boldsymbol{u} \bullet \boldsymbol{v})^2.$$

定理

Proof

This is a calculation using components.

$$\text{LHS} = (u_2 v_3 - u_3 v_2)^2 + (u_3 v_1 - u_1 v_3)^2 + (u_1 v_2 - u_2 v_1)^2.$$

$$\text{RHS} = (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1 v_1 + u_2 v_2 + u_3 v_3)^2.$$

Expanding both sides reveals they are identical.

■

Substituting $\boldsymbol{u} \bullet \boldsymbol{v} = \|\boldsymbol{u}\| \|\boldsymbol{v}\| \cos\theta$ into theorem 0.3:

$$\begin{aligned} \|\boldsymbol{u} \times \boldsymbol{v}\|^2 &= \|\boldsymbol{u}\|^2 \|\boldsymbol{v}\|^2 - \|\boldsymbol{u}\|^2 \|\boldsymbol{v}\|^2 \cos^2\theta \\ &= \|\boldsymbol{u}\|^2 \|\boldsymbol{v}\|^2 (1 - \cos^2\theta) \\ &= \|\boldsymbol{u}\|^2 \|\boldsymbol{v}\|^2 \sin^2\theta. \end{aligned}$$

Taking the square root (since $\theta \in [0, \pi]$, $\sin\theta \geq 0$):

$$\|\boldsymbol{u} \times \boldsymbol{v}\| = \|\boldsymbol{u}\| \|\boldsymbol{v}\| \sin\theta.$$

This magnitude has a precise geometric interpretation: it is the **area of the parallelogram** spanned by $\boldsymbol{u}$ and $\boldsymbol{v}$. The direction is determined by the **Right-Hand Rule**: if you curl the fingers of your right hand from $\boldsymbol{u}$ towards $\boldsymbol{v}$, your thumb points along $\boldsymbol{u} \times \boldsymbol{v}$.

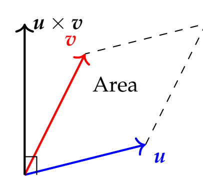


Figure 10: The cross product $\boldsymbol{u} \times \boldsymbol{v}$ is orthogonal to the plane spanned by $\boldsymbol{u}$ and $\boldsymbol{v}$. Its magnitude equals the area of the parallelogram formed by the vectors.

Example 0.6. Calculating Cross Products. Let $\boldsymbol{a} = \langle 1, 2, 3 \rangle$ and $\boldsymbol{b} = \langle 4, 5, 6 \rangle$.

$$\boldsymbol{a} \times \boldsymbol{b} = \det \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$= \hat{x}(12 - 15) - \hat{y}(6 - 12) + \hat{z}(5 - 8)$$

$$= -3\hat{x} + 6\hat{y} - 3\hat{z} = \langle -3, 6, -3 \rangle.$$

Check: $(\boldsymbol{a} \times \boldsymbol{b}) \bullet \boldsymbol{a} = -3(1) + 6(2) - 3(3) = -3 + 12 - 9 = 0.$

範例

Scalar Triple Product and Volume

Combining the dot and cross products yields the scalar triple product, which computes volumes.

Definition 0.11. ***Scalar Triple Product.***
For vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^3$, the scalar triple product is $\boldsymbol{u} \bullet (\boldsymbol{v} \times \boldsymbol{w})$. It is calculated by the determinant:

$$\boldsymbol{u} \bullet (\boldsymbol{v} \times \boldsymbol{w}) = \det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}.$$

定義

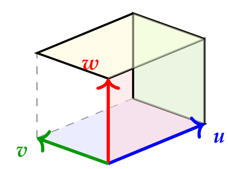


Figure 11: The scalar triple product $\boldsymbol{u} \bullet (\boldsymbol{v} \times \boldsymbol{w})$ gives the signed volume of the parallelepiped spanned by $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$.

The absolute value $|\boldsymbol{u} \bullet (\boldsymbol{v} \times \boldsymbol{w})|$ represents the **volume of the parallelepiped** determined by the three vectors.

- The triple product is **cyclic**: $\boldsymbol{u} \bullet (\boldsymbol{v} \times \boldsymbol{w}) = \boldsymbol{v} \bullet (\boldsymbol{w} \times \boldsymbol{u}) = \boldsymbol{w} \bullet (\boldsymbol{u} \times \boldsymbol{v})$.
- If the triple product is zero, the vectors are **coplanar** (the volume is zero).

Example 0.7. Volume Calculation. Find the volume of the parallelepiped defined by $\boldsymbol{u} = \langle 1, 0, 0 \rangle$, $\boldsymbol{v} = \langle 0, 1, 0 \rangle$, and $\boldsymbol{w} = \langle 1, 1, 1 \rangle$.
The volume is:

$$V = \left| \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \right| = |1(1 - 0) - 0 + 0| = 1.$$

This represents a unit cube "sheared" in the z-direction; shearing preserves volume.

範例

0.5 Lines and Planes

Analytic geometry provides two complementary ways to describe geometric objects: *parametric equations*, which label points using independent variables, and *implicit equations* (or level sets), which define objects as the solution set of algebraic constraints.

Lines in $\mathbb{R}^n$

A line is determined by a point and a direction.

> **Definition 0.12.** ***Parametric Equation of a Line.***
> The line L passing through a base point P_0 in the direction of a non-zero vector $\boldsymbol{v}$ is the set of points $\boldsymbol{r}(t)$ given by:
>
> $$\boldsymbol{r}(t) = \boldsymbol{r}_0 + t\boldsymbol{v}, \quad t \in \mathbb{R},$$
>
> where $\boldsymbol{r}_0$ is the position vector of P_0.
>
> 定義

In $\mathbb{R}^3$, if $\boldsymbol{r}_0 = \langle x_0, y_0, z_0 \rangle$ and $\boldsymbol{v} = \langle a, b, c \rangle$, the component equations are:

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$

If $a, b, c \neq 0$, we can eliminate t to form the **symmetric equations**:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

> **Example 0.8.** Line Between Two Points. Find the line passing through $P = (1, 3)$ and $Q = (5, 2)$.
> The direction vector is $\boldsymbol{v} = Q - P = \langle 4, -1 \rangle$. The parametric equation is $\boldsymbol{r}(t) = \langle 1, 3 \rangle + t\langle 4, -1 \rangle = \langle 1 + 4t, 3 - t \rangle$. Restricting $t \in [0, 1]$ parameterises the *segment* PQ.
>
> 範例

Planes in $\mathbb{R}^3$

A plane is a two-dimensional surface. It can be described parametrically by two direction vectors, or implicitly by a single normal vector.

> **Definition 0.13.** ***Parametric Equation of a Plane.***
> The plane containing a point P_0 and two non-collinear vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is given by:
>
> $$\boldsymbol{r}(s, t) = \boldsymbol{r}_0 + s\boldsymbol{u} + t\boldsymbol{v}, \quad s, t \in \mathbb{R}.$$
>
> 定義

Definition 0.14. ***Implicit (Scalar) Equation of a Plane.***
A plane is the set of all points $\boldsymbol{r}$ such that the vector $\boldsymbol{r} - \boldsymbol{r}_0$ is orthogonal to a non-zero **normal vector** $\boldsymbol{n}$.

$$\boldsymbol{n} \bullet (\boldsymbol{r} - \boldsymbol{r}_0) = 0.$$

If $\boldsymbol{n} = \langle a, b, c \rangle$ and $\boldsymbol{r}_0 = \langle x_0, y_0, z_0 \rangle$, this expands to:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \implies ax + by + cz = d,$$

where $d = ax_0 + by_0 + cz_0$.

定義

Example 0.9. From Points to Plane. Find the equation of the plane passing through $P = (2, 1, 0)$, $Q = (3, 4, 1)$, and $R = (4, 5, 6)$.
First, find two tangent vectors:

$$\boldsymbol{u} = \boldsymbol{PQ} = \langle 1, 3, 1 \rangle, \quad \boldsymbol{v} = \boldsymbol{PR} = \langle 2, 4, 6 \rangle.$$

The normal vector is their cross product:

$$\boldsymbol{n} = \boldsymbol{u} \times \boldsymbol{v} = \det \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 3 & 1 \\ 2 & 4 & 6 \end{bmatrix} = \langle 14, -4, -2 \rangle.$$

Using P as the base point:

$$14(x - 2) - 4(y - 1) - 2(z - 0) = 0.$$

範例

Intersection of Planes

Two non-parallel planes intersect in a line. The direction of this line is orthogonal to the normals of both planes. Thus, if the normals are $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$, the line of intersection is parallel to $\boldsymbol{v} = \boldsymbol{n}_1 \times \boldsymbol{n}_2$.

Example 0.10. Line of Intersection. Find the line of intersection of $x + y + z = 10$ and $2x + 3y + z = 20$.
Normals are $\boldsymbol{n}_1 = \langle 1, 1, 1 \rangle$ and $\boldsymbol{n}_2 = \langle 2, 3, 1 \rangle$. Direction vector:

$$\boldsymbol{v} = \boldsymbol{n}_1 \times \boldsymbol{n}_2 = \langle 1(1) - 1(3), 1(2) - 1(1), 1(3) - 1(2) \rangle = \langle -2, 1, 1 \rangle.$$

To find a point on the line, set $z = 0$ and solve the system:

$$x + y = 10, \quad 2x + 3y = 20.$$

From the first, $y = 10 - x$. Substitute into the second: $2x + 3(10 - x) = 20 \implies -x + 30 = 20 \implies x = 10$. Then $y = 0$. So $P_0 = (10, 0, 0)$ is on the line. The line is $\boldsymbol{r}(t) = \langle 10, 0, 0 \rangle + t \langle -2, 1, 1 \rangle$.

範例

Distance Problems

The projection formulas from the previous section provide elegant solutions for distance problems.

Proposition 0.4. ***Distance from a Point to a Plane.***
The distance D from a point P to the plane $ax + by + cz + d = 0$ is given by the projection of any vector from the plane to P onto the normal direction $\boldsymbol{n} = \langle a, b, c \rangle$. Let Q be any point on the plane. Then:

$$D = \|\text{proj}_{\boldsymbol{n}}(\boldsymbol{PQ})\| = \frac{|\boldsymbol{PQ} \bullet \boldsymbol{n}|}{\|\boldsymbol{n}\|}.$$

Explicitly, if $P = (x_1, y_1, z_1)$:

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

命題

Example 0.11. Closest Point on a Plane. Find the point on the plane $x - y + 10z = 10$ closest to $P(1, 2, 3)$.
The normal is $\boldsymbol{n} = \langle 1, -1, 10 \rangle$. Pick a point on the plane, say $Q(0, 0, 1)$. The vector from the plane to P is $\boldsymbol{v} = \boldsymbol{QP} = \langle 1, 2, 2 \rangle$. The vector offset from the plane is the projection of $\boldsymbol{v}$ onto $\boldsymbol{n}$:

$$\boldsymbol{w} = \text{proj}_{\boldsymbol{n}}\boldsymbol{v} = \frac{\langle 1, 2, 2 \rangle \bullet \langle 1, -1, 10 \rangle}{1^2 + (-1)^2 + 10^2}\boldsymbol{n} = \frac{19}{102}\langle 1, -1, 10 \rangle.$$

The closest point R is $P - \boldsymbol{w}$:

$$R = (1, 2, 3) - \left(\frac{19}{102}, \frac{-19}{102}, \frac{190}{102}\right) = \left(\frac{83}{102}, \frac{223}{102}, \frac{116}{102}\right).$$

範例

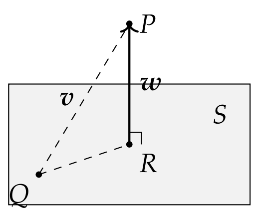


Figure 12: The closest point R on plane S to point P is found by subtracting the normal projection $\boldsymbol{w}$ from P.

1

Curves and Vector-Valued Functions

While elementary calculus typically models curves as graphs of functions $y = f(x)$, this perspective is insufficient for studying geometry in higher dimensions or objects with complex topology (such as self-intersecting loops).

There are three primary frameworks for describing a curve:

1. **Implicit (Level Set):** The set of points satisfying an equation $F(x_1, \ldots, x_n) = c$.
2. **Graph:** The set of points $(x, f(x))$ for a function f.
3. **Parametric:** The image of a map from an interval $I \subseteq \mathbb{R}$ into $\mathbb{R}^n$.

For the purposes of multivariable calculus and differential geometry, the parametric viewpoint is the most robust. It allows us to describe motion, direction, and velocity intrinsically, without reliance on a specific coordinate grid.

1.1 *Parametrised Differentiable Curves*

We define a curve not merely as a set of points, but as a mapping. This distinction allows us to discuss properties such as velocity and acceleration.

> **Definition 1.1.** *Parametrised Curve.*
> A **parametrised differentiable curve** is a smooth (infinitely differentiable) map
> $$\gamma : I \to \mathbb{R}^n,$$
> where $I \subseteq \mathbb{R}$ is an open interval. The variable $t \in I$ is called the **parameter**. The image set $\gamma(I) \subset \mathbb{R}^n$ is called the **trace** of the curve.
> 定義

If $\gamma(t) = (x_1(t), \ldots, x_n(t))$, differentiability implies that each component function $x_j(t)$ possesses derivatives of all orders. We denote the first derivative with respect to t by $\gamma'(t)$ or $\dot{\gamma}(t)$.

Note

We distinguish strictly between the map γ (the path) and its trace (the geometric locus). Different paths may have the same trace but distinct dynamic properties.

Velocity and Tangent Vectors

The derivative of a vector-valued function is defined component-wise.

Definition 1.2. ***Velocity Vector.***
Let $\gamma : I \to \mathbb{R}^n$ be a smooth curve. The **velocity vector** (or tangent vector) at t is defined as:

$$\gamma'(t) = \lim_{h \to 0} \frac{\gamma(t+h) - \gamma(t)}{h} = \langle x_1'(t), \ldots, x_n'(t) \rangle.$$

定義

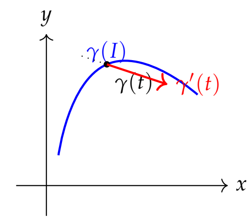


Figure 1.1: The velocity vector $\gamma'(t)$ is tangent to the path.

Geometrically, the vector $\gamma'(t)$ is tangent to the trace of the curve at the point $\gamma(t)$, pointing in the direction of increasing parameter values.

Example 1.1. The Helix. Consider the map $\gamma : \mathbb{R} \to \mathbb{R}^3$ defined by:

$$\gamma(t) = (a \cos t, a \sin t, bt),$$

where $a, b > 0$. The trace of this curve lies on the cylinder $x^2 + y^2 = a^2$. The velocity vector is:

$$\gamma'(t) = \langle -a \sin t, a \cos t, b \rangle.$$

Since $\|\gamma'(t)\|^2 = (-a \sin t)^2 + (a \cos t)^2 + b^2 = a^2 + b^2$, the speed $\|\gamma'(t)\|$ is constant.

範例

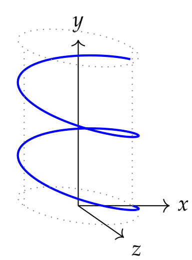


Figure 1.2: The helix $\gamma(t) = (a \cos t, a \sin t, bt)$ winds around the z-axis on the cylinder $x^2 + y^2 = a^2$.

Regularity and Singularities

A curve is said to be **regular** at t if $\gamma'(t) \neq \mathbf{0}$. Points where $\gamma'(t) = \mathbf{0}$ are called **singularities** of the parametrisation. At a singular point, the curve may have a sharp corner or a cusp, although it is also possible for the trace to be smooth while the velocity vanishes due to the parametrisation "stopping" momentarily.

Example 1.2. The Cusp. Let $\gamma(t) = (t^3, t^2)$ for $t \in \mathbb{R}$. The derivative is $\gamma'(t) = (3t^2, 2t)$. At $t = 0$, $\gamma'(0) = (0, 0)$. The trace exhibits a sharp cusp at the origin, satisfying $y = x^{2/3}$. This sharp turn prevents the existence of a unique tangent line at $(0, 0)$.

範例

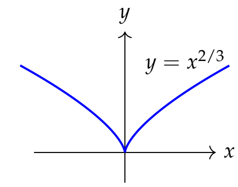


Figure 1.3: A cusp at the origin generated by $\gamma(t) = (t^3, t^2)$.

1.2 Implicit vs. Parametric Representations

While parametric equations are ideal for calculus, geometric loci are often defined implicitly. We compare these viewpoints in $\mathbb{R}^2$.

Graphs as Curves

The graph of a function $f : I \to \mathbb{R}$ is the set $\{(x, f(x)) \mid x \in I\}$. This can be trivially parametrised by choosing x as the parameter:

$$\gamma(t) = (t, f(t)), \quad t \in I.$$

This parametrisation is always regular, as $\gamma'(t) = (1, f'(t)) \neq \mathbf{0}$.

Level Curves

A level curve is the set of solutions to $F(x, y) = c$. Unlike graphs, level curves can be closed loops (like circles) or self-intersecting figures. Finding a parametrisation for a level curve often requires exploiting trigonometric or hyperbolic identities.

Example 1.3. The Hyperbola. Consider the level curve $x^2 - y^2 = 1$.

1. **Right Branch ($x > 0$):** We use the identity $\cosh^2 t - \sinh^2 t = 1$. A natural parametrisation is:

$$\gamma(t) = (\cosh t, \sinh t), \quad t \in \mathbb{R}.$$

Since $\cosh t \geq 1$, this covers only the region $x \geq 1$.

2. **Left Branch ($x < 0$):** We reflect the x-coordinate:

$$\beta(t) = (-\cosh t, \sinh t), \quad t \in \mathbb{R}.$$

Unlike the circle, the hyperbola is disconnected, requiring separate parametrisations for each component (or a discontinuous domain).

範例

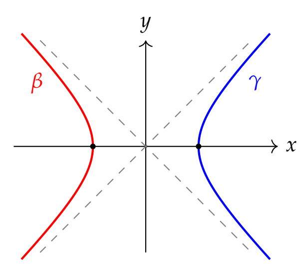


Figure 1.4: The hyperbola $x^2 - y^2 = 1$ has two branches: right branch $\gamma(t) = (\cosh t, \sinh t)$ and left branch $\beta(t) = (-\cosh t, \sinh t)$.

Non-Uniqueness of Parametrisation

A single geometric set can be covered by infinitely many different paths. These paths may differ in speed or orientation.

Example 1.4. Parametrisations of the Unit Circle. The level set $x^2 + y^2 = 1$ can be parametrised by:

1. $\gamma_1(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi)$. Velocity $\|\gamma_1'\| = 1$. Orienta-

tion is counter-clockwise (CCW).

2. $\gamma_2(t) = (\cos 2t, \sin 2t)$ for $t \in [0, \pi)$. Velocity $\|\gamma_2'\| = 2$. The particle moves twice as fast.
3. $\gamma_3(t) = (\cos(-t), \sin(-t)) = (\cos t, -\sin t)$. Orientation is clockwise (CW).

While the traces are identical, the maps $\gamma_1, \gamma_2, \gamma_3$ are distinct mathematical objects.

範例

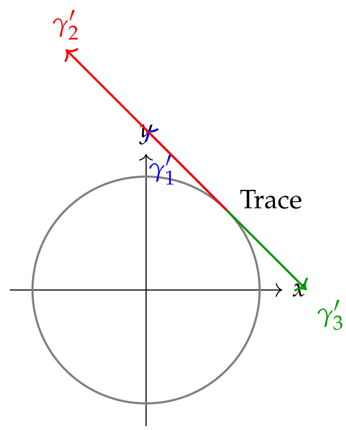


Figure 1.5: Vectors tangent to the same point on the circle, generated by different parametrisations. γ_1 is unit speed CCW, γ_2 is double speed CCW, γ_3 is unit speed CW.

1.3 *Calculus Properties*

The operations of calculus extend linearly to vector-valued functions.

Proposition 1.1. ***Differentiation Rules.***
Let $\boldsymbol{u}, \boldsymbol{v} : I \to \mathbb{R}^n$ be differentiable curves and let $\lambda : I \to \mathbb{R}$ be a differentiable scalar function.

1. **Sum Rule:** $\frac{d}{dt}(\boldsymbol{u}(t) + \boldsymbol{v}(t)) = \boldsymbol{u}'(t) + \boldsymbol{v}'(t)$.
2. **Scalar Product Rule:** $\frac{d}{dt}(\lambda(t)\boldsymbol{u}(t)) = \lambda'(t)\boldsymbol{u}(t) + \lambda(t)\boldsymbol{u}'(t)$.
3. **Dot Product Rule:** $\frac{d}{dt}(\boldsymbol{u}(t) \bullet \boldsymbol{v}(t)) = \boldsymbol{u}'(t) \bullet \boldsymbol{v}(t) + \boldsymbol{u}(t) \bullet \boldsymbol{v}'(t)$.

命題

Proof

These follow immediately from the component-wise definitions and the standard product rule for real-valued functions. For the dot product:

$$\frac{d}{dt}\sum_{i=1}^{n} u_i(t)v_i(t) = \sum_{i=1}^{n} \left(u_i'(t)v_i(t) + u_i(t)v_i'(t)\right)$$
$$= \sum_{i=1}^{n} u_i'(t)v_i(t) + \sum_{i=1}^{n} u_i(t)v_i'(t) = \boldsymbol{u}' \bullet \boldsymbol{v} + \boldsymbol{u} \bullet \boldsymbol{v}'.$$

■

Proposition 1.2. ***Orthogonality of Constant Norm Curves.***
If a curve $\gamma(t)$ has constant norm $\|\gamma(t)\| = c$ for all t, then the velocity vector $\gamma'(t)$ is orthogonal to the position vector $\gamma(t)$ for all t.

$$\|\gamma(t)\| = c \implies \gamma(t) \bullet \gamma'(t) = 0.$$

命題

Proof
Since $\|\gamma(t)\|^2 = \gamma(t) \bullet \gamma(t) = c^2$, we differentiate both sides with respect to t:

$$\frac{d}{dt}(\gamma(t) \bullet \gamma(t)) = \frac{d}{dt}(c^2) = 0.$$

By the dot product rule:

$$\gamma'(t) \bullet \gamma(t) + \gamma(t) \bullet \gamma'(t) = 0 \implies 2\gamma(t) \bullet \gamma'(t) = 0.$$

Thus, $\gamma(t) \bullet \gamma'(t) = 0$.

■

This proposition explains why the velocity vector of a circle (where $\|\gamma(t)\| = R$) is always tangent to the circle and perpendicular to the radius.

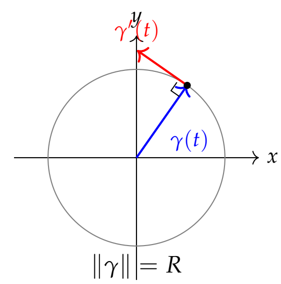


Figure 1.6: For a curve of constant norm (e.g., a circle), the velocity $\gamma'(t)$ is always perpendicular to the position vector $\gamma(t)$.

Example 1.5. Parametric Intersection. Find the points where the curve $\gamma(t) = (t, t^2, t^3)$ intersects the plane $6x - 3y + 2z = 5$.
Substitute the components of $\gamma(t)$ into the plane equation:

$$6(t) - 3(t^2) + 2(t^3) = 5 \implies 2t^3 - 3t^2 + 6t - 5 = 0.$$

To solve for t, we check for integer roots. If $t = 1$:

$$2(1)^3 - 3(1)^2 + 6(1) - 5 = 2 - 3 + 6 - 5 = 0.$$

Thus, $t = 1$ is a solution. The intersection point is $\gamma(1) = (1, 1, 1)$.

範例

1.4 *Curves in Three Dimensions*

In $\mathbb{R}^2$, a curve can often be described by a single equation $F(x, y) = c$. In $\mathbb{R}^3$, a single scalar equation $F(x, y, z) = c$ typically describes a *surface* (a two-dimensional object), not a curve. To describe a curve implicitly in three dimensions, we require the intersection of two surfaces.

Intersection of Surfaces

Geometrically, restricting a point to satisfy two independent equations $F(x, y, z) = c_1$ and $G(x, y, z) = c_2$ removes two degrees of freedom, leaving a one-dimensional locus (a curve). Converting this implicit description into a parametric one $\boldsymbol{r}(t)$ is a fundamental task in analytic geometry.

Example 1.6. Intersection of a Sphere and a Plane. Consider the curve defined by the intersection of the sphere $x^2 + y^2 + z^2 = 4$ and the plane $x = \sqrt{3}$.
Substituting $x = \sqrt{3}$ into the sphere's equation:

$$(\sqrt{3})^2 + y^2 + z^2 = 4 \implies 3 + y^2 + z^2 = 4 \implies y^2 + z^2 = 1.$$

This describes a unit circle in the plane $x = \sqrt{3}$. A natural parametrisation is:

$$\boldsymbol{r}(t) = \langle \sqrt{3}, \cos t, \sin t \rangle, \quad t \in [0, 2\pi).$$

範例

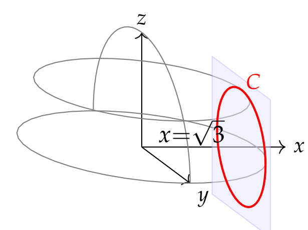


Figure 1.7: The intersection of sphere $x^2 + y^2 + z^2 = 4$ and plane $x = \sqrt{3}$ is a circle C of radius 1.

Example 1.7. Intersection of Quadric Surfaces. Find a parametrisation for the curve of intersection between the hyperbolic paraboloid $z = x^2 - y^2$ and the plane $z = 2x$.
Equating the expressions for z:

$$x^2 - y^2 = 2x \implies x^2 - 2x - y^2 = 0.$$

Completing the square for x:

$$(x-1)^2 - 1 - y^2 = 0 \implies (x-1)^2 - y^2 = 1.$$

This describes a hyperbola in the xy-plane (specifically, the projection of the curve). We parametrise this hyperbola using hyperbolic functions. For the right branch ($x > 1$):

$$x(t) - 1 = \cosh t \implies x(t) = 1 + \cosh t, \quad y(t) = \sinh t.$$

Substituting back into $z = 2x$ to find the third component:

$$z(t) = 2(1 + \cosh t).$$

Thus, a parametrisation is $\boldsymbol{r}(t) = \langle 1+\cosh t, \sinh t, 2+2\cosh t\rangle$ for $t \in \mathbb{R}$.

範例

1.5 Surfaces in Euclidean Space

A surface is a two-dimensional subset of $\mathbb{R}^3$. Just as with curves, we distinguish between three primary modes of description.

Surfaces as Graphs

The graph of a function $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is the set

$$S = \{(x, y, f(x, y)) \mid (x, y) \in D\}.$$

This is the most restrictive definition, as it cannot represent surfaces that fail the "vertical line test" (such as a sphere). However, any graph can be trivially parametrised by letting $x = u$ and $y = v$.

Parametrised Surfaces

A general surface is the image of a map from a region $D \subseteq \mathbb{R}^2$ into $\mathbb{R}^3$.

Definition 1.3. ***Parametrised Surface.***
A **parametrised surface** (or patch) is a smooth map $\boldsymbol{r} : D \to \mathbb{R}^3$ defined by

$$\boldsymbol{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle, \quad (u, v) \in D.$$

The variables u and v are the **parameters**, and D is the **parameter domain**.

定義

Example 1.8. The Sphere. The sphere of radius R is naturally parametrised by spherical coordinates (fixing $\rho = R$). Let $u = \theta$ (azimuthal angle) and $v = \phi$ (polar angle).

$$\boldsymbol{r}(u, v) = \langle R\cos u \sin v, R\sin u \sin v, R\cos v\rangle,$$

with domain $D = [0, 2\pi] \times [0, \pi]$.

範例

Example 1.9. The Cylinder. A cylinder of radius R aligned with the z-axis can be parametrised by:

$$\boldsymbol{r}(u, v) = \langle R\cos u, R\sin u, v\rangle,$$

where $u \in [0, 2\pi]$ is the angle and $v \in \mathbb{R}$ represents the height z.

範例

Level Surfaces

A level surface is the set of solutions to an implicit equation $F(x, y, z) = k$. This is the three-dimensional analogue of a level curve.

Proposition 1.3. ***Graphs as Level Surfaces.***
Any graph $z = f(x, y)$ can be represented as the level surface $F(x, y, z) = 0$ by defining $F(x, y, z) = f(x, y) - z$.

命題

Implicit equations are particularly useful for describing **quadric surfaces**, which are the zero sets of quadratic polynomials in three variables.

Classification of Standard Quadric Surfaces

The general quadratic equation in $\mathbb{R}^3$ can often be reduced to one of the following standard forms by translation and rotation of coordinates.

1. **Ellipsoid:** $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.
2. **Elliptic Paraboloid:** $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$.
3. **Hyperbolic Paraboloid:** $\frac{z}{c} = \frac{y^2}{b^2} - \frac{x^2}{a^2}$.
4. **Cone:** $\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$.
5. **Hyperboloid of One Sheet:** $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$.
6. **Hyperboloid of Two Sheets:** $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$.

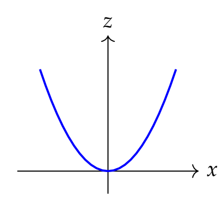
To identify a surface from its equation, one employs the **method of traces** (or slices): fixing one variable (e.g., $z = k$) and analysing the resulting curve in the remaining variables.

Example 1.10. Analysing a Cone. Consider the surface $z^2 = x^2 + y^2$.

- **Trace $z = k$:** $x^2 + y^2 = k^2$. For $k \neq 0$, these are circles of radius $|k|$.
- **Trace $x = 0$:** $z^2 = y^2 \implies z = \pm y$. These are intersecting lines.
- **Trace $y = 0$:** $z^2 = x^2 \implies z = \pm x$. These are intersecting lines.

This structure characterises a circular cone with vertex at the origin.

範例



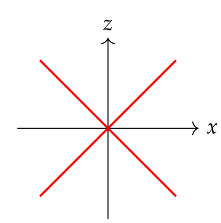


Figure 1.8: Traces in the xz-plane ($y = 0$) help identify the surface type.

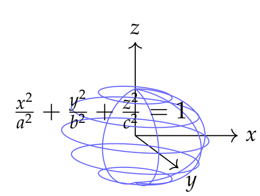


Figure 1.9: An ellipsoid with semi-axes a, b, c along the coordinate directions.

1.6 *Intersections and Parametric Solutions*

We now apply our parametric tools to solve geometric intersection problems.

Intersection of a Parametrised Curve and a Surface

To find where a path $\boldsymbol{r}(t)$ intersects a surface defined implicitly by $F(x, y, z) = 0$, we substitute the components of $\boldsymbol{r}(t)$ into the surface equation. This yields a scalar equation in t.

Example 1.11. Path and Paraboloid. Let $\boldsymbol{r}(t) = \langle t, 0, 2t - t^2 \rangle$. Find the intersection with the paraboloid $z = x^2 + y^2$.
Substituting $x = t, y = 0, z = 2t - t^2$ into the surface equation:

$$2t - t^2 = (t)^2 + (0)^2 \implies 2t - 2t^2 = 0 \implies 2t(1 - t) = 0.$$

The solutions are $t = 0$ and $t = 1$.

- At $t = 0$, intersection is at $\boldsymbol{r}(0) = (0, 0, 0)$.
- At $t = 1$, intersection is at $\boldsymbol{r}(1) = (1, 0, 1)$.

範例

Constructing Parametrisations for Intersection Curves

When two surfaces intersect, we can often parametrise the resulting curve by using one of the variables as a parameter or by exploiting trigonometric identities.

Example 1.12. Viviani-Style Curve. Find a parametrisation for the intersection of the cylinder $x^2 + y^2 = 4$ and the surface $z = xy$.

1. **Step 1:** Parametrise the "base" projection. The cylinder constraint suggests polar coordinates:

$$x = 2\cos t, \quad y = 2\sin t, \quad t \in [0, 2\pi).$$

2. **Step 2:** Lift to the second surface. Substitute x and y into $z = xy$:

$$z = (2\cos t)(2\sin t) = 4\cos t \sin t = 2\sin(2t).$$

The intersection curve is $\boldsymbol{r}(t) = \langle 2\cos t, 2\sin t, 2\sin(2t) \rangle$.

範例

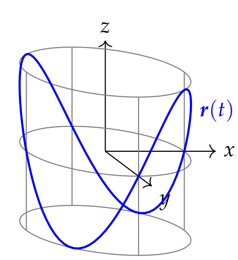


Figure 1.10: The curve $\boldsymbol{r}(t) = (2\cos t, 2\sin t, 2\sin 2t)$ lies on the cylinder $x^2 + y^2 = 4$ and the saddle surface $z = xy$.

Collisions vs. Intersections

When dealing with moving particles, we must distinguish between:

1. **Intersection:** The geometric paths cross (the traces share a point). The particles may occupy this point at different times.
2. **Collision:** The particles occupy the same point at the *same* time.

Example 1.13. The Cat and Mouse Problem. Let the mouse's path be $\boldsymbol{r}_m(t) = \langle t, t^2, t^3 \rangle$ and the cat's path be $\boldsymbol{r}_c(s) = \langle 1 + 2s, 1 + 6s, 1 + 14s \rangle$. Note that we use different parameters (t and s) to test for geometric intersection.
We solve $\boldsymbol{r}_m(t) = \boldsymbol{r}_c(s)$:

$$t = 1 + 2s \tag{1.1}$$
$$t^2 = 1 + 6s \tag{1.2}$$
$$t^3 = 1 + 14s \tag{1.3}$$

From (1.1), $2s = t - 1$. Substitute into (1.2):

$$t^2 = 1 + 3(2s) = 1 + 3(t-1) = 3t - 2 \implies t^2 - 3t + 2 = 0.$$

Factorising: $(t-1)(t-2) = 0$. The potential solution times for the mouse are $t = 1$ and $t = 2$.

- If $t = 1$: $2s = 1 - 1 = 0 \implies s = 0$. Checking (1.3): $1^3 = 1 + 14(0)$, which holds.
- If $t = 2$: $2s = 2 - 1 = 1 \implies s = 1/2$. Checking (1.3): $2^3 = 8$ and $1 + 14(0.5) = 8$. This also holds.

Conclusion:

- The paths intersect at point $A = (1, 1, 1)$ (mouse at $t = 1$, cat at $s = 0$).
- The paths intersect at point $B = (2, 4, 8)$ (mouse at $t = 2$, cat at $s = 0.5$).

Do they collide? This requires $t = s$. At A, $t = 1 \neq s = 0$. No collision. At B, $t = 2 \neq s = 0.5$. No collision. The geometric paths cross twice, but the animals are never at the same place at the same time.

範例

1.7 *Curvilinear Coordinates*

While Cartesian coordinates (x, y, z) are universal, they are often ill-suited for problems possessing radial or rotational symmetry. To simplify the description of such systems, we introduce *curvilinear coordinates*. Unlike the Cartesian basis $\{\hat{x}, \hat{y}, \hat{z}\}$, which is constant throughout space, curvilinear systems employ a local basis (or frame) that varies from point to point.
A coordinate system is **right-handed** if its associated local unit vec-

tors $\{\hat{u}_1, \hat{u}_2, \hat{u}_3\}$ satisfy the cyclic cross-product relations:

$$\hat{u}_1 \times \hat{u}_2 = \hat{u}_3, \quad \hat{u}_2 \times \hat{u}_3 = \hat{u}_1, \quad \hat{u}_3 \times \hat{u}_1 = \hat{u}_2.$$

Equivalently, the determinant of the matrix formed by these vectors is $+1$.

Polar Coordinates

In the Euclidean plane $\mathbb{R}^2$, the polar coordinate system (r, θ) is defined by the transformation:

$$x = r\cos\theta, \quad y = r\sin\theta,$$

where $r \geq 0$ represents the distance from the origin, and $\theta \in [0, 2\pi)$ represents the angle from the positive x-axis.
The inverse transformation for $x \neq 0$ is given by:

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right) + k\pi,$$

where k is chosen based on the quadrant of (x, y).
The local basis vectors $\{\hat{r}, \hat{\theta}\}$ are obtained by normalising the partial derivatives of the position vector $\boldsymbol{r} = x\hat{x} + y\hat{y}$.

$$\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$$
$$\hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$$

This frame is orthonormal ($\hat{r} \bullet \hat{\theta} = 0$) and rotates as the point moves around the origin.

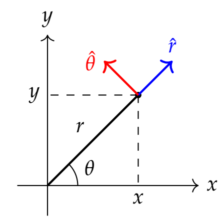


Figure 1.11: Polar coordinates and the local frame $\{\hat{r}, \hat{\theta}\}$.

Example 1.14. Polar Curves.

1. **Circle:** The equation $x^2 + y^2 = R^2$ simplifies to $r = R$.
2. **Line:** The line $y = mx + b$ becomes $r\sin\theta = mr\cos\theta + b$, or

$$r = \frac{b}{\sin\theta - m\cos\theta}.$$

範例

Cylindrical Coordinates

Cylindrical coordinates (r, θ, z) extend polar coordinates to $\mathbb{R}^3$ by retaining the Cartesian z-coordinate.

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

Here $r \geq 0$, $\theta \in [0, 2\pi)$, and $z \in \mathbb{R}$.
The basis vectors are:

$$\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}, \quad \hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}, \quad \hat{z} = \hat{z}.$$

These form a right-handed orthonormal system: $\hat{r} \times \hat{\theta} = \hat{z}$.

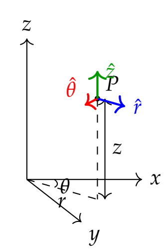


Figure 1.12: Cylindrical coordinates (r, θ, z): radial distance r in the xy-plane, azimuth θ from the x-axis, and height z.

Example 1.15. Surfaces in Cylindrical Coordinates.

1. The cylinder $x^2 + y^2 = R^2$ is simply $r = R$.
2. The paraboloid $z = 4 - x^2 - y^2$ becomes $z = 4 - r^2$.
3. The half-plane $y = x$ $(x > 0)$ is defined by $\theta = \pi/4$.

範例

This system is ideal for problems with axial symmetry, such as calculating the magnetic field of a current-carrying wire ($\boldsymbol{B} \propto \frac{1}{r}\hat{\theta}$).

Spherical Coordinates

Spherical coordinates (ρ, ϕ, θ) exploit the full rotational symmetry of $\mathbb{R}^3$. We define:

- ρ: Distance from the origin ($\rho \geq 0$).
- ϕ: Polar angle from the positive z-axis ($\phi \in [0, \pi]$).
- θ: Azimuthal angle in the xy-plane ($\theta \in [0, 2\pi)$).

The coordinate transformation is:

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi.$$

Note

Caution: In physics literature, the roles of θ and ϕ are often reversed (with θ as the polar angle). We adhere to the ISO 80000-2 standard common in mathematics.

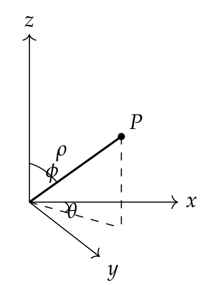


Figure 1.13: Spherical coordinates (ρ, ϕ, θ): radius ρ, polar angle ϕ from the z-axis, azimuth θ from the x-axis.

The inverse relations are:

$$\rho = \sqrt{x^2 + y^2 + z^2}, \quad \phi = \arccos\left(\frac{z}{\rho}\right), \quad \theta = \arctan\left(\frac{y}{x}\right).$$

The local orthonormal frame $\{\hat{\rho}, \hat{\phi}, \hat{\theta}\}$ is given by:

$$\begin{aligned} \hat{\rho} &= \sin\phi\cos\theta\,\hat{x} + \sin\phi\sin\theta\,\hat{y} + \cos\phi\,\hat{z} \\ \hat{\phi} &= \cos\phi\cos\theta\,\hat{x} + \cos\phi\sin\theta\,\hat{y} - \sin\phi\,\hat{z} \\ \hat{\theta} &= -\sin\theta\,\hat{x} + \cos\theta\,\hat{y} \end{aligned}$$

This system is right-handed: $\hat{\rho} \times \hat{\phi} = \hat{\theta}$. The vector $\hat{\rho}$ points radially outward, $\hat{\phi}$ points tangent to a meridian (downward), and $\hat{\theta}$ points tangent to a parallel (eastward).

Example 1.16. Spherical Surfaces.

1. **Sphere:** $\rho = R$.
2. **Cone:** The equation $z^2 = \frac{1}{3}(x^2 + y^2)$ represents a cone. In spherical coordinates:

$$(\rho\cos\phi)^2 = \frac{1}{3}(\rho^2\sin^2\phi) \implies \cos^2\phi = \frac{1}{3}\sin^2\phi \implies \tan\phi = \pm\sqrt{3}.$$

Thus, $\phi = \pi/3$ or $\phi = 2\pi/3$, describing a cone with a fixed opening angle.

3. **Vertical Plane:** The plane $y = x$ corresponds to $\theta = \pi/4$ (and $\theta = 5\pi/4$).

範例

Example 1.17. Converting a Plane to Spherical Coordinates. Convert the equation of the plane $z = 2$ into spherical coordinates.

$$\rho \cos \phi = 2 \implies \rho = 2 \sec \phi.$$

In spherical coordinates, the plane $z = 2$ is described by

$$\rho = 2 \sec \phi, \quad 0 \le \phi < \frac{\pi}{2}, \quad 0 \le \theta < 2\pi.$$

Note that $\phi < \pi/2$ corresponds to $z > 0$, which matches the fact that the plane $z = 2$ lies entirely above the origin. For each fixed direction (ϕ, θ) with $\phi < \pi/2$, there is exactly one point of the plane along that ray, at distance $\rho = 2/\cos \phi$.

範例

1.8 Exercises

1. **Circular Orientation.** Find a smooth parametrisation $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$ such that the trace of $\mathbf{r}$ is the unit circle $x^2 + y^2 = 1$, the curve is traversed clockwise, and $\mathbf{r}(0) = (0, 1)$.
2. **Orthogonality at Extrema.** Let $\mathbf{r} : I \to \mathbb{R}^n$ be a differentiable curve whose trace does not contain the origin. Suppose $t_0 \in I$ is a value such that the distance $\|\mathbf{r}(t)\|$ achieves a local minimum at t_0. Prove that if $\mathbf{r}'(t_0) \neq \mathbf{0}$, then the position vector $\mathbf{r}(t_0)$ is orthogonal to the velocity vector $\mathbf{r}'(t_0)$.
3. **Vanishing Acceleration.** A parametrised curve $\mathbf{r}(t)$ satisfies $\mathbf{r}''(t) = \mathbf{0}$ for all $t \in I$. Characterise the geometric nature of the trace of $\mathbf{r}$.
4. **Planar Confinement.** Let $\mathbf{r} : I \to \mathbb{R}^3$ be a differentiable curve and let $\mathbf{v} \in \mathbb{R}^3$ be a fixed non-zero vector. Suppose that $\mathbf{r}'(t) \perp \mathbf{v}$ for all $t \in I$ and that the initial point $\mathbf{r}(0)$ is orthogonal to $\mathbf{v}$. Prove that $\mathbf{r}(t)$ is orthogonal to $\mathbf{v}$ for all $t \in I$. Geometrically, what does this imply about the curve?
5. **Characterisation of Spherical Curves.** Let $\mathbf{r} : I \to \mathbb{R}^n$ be a differentiable curve with non-vanishing velocity. Prove that $\|\mathbf{r}(t)\|$ is constant (i.e., the curve lies on a sphere centred at the origin) if and only if $\mathbf{r}(t) \perp \mathbf{r}'(t)$ for all $t \in I$.
6. **Cartesian Products.** The Cartesian product of three sets is defined

as $A \times B \times C = \{(a,b,c) \mid a \in A, b \in B, c \in C\}$. If $(x,y,z) \in [0,1] \times \mathbb{R} \times \mathbb{Z}$, describe the possible values for the coordinates x, y, and z.

7. **Centroids.** Let $P = (1,2)$, $Q = (-1,-2)$, and $R = (0,3)$ be points in $\mathbb{R}^2$. Determine the centroid M of the triangle PQR.

The centroid of a finite set of points $\{P_1, \ldots, P_k\}$ is given by the vector average $\frac{1}{k}\sum P_i$.

8. **Triangle Geometry.** Let $A = (1,2,3)$, $B = (1,1,-2)$, and $C = (4,4,4)$ be points in $\mathbb{R}^3$.
   (a) Compute the displacement vectors $\mathbf{u} = \overrightarrow{AB}$, $\mathbf{v} = \overrightarrow{BC}$, and $\mathbf{w} = \overrightarrow{CA}$.
   (b) Verify that $\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$.
   (c) Calculate the internal angles of the triangle ABC using the dot product.
   (d) Sum these angles to verify the Euclidean plane triangle sum theorem.
9. **Orthogonality Check.** Determine whether the vectors $\mathbf{v} = \langle 1,0,4 \rangle$ and $\mathbf{w} = \langle 0,2,0 \rangle$ are orthogonal, collinear, or neither. Justify your answer.
10. **Projection and Decomposition.** Let $\mathbf{v} = \langle 1,1,1 \rangle$ and $\mathbf{w} = 2\hat{y} - \hat{z}$. Calculate $\text{proj}_{\mathbf{w}}(\mathbf{v})$. Hence, express $\mathbf{v}$ as the sum of a vector parallel to $\mathbf{w}$ and a vector orthogonal to $\mathbf{w}$.
11. **Scalar Triple Product.** Let $\mathbf{a} = \hat{x} + \hat{y}$, $\mathbf{b} = \hat{z}$, and $\mathbf{c} = \hat{y}$.
    (a) Calculate $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})$.
    (b) Calculate $\mathbf{b} \bullet (\mathbf{a} \times \mathbf{c})$.
    (c) Interpret these results in terms of the volume of the parallelepiped spanned by $\mathbf{a}, \mathbf{b}, \mathbf{c}$.
12. **Plane Construction.** Find the scalar equation of the plane containing the line $\mathbf{r}(t) = \langle 1+t, 2-t, 3 \rangle$ and the vector $\mathbf{w} = \langle 1,2,3 \rangle$ (assuming $\mathbf{w}$ is not parallel to the line).
13. **Work and Displacement.** A constant force $\mathbf{F} = 100\hat{x}$ is applied to a mass, displacing it from $P(1,2,3)$ to $Q(4,4,4)$. Calculate the work done by $\mathbf{F}$.
14. **Planar Intersections.** Find the direction vector of the line of intersection of the planes given by $x + y + z = 3$ and $2x - 3y - 4z = 7$.
15. **Parametric to Cartesian.** Interpret the geometric object described by the parametric equations $x = u + v$, $y = u - v$, and $z = 1 + u$. Convert this description into a Cartesian equation $F(x,y,z) = 0$.
16. **Minimising Distance.** Let S be the plane containing the points $A(1,0,2)$, $B(3,4,1)$, and $C(0,0,1)$. Find the unique point on S closest to the origin.
17. **Triangular Parametrisation.** Using the points A, B, C from the previous exercise, provide a parametrisation for the triangular region ΔABC. Explicitly state the domain of the parameters.
18. **Non-Cancellation of Dot Products.** Suppose $\mathbf{a} \bullet \mathbf{b} = \mathbf{a} \bullet \mathbf{c}$ for a non-zero vector $\mathbf{a}$. Does it necessarily follow that $\mathbf{b} = \mathbf{c}$? Provide a

proof or a counter-example.

19. **Uniqueness via Dot Products.** Suppose $\mathbf{a} \bullet \mathbf{b} = \mathbf{a} \bullet \mathbf{c}$ holds for *all* vectors $\mathbf{a} \in \mathbb{R}^n$. Prove that $\mathbf{b} = \mathbf{c}$.
20. **Uniqueness via Cross Products.** Suppose $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$ holds for *all* vectors $\mathbf{a} \in \mathbb{R}^3$. Prove that $\mathbf{b} = \mathbf{c}$.
21. **Rhombus Diagonals.** Use vector algebra to prove that the diagonals of a parallelogram are orthogonal if and only if the parallelogram is a rhombus (i.e., all sides have equal length).
22. **The Varignon Parallelogram.** Let A, B, C, D be vertices of a quadrilateral in $\mathbb{R}^3$ (not necessarily coplanar). Let M_1, M_2, M_3, M_4 be the midpoints of the consecutive sides. Prove that $M_1M_2M_3M_4$ forms a parallelogram.
23. **Thales' Theorem.** Use the dot product to prove that a triangle inscribed in a semicircle is a right-angled triangle.

Set the origin at the centre of the semicircle.

24. **Skew Lines Distance.** Calculate the minimum distance between the skew lines $\mathbf{r}_1(t) = \langle 1+t, 2-3t, 3+4t \rangle$ and $\mathbf{r}_2(s) = \langle 1+2s, 2+s, 3s \rangle$.
25. **Difference of Squares.** Simplify the expression $[\mathbf{a} - \mathbf{b}] \bullet [\mathbf{a} + \mathbf{b}]$. Under what condition are the sum and difference of two vectors orthogonal?
26. **Curve Identification.** Identify the geometric nature of the traces of the following curves:
    (a) $\mathbf{r}(t) = \langle \cos t, \sin t, \cos t \rangle$ for $t \in [0, 2\pi]$.
    (b) $\mathbf{r}(t) = \langle \cosh t, 4 \sinh t \rangle$ for $t \in \mathbb{R}$.
    (c) $\mathbf{r}(t) = \langle e^t, 2e^t, 3e^t \rangle$ for $t \in [0, \ln 3]$.
27. **Surface Identification.** Identify the following surfaces given by their Cartesian equations:
    (a) $z = x^2 + y^2$
    (b) $x^2 + y^2 - 3z^2 = 1$
    (c) $(x-1)^2 + (y+2)^2 + (z-3)^2 = 1$
    (d) $x^2 + 2y^2 = 1$
28. **Surface Parametrisation.** Provide explicit parametrisations (including domains) for the surfaces listed in the previous exercise.
29. **Rectangular Patch.** Parametrise the subset of the plane $x + 3y - z = 10$ defined by the bounds $1 \leq x \leq 3$ and $2 \leq y \leq 4$.
30. **Inequalities in Curvilinear Coordinates.** Describe the regions defined by the following inequalities, converting to cylindrical or spherical coordinates where appropriate:
    (a) $1 \leq x^2 + y^2 + z^2 \leq 3$
    (b) $0 \leq x^2 + y^2 \leq 4$
31. **Coordinate Conversion.** Let $P = (\sqrt{3}, 1, 2)$ be a point in Cartesian coordinates. Find the coordinates of P in:
    (a) Cylindrical coordinates (r, θ, z).
    (b) Spherical coordinates (ρ, ϕ, θ).

32. **Viviani's Curve.** Find and parametrise the curve of intersection of the cylinder $x^2 + y^2 = 4$ and the hyperbolic paraboloid $z = x^2 - y^2$.
33. **Elliptic Intersection.** Find and parametrise the curve of intersection of the plane $x + y + z = 10$ and the paraboloid $z = x^2 + y^2$.
34. **Coordinate Surfaces.** Convert the equation $4 = \rho \sin \phi$ into:
    (a) Cylindrical coordinates.
    (b) Cartesian coordinates.
35. **Spherical Curves.** Find the intersection of the cone $\phi = \pi/3$ and the plane $z = 4$. Provide a parametrisation for this curve.
36. **Vertical Intersection.** Find the intersection of the half-plane $\theta = \pi/4$ and the sphere $\rho = 4$. Parametrise the resulting curve.
37. **Polar Forms.** Derive the polar coordinate equations $r(\theta)$ for:
    (a) The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
    (b) The line $y = 1 - 2x$.
38. **Vector inversion.** Let $\mathbf{v}, \mathbf{b}$ be non-zero vectors and c be a scalar. Prove that if a vector $\mathbf{x}$ satisfies both $\mathbf{v} \bullet \mathbf{x} = c$ and $\mathbf{v} \times \mathbf{x} = \mathbf{b}$, then $\mathbf{x}$ is uniquely determined. Find an explicit formula for $\mathbf{x}$ in terms of $\mathbf{v}, \mathbf{b}, c$.

For a solution to exist, we must have $\mathbf{v} \perp \mathbf{b}$. Use the identity $\mathbf{v} \times (\mathbf{v} \times \mathbf{x})$.

39. **The Jacobi Identity.** For any vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$, prove the Jacobi Identity:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}.$$

Use the vector triple product expansion $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \bullet \mathbf{c})\mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\mathbf{c}$.

40. **Stereographic Projection.** Consider the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. Let $N = (0, 0, 1)$ be the north pole. For any point $P \in \mathbb{S}^2 \setminus \{N\}$, let $\pi(P)$ be the intersection of the line passing through N and P with the xy-plane ($z = 0$).
    (a) Derive the formula for $\pi(x, y, z)$ in terms of Cartesian coordinates.
    (b) Show that this map is a bijection between $\mathbb{S}^2 \setminus \{N\}$ and $\mathbb{R}^2$.
41. **Kepler's Law of Areas.** Let $\mathbf{r}(t)$ be the position of a particle moving in $\mathbb{R}^3$ under a central force field (where the force $\mathbf{F}$ is parallel to $\mathbf{r}$).
    (a) Show that the quantity $\mathbf{L} = \mathbf{r} \times \mathbf{r}'$ (angular momentum per unit mass) is constant.
    (b) Deduce that the particle moves in a fixed plane.
    (c) Show that the rate at which the position vector sweeps out area is constant (Kepler's Second Law).
42. **The Gram-Schmidt Process.** Let $\mathbf{a}_1 = \langle 1, 1, 0 \rangle$, $\mathbf{a}_2 = \langle 1, 0, 1 \rangle$, and $\mathbf{a}_3 = \langle 0, 1, 1 \rangle$.
    (a) Construct a unit vector $\mathbf{u}_1$ parallel to $\mathbf{a}_1$.
    (b) Construct a unit vector $\mathbf{u}_2$ orthogonal to $\mathbf{u}_1$ that lies in the plane spanned by $\mathbf{a}_1$ and $\mathbf{a}_2$.
    (c) Construct a unit vector $\mathbf{u}_3$ orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$.

2

Calculus and Geometry of Curves

In this chapter, we extend the calculus of real-valued functions to vector-valued functions mapping an interval $I \subseteq \mathbb{R}$ to $\mathbb{R}^n$. While a *curve* represents a geometric set of points in space, a *path* is a specific parametrisation of that curve. We establish the machinery for differentiation and integration of paths, which allows us to analyse dynamic properties such as velocity and acceleration, as well as geometric properties like tangency and curvature.

2.1 Differentiation and Integration

We define the calculus of vector-valued functions component-wise. This pragmatic approach relies on the standard limit definitions from single-variable calculus.

Definition 2.1. ***Derivative and Integral.***
Let $\boldsymbol{F} : I \to \mathbb{R}^n$ be a vector-valued function with components $\boldsymbol{F}(t) = \langle F_1(t), \ldots, F_n(t) \rangle$.

1. **Differentiation:** If the components F_j are differentiable at t, the derivative of $\boldsymbol{F}$ is:
$$\frac{d\boldsymbol{F}}{dt} = \boldsymbol{F}'(t) = \left\langle \frac{dF_1}{dt}, \ldots, \frac{dF_n}{dt} \right\rangle.$$
2. **Integration:** If the components F_j are integrable on $[a, b]$, the definite integral is:
$$\int_a^b \boldsymbol{F}(t)\, dt = \left\langle \int_a^b F_1(t)\, dt, \ldots, \int_a^b F_n(t)\, dt \right\rangle.$$
3. **Antiderivative:** The indefinite integral is defined as:
$$\int \boldsymbol{F}(t)\, dt = \boldsymbol{G}(t) + \boldsymbol{c} \iff \boldsymbol{G}'(t) = \boldsymbol{F}(t),$$
where $\boldsymbol{c} \in \mathbb{R}^n$ is a constant vector.

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Geometrically, if $\boldsymbol{r}(t)$ represents the position of a particle at time t, then $\boldsymbol{r}'(t)$ represents the velocity vector, tangent to the trajectory.

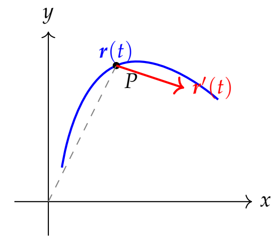


Figure 2.1: The derivative $\boldsymbol{r}'(t)$ is tangent to the curve at P.

Example 2.1. Calculus of a Space Curve. Let $\boldsymbol{F}(t) = \langle \cos^2 t, (1+3t)^{-1}, t^2 \sin(t/3) \rangle$.

1. **Derivative:** Differentiating component-wise using the chain and product rules:

$$\begin{aligned} \frac{d\boldsymbol{F}}{dt} &= \left\langle \frac{d}{dt}(\cos^2 t), \frac{d}{dt}((1+3t)^{-1}), \frac{d}{dt}\left(t^2 \sin\frac{t}{3}\right) \right\rangle \\ &= \left\langle -2\cos t \sin t, \frac{-3}{(1+3t)^2}, 2t\sin\frac{t}{3} + \frac{t^2}{3}\cos\frac{t}{3} \right\rangle. \end{aligned}$$

2. **Integral:** Using the identity $\cos^2 t = \frac{1}{2}(1+\cos 2t)$, we integrate:

$$\int \boldsymbol{F}(t)\,dt = \left\langle \frac{t}{2} + \frac{\sin 2t}{4}, \frac{1}{3}\ln|1+3t|, -3t^2\cos\frac{t}{3} + 18t\sin\frac{t}{3} + 54\cos\frac{t}{3} \right\rangle + \boldsymbol{c}.$$

The third component is obtained via integration by parts (twice).

範例

The Fundamental Theorem of Calculus extends naturally to this setting.

Theorem 2.1. ***Fundamental Theorems of Calculus for Curves.***
Let $\boldsymbol{F}$ and $\boldsymbol{G}$ be vector-valued functions.

1. If $\boldsymbol{F}$ is continuous on $[a,b]$, then $\frac{d}{dt}\int_a^t \boldsymbol{F}(\tau)\,d\tau = \boldsymbol{F}(t)$.
2. If $\boldsymbol{G}$ is differentiable on $[a,b]$ and $\boldsymbol{G}'$ is integrable, then

$$\int_a^b \boldsymbol{G}'(t)\,dt = \boldsymbol{G}(b) - \boldsymbol{G}(a).$$

定理

Proof

We prove the second statement; the first follows similarly. Let $\boldsymbol{G}(t) = \sum_{j=1}^n G_j(t)\hat{x}_j$. By linearity of the integral and the standard scalar Fundamental Theorem of Calculus:

$$\begin{aligned} \int_a^b \boldsymbol{G}'(t)\,dt &= \sum_{j=1}^n \left(\int_a^b G_j'(t)\,dt\right)\hat{x}_j \\ &= \sum_{j=1}^n (G_j(b) - G_j(a))\hat{x}_j \\ &= \sum_{j=1}^n G_j(b)\hat{x}_j - \sum_{j=1}^n G_j(a)\hat{x}_j = \boldsymbol{G}(b) - \boldsymbol{G}(a). \end{aligned}$$

■

2.2 *Algebraic Properties and Rules*

The linear structure of differentiation preserves vector addition and scalar multiplication. More interestingly, the product rule adapts to the various vector products defined in analytic geometry.

Theorem 2.2. ***Vector Product Rules.***
Let $\boldsymbol{u}, \boldsymbol{v} : I \to \mathbb{R}^n$ be differentiable vector functions, and let $f : I \to \mathbb{R}$ be a differentiable scalar function.

1. **Scalar Product Rule:** $\frac{d}{dt}(f\boldsymbol{u}) = \frac{df}{dt}\boldsymbol{u} + f\frac{d\boldsymbol{u}}{dt}$.
2. **Dot Product Rule:** $\frac{d}{dt}(\boldsymbol{u} \bullet \boldsymbol{v}) = \boldsymbol{u}' \bullet \boldsymbol{v} + \boldsymbol{u} \bullet \boldsymbol{v}'$.
3. **Cross Product Rule (for $n = 3$):** $\frac{d}{dt}(\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{u}' \times \boldsymbol{v} + \boldsymbol{u} \times \boldsymbol{v}'$.

定理

Note

The order of vectors in the cross product rule must be preserved due to anti-commutativity.

Proof

We demonstrate the proof for the cross product (3) using summation notation and the Levi-Civita symbol ϵ_{ijk}. Let $\boldsymbol{u} \times \boldsymbol{v} = \sum_{k=1}^{3}(\sum_{i,j=1}^{3} \epsilon_{ijk} u_i v_j)\hat{x}_k$.

$$\begin{aligned}
\frac{d}{dt}(\boldsymbol{u} \times \boldsymbol{v}) &= \sum_{k=1}^{3} \frac{d}{dt}\left(\sum_{i,j=1}^{3} \epsilon_{ijk} u_i v_j\right)\hat{x}_k \\
&= \sum_{k=1}^{3}\sum_{i,j=1}^{3} \epsilon_{ijk}\left(\frac{du_i}{dt}v_j + u_i\frac{dv_j}{dt}\right)\hat{x}_k \\
&= \sum_{k=1}^{3}\left(\sum_{i,j=1}^{3} \epsilon_{ijk} u_i' v_j\right)\hat{x}_k + \sum_{k=1}^{3}\left(\sum_{i,j=1}^{3} \epsilon_{ijk} u_i v_j'\right)\hat{x}_k \\
&= (\boldsymbol{u}' \times \boldsymbol{v}) + (\boldsymbol{u} \times \boldsymbol{v}').
\end{aligned}$$

The proofs for scalar and dot products follow equivalent logic using the standard product rule on components.

■

Theorem 2.3. ***Chain Rule for Curves.***
Let $\boldsymbol{u} : J \to \mathbb{R}^n$ be differentiable and $g : I \to J$ be a differentiable scalar function. Then the composition $\boldsymbol{u} \circ g$ is differentiable, and:

$$\frac{d}{dt}(\boldsymbol{u}(g(t))) = \boldsymbol{u}'(g(t))\, g'(t).$$

定理

Note

In more general contexts, this derivative can be viewed as the matrix product of the Jacobian $D\boldsymbol{u}(g(t))$ (a column vector) and the scalar $g'(t)$.

Proof

Let $\boldsymbol{u}(s) = \langle u_1(s), \ldots, u_n(s) \rangle$. By the scalar chain rule applied to each component, $u_j'(g(t))$ represents the derivative of u_j with respect to its argument s, evaluated at $s = g(t)$:

$$\frac{d}{dt} u_j(g(t)) = \left. \frac{du_j}{ds} \right|_{s=g(t)} \frac{dg}{dt} = u_j'(g(t)) g'(t).$$

Assembling the vector yields the result.

■

2.3 *Geometric Applications*

The calculus of curves provides rigorous tools for analysing the geometry of paths in $\mathbb{R}^n$.

Tangent Lines

Given a parametrisation $\boldsymbol{r}(t)$, the vector $\boldsymbol{r}'(t_0)$ defines the direction of the tangent line at t_0, provided $\boldsymbol{r}'(t_0) \neq \boldsymbol{0}$.

Definition 2.2. ***Parametric Tangent Line.***
The tangent line to a regular curve $\boldsymbol{r}(t)$ at $t = t_0$ is given by:

$$\boldsymbol{l}(s) = \boldsymbol{r}(t_0) + s\boldsymbol{r}'(t_0), \quad s \in \mathbb{R}.$$

定義

Example 2.2. Tangent to a Helix. Consider the curve $\boldsymbol{r}(t) = \langle \cos t, \sin t, t \rangle$. At $t = \pi/2$:

1. **Point:** $\boldsymbol{r}(\pi/2) = \langle 0, 1, \pi/2 \rangle$.
2. **Tangent Vector:** $\boldsymbol{r}'(t) = \langle -\sin t, \cos t, 1 \rangle \implies \boldsymbol{r}'(\pi/2) = \langle -1, 0, 1 \rangle$.
3. **Line:** $\boldsymbol{l}(s) = \langle 0, 1, \pi/2 \rangle + s\langle -1, 0, 1 \rangle = \langle -s, 1, \pi/2 + s \rangle$.

範例

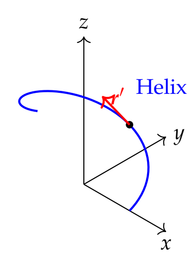


Figure 2.2: A helix $\langle \cos t, \sin t, t \rangle$ with tangent at $t = \pi/2$. The tangent has components in both the xy-plane and z-direction.

Intersection of Curves

Two curves intersect if they share a common point in space. The *angle of intersection* is defined as the angle between their tangent vectors at that point.

Example 2.3. Angle of Intersection. Let $\boldsymbol{r}_1(t) = \langle t, t^2, t^3 \rangle$ and $\boldsymbol{r}_2(u) = \langle \sin u, \sin 2u, u \rangle$.
Intersection occurs at the origin, where $t = 0$ and $u = 0$. Compute the tangent vectors:

$$\boldsymbol{r}_1'(t) = \langle 1, 2t, 3t^2 \rangle \implies \boldsymbol{v}_1 = \boldsymbol{r}_1'(0) = \langle 1, 0, 0 \rangle.$$

$$\boldsymbol{r}_2'(u) = \langle \cos u, 2\cos 2u, 1 \rangle \implies \boldsymbol{v}_2 = \boldsymbol{r}_2'(0) = \langle 1, 2, 1 \rangle.$$

The angle θ satisfies:

$$\cos\theta = \frac{\boldsymbol{v}_1 \bullet \boldsymbol{v}_2}{\|\boldsymbol{v}_1\|\|\boldsymbol{v}_2\|} = \frac{1(1)+0+0}{(1)(\sqrt{1^2+2^2+1^2})} = \frac{1}{\sqrt{6}}.$$

Therefore, $\theta = \arccos(1/\sqrt{6}) \approx 66°$.

範例

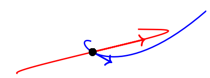


Figure 2.3: Two curves $\langle t, t^2, t^3 \rangle$ (blue) and $\langle \sin u, \sin 2u, u \rangle$ (red) intersecting at the origin. The angle between their tangent vectors is $\theta = \arccos(1/\sqrt{6})$.

Orthogonality of Constant Norm Curves

A fundamental result in the geometry of curves (and mechanics) relates the magnitude of a vector function to its direction of change. By *proposition 1.2*, if $\|\boldsymbol{r}(t)\|$ is constant then $\boldsymbol{r}(t) \perp \boldsymbol{r}'(t)$ for all t.
This result explains why the velocity of a particle moving on a sphere is always tangent to the sphere's surface.

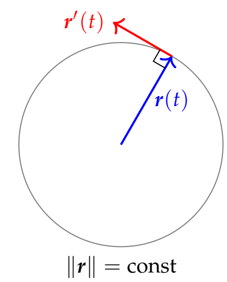


Figure 2.4: For a curve on a sphere (constant norm), the velocity vector is always tangent to the surface, hence perpendicular to the radius vector.

Example 2.4. Angular Momentum and Torque. In mechanics, the angular momentum $\boldsymbol{L}$ of a particle with position $\boldsymbol{r}$ and momentum $\boldsymbol{p} = m\boldsymbol{v}$ is defined as $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = m(\boldsymbol{r} \times \boldsymbol{v})$.
The torque $\boldsymbol{\tau}$ is the rate of change of angular momentum. Using *Theorem 2.2*:

$$\frac{d\boldsymbol{L}}{dt} = \frac{d}{dt}(m\boldsymbol{r} \times \boldsymbol{v}) = m(\boldsymbol{r}' \times \boldsymbol{v} + \boldsymbol{r} \times \boldsymbol{v}').$$

Since $\boldsymbol{r}' = \boldsymbol{v}$ and $\boldsymbol{v} \times \boldsymbol{v} = \boldsymbol{0}$, the first term vanishes. Thus:

$$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{r} \times (m\boldsymbol{a}) = \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{\tau},$$

where $\boldsymbol{F}$ is the net force. If the force is central (parallel to $\boldsymbol{r}$), then $\boldsymbol{\tau} = \boldsymbol{0}$ and angular momentum is conserved.

範例

2.4 *Geometry of Smooth Oriented Curves*

An *oriented curve* is a geometric curve endowed with a specific direction of traversal. While a curve may be the image of infinitely many different functions, we often identify the curve with a particular regular parametrisation.

Definition 2.3. ***Smooth Path.***
Let $C \subseteq \mathbb{R}^n$ be an oriented curve starting at P and ending at Q. A **smooth non-stop path** parametrising C is a map $\gamma : [a,b] \to \mathbb{R}^n$ such that:

1. γ is smooth (C^∞).
2. γ is regular: $\gamma'(t) \neq \mathbf{0}$ for all $t \in [a,b]$.
3. $\gamma([a,b]) = C$ with $\gamma(a) = P$ and $\gamma(b) = Q$.

定義

Arclength Parametrisation

The length of a curve is independent of its parametrisation. To see this, consider the infinitesimal displacement ds along the curve $\gamma(t) = \langle x(t), y(t), z(t) \rangle$:

$$ds = \|\gamma'(t)\| dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

Integrating this magnitude yields the total arclength.

Definition 2.4. ***Arclength Function.***
The **arclength function** $s(t)$ measuring the distance along γ from the start point $t = a$ is:

$$s(t) = \int_a^t \|\gamma'(\tau)\|\, d\tau.$$

If $\|\gamma'(t)\| = 1$ for all t, the curve is said to be **unit-speed** or **parametrised by arclength**.

定義

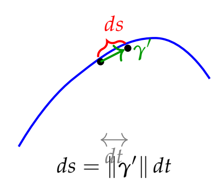


Figure 2.5: The infinitesimal arc length ds relates to the parameter increment dt through the speed $\|\gamma'\|$.

Proposition 2.1. ***Properties of Arclength.***
The function $s(t)$ satisfies:

1. $\frac{ds}{dt} = \|\gamma'(t)\|$ (speed).
2. Since γ is regular, $s'(t) > 0$, so $s(t)$ is strictly increasing and invertible.
3. The inverse function $t(s)$ allows reparametrisation: $\tilde{\gamma}(s) = \gamma(t(s))$ is a unit-speed path.

命題

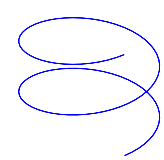


Figure 2.6: A helix $\langle \cos t, \sin t, bt \rangle$. The arclength is $s = t\sqrt{1+b^2}$, reflecting constant speed along the curve.

Example 2.5. Helix Arclength. Consider the helix $\boldsymbol{r}(t) = \langle R\cos t, R\sin t, bt \rangle$ for $t \in [0, 2\pi]$.
The velocity is $\boldsymbol{r}'(t) = \langle -R\sin t, R\cos t, b \rangle$. The speed is constant:

$$\|\boldsymbol{r}'(t)\| = \sqrt{R^2\sin^2 t + R^2\cos^2 t + b^2} = \sqrt{R^2 + b^2}.$$

The arclength function is $s(t) = \int_0^t \sqrt{R^2+b^2}\,du = t\sqrt{R^2+b^2}$. Solving for t, we get $t(s) = \frac{s}{\sqrt{R^2+b^2}}$. The unit-speed reparametrisation is:

$$\boldsymbol{r}(s) = \left\langle R\cos\left(\frac{s}{\sqrt{R^2+b^2}}\right), R\sin\left(\frac{s}{\sqrt{R^2+b^2}}\right), \frac{bs}{\sqrt{R^2+b^2}}\right\rangle.$$

範例

The Frenet-Serret Frame

To study the intrinsic geometry of a curve in $\mathbb{R}^3$, we attach a moving orthonormal basis $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ to each point on the curve. This frame is well-defined for any curve that is non-linear (so the curvature is non-zero) and regular.

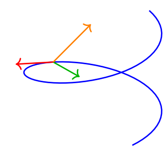


Figure 2.7: The Frenet frame $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ on a helix. $\boldsymbol{T}$ (red) is tangent, $\boldsymbol{N}$ (green) points toward the axis, $\boldsymbol{B}$ (orange) completes the right-handed frame.

Definition 2.5. ***Frenet Frame.***
Let $\gamma(t)$ be a regular curve of class C^3 with non-vanishing curvature.

1. **Unit Tangent:** $\boldsymbol{T}(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|}$.
2. **Principal Normal:** $\boldsymbol{N}(t) = \frac{\boldsymbol{T}'(t)}{\|\boldsymbol{T}'(t)\|}$.
3. **Binormal:** $\boldsymbol{B}(t) = \boldsymbol{T}(t) \times \boldsymbol{N}(t)$.

定義

Note

$\boldsymbol{N}$ is well-defined only if $\boldsymbol{T}'(t) \neq \boldsymbol{0}$ (i.e., the curve is not a straight line). By *Proposition 1.2*, since $\|\boldsymbol{T}\| = 1$, $\boldsymbol{T}'$ is orthogonal to $\boldsymbol{T}$, ensuring $\boldsymbol{N} \perp \boldsymbol{T}$. The definition of $\boldsymbol{B}$ ensures the frame is orthonormal and right-handed.

Theorem 2.4. ***Frenet-Serret Equations.***
For a unit-speed curve $\gamma(s)$, the derivatives of the frame vectors are given by:

$$\begin{bmatrix} \boldsymbol{T}'(s) \\ \boldsymbol{N}'(s) \\ \boldsymbol{B}'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T}(s) \\ \boldsymbol{N}(s) \\ \boldsymbol{B}(s) \end{bmatrix}.$$

Here $\kappa(s)$ is the **curvature** and $\tau(s)$ is the **torsion**.

定理

Proof

Since $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ is a basis, any vector derivative can be expanded in this basis.

1. By definition, $\boldsymbol{T}'(s)$ is parallel to $\boldsymbol{N}(s)$. We define $\kappa(s) = \|\boldsymbol{T}'(s)\|$. Thus $\boldsymbol{T}' = \kappa\boldsymbol{N}$.
2. Since $\boldsymbol{B} = \boldsymbol{T} \times \boldsymbol{N}$, differentiating yields $\boldsymbol{B}' = \boldsymbol{T}' \times \boldsymbol{N} + \boldsymbol{T} \times \boldsymbol{N}' =$

$(\kappa \boldsymbol{N}) \times \boldsymbol{N} + \boldsymbol{T} \times \boldsymbol{N}' = \boldsymbol{T} \times \boldsymbol{N}'$. Thus $\boldsymbol{B}'$ is orthogonal to $\boldsymbol{T}$. Also, since $\|\boldsymbol{B}\| = 1$, $\boldsymbol{B}'$ is orthogonal to $\boldsymbol{B}$. Hence $\boldsymbol{B}'$ must be parallel to $\boldsymbol{N}$. We define the torsion τ such that $\boldsymbol{B}' = -\tau \boldsymbol{N}$.

3. For $\boldsymbol{N}'$, differentiate $\boldsymbol{N} = \boldsymbol{B} \times \boldsymbol{T}$:

$$\boldsymbol{N}' = \boldsymbol{B}' \times \boldsymbol{T} + \boldsymbol{B} \times \boldsymbol{T}' = (-\tau \boldsymbol{N}) \times \boldsymbol{T} + \boldsymbol{B} \times (\kappa \boldsymbol{N}) = -\tau(\boldsymbol{N} \times \boldsymbol{T}) + \kappa(\boldsymbol{B} \times \boldsymbol{N}).$$

Using cross product identities ($\boldsymbol{N} \times \boldsymbol{T} = -\boldsymbol{B}$ and $\boldsymbol{B} \times \boldsymbol{N} = -\boldsymbol{T}$), we get $\boldsymbol{N}' = \tau \boldsymbol{B} - \kappa \boldsymbol{T}$.

■

Figure 2.8: The twisted cubic $\langle t, t^2, t^3 \rangle$. Non-zero torsion causes the curve to leave any fixed plane.

Curvature and Torsion

- **Curvature (κ)** measures the rate at which the curve turns (deviates from a line).
- **Torsion (τ)** measures the rate at which the curve twists out of the plane defined by $\boldsymbol{T}$ and $\boldsymbol{N}$ (the *osculating plane*).

If the parameter t is arbitrary (not necessarily arclength), we use the chain rule (via speed $\dot{s} = \|\gamma'(t)\|$):

$$\kappa(t) = \frac{\|\boldsymbol{T}'(t)\|}{\dot{s}} = \frac{\|\gamma' \times \gamma''\|}{\|\gamma'\|^3},$$
$$\tau(t) = -\frac{\boldsymbol{B}'(t) \bullet \boldsymbol{N}(t)}{\dot{s}} = \frac{(\gamma' \times \gamma'') \bullet \gamma'''}{\|\gamma' \times \gamma''\|^2}.$$

Example 2.6. Curvature of a Circle. For a circle of radius R, we have $\gamma(t) = \langle R\cos t, R\sin t, 0 \rangle$.

$$\gamma' = \langle -R\sin t, R\cos t, 0 \rangle, \quad \gamma'' = \langle -R\cos t, -R\sin t, 0 \rangle.$$

$$\gamma' \times \gamma'' = \langle 0, 0, R^2 \sin^2 t + R^2 \cos^2 t \rangle = \langle 0, 0, R^2 \rangle.$$

$$\kappa = \frac{R^2}{(R)^3} = \frac{1}{R}.$$

The curvature is constant and reciprocal to the radius. The torsion is zero, as the curve is planar.

範例

Example 2.7. Frenet Frame of a Helix. Let $\gamma(t) = \langle \cos t, \sin t, t \rangle$.

1. **Derivatives:**

$$\gamma'(t) = \langle -\sin t, \cos t, 1 \rangle, \quad \|\gamma'\| = \sqrt{2}.$$
$$\boldsymbol{T}(t) = \frac{1}{\sqrt{2}} \langle -\sin t, \cos t, 1 \rangle.$$

2. **Normal:**

$$T'(t) = \frac{1}{\sqrt{2}}\langle -\cos t, -\sin t, 0\rangle, \quad \|T'\| = \frac{1}{\sqrt{2}}.$$

$$N(t) = \langle -\cos t, -\sin t, 0\rangle.$$

3. **Binormal:**

$$B = T \times N = \frac{1}{\sqrt{2}} \det \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\langle \sin t, -\cos t, 1\rangle.$$

4. **Curvature and Torsion:** From $T' = \dot{s}\kappa N$, we have $\frac{1}{\sqrt{2}} = \sqrt{2}\kappa \implies \kappa = \frac{1}{2}$. From $B' = -\dot{s}\tau N$:

$$B' = \frac{1}{\sqrt{2}}\langle \cos t, \sin t, 0\rangle = -\frac{1}{\sqrt{2}} N.$$

Then $-\frac{1}{\sqrt{2}} N = -\sqrt{2}\tau N \implies \tau = \frac{1}{2}$.

範例

Osculating Plane and Circle

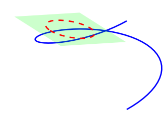


Figure 2.9: The osculating plane (green) at a point on a helix. The osculating circle (red, dashed) lies in this plane with radius $\rho = 1/\kappa$.

At any point P on the curve, the vectors T and N span the **osculating plane**. This is the plane that "best fits" the curve locally. The deviation of the curve from this plane is measured by torsion.
Inside the osculating plane, the best-fitting circle to the curve at P is the **osculating circle**.

- **Radius:** $\rho = 1/\kappa$ (radius of curvature).
- **Centre:** $C = P + \rho N$.

Example 2.8. Finding the Osculating Plane. Find the equation of the osculating plane for $r(t) = \langle 2\sin(3t), t, 2\cos(3t)\rangle$ at $t = \pi$.

1. Point: $r(\pi) = \langle 0, \pi, -2\rangle$.
2. Velocity: $r'(t) = \langle 6\cos(3t), 1, -6\sin(3t)\rangle \implies r'(\pi) = \langle -6, 1, 0\rangle$.
3. Acceleration: $r''(t) = \langle -18\sin(3t), 0, -18\cos(3t)\rangle \implies r''(\pi) = \langle 0, 0, 18\rangle$.
4. Normal to Plane: The binormal B is parallel to $r' \times r''$.

$$n = \langle -6, 1, 0\rangle \times \langle 0, 0, 18\rangle = \langle 18, 108, 0\rangle = 18\langle 1, 6, 0\rangle.$$

Using $\langle 1, 6, 0\rangle$ as the normal direction:

$$1(x-0) + 6(y-\pi) + 0(z+2) = 0 \implies x + 6y = 6\pi.$$

範例

2.5 Physics of Motion

In kinematics, we study the motion of objects without necessarily invoking the forces that cause them, though the framework is built upon Newton's Second Law, $\boldsymbol{F} = m\boldsymbol{a}$. We treat the trajectory of a particle as a path $t \mapsto \boldsymbol{r}(t)$ in $\mathbb{R}^3$, where t denotes time.

Kinematic Quantities

We define the fundamental vector quantities of motion.

Definition 2.6. ***Position, Velocity, and Acceleration.***
Let $\boldsymbol{r}(t)$ be the position vector of a particle at time t.

1. **Velocity:** $\boldsymbol{v}(t) = \frac{d\boldsymbol{r}}{dt}$. The speed is $v(t) = \|\boldsymbol{v}(t)\|$.
2. **Acceleration:** $\boldsymbol{a}(t) = \frac{d\boldsymbol{v}}{dt} = \frac{d^2\boldsymbol{r}}{dt^2}$.

定義

Using the Frenet frame $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$, we can decompose the acceleration vector into components that have direct physical interpretations.

Proposition 2.2. ***Tangential and Normal Acceleration.***
The acceleration vector $\boldsymbol{a}$ lies entirely in the osculating plane (the plane spanned by $\boldsymbol{T}$ and $\boldsymbol{N}$):

$$\boldsymbol{a} = a_T\boldsymbol{T} + a_N\boldsymbol{N}.$$

The components are given by:

· **Tangential Acceleration (a_T):** $a_T = \frac{dv}{dt}$. This measures the rate of change of speed.

· **Normal Acceleration (a_N):** $a_N = \kappa v^2$. This measures the change in direction (centripetal acceleration).

命題

Proof

Since $\boldsymbol{v} = v\boldsymbol{T}$, differentiation yields:

$$\boldsymbol{a} = \frac{d}{dt}(v\boldsymbol{T}) = \frac{dv}{dt}\boldsymbol{T} + v\frac{d\boldsymbol{T}}{dt}.$$

Recall that $\frac{d\boldsymbol{T}}{dt} = \frac{d\boldsymbol{T}}{ds}\frac{ds}{dt} = (\kappa\boldsymbol{N})v$. Substituting this back:

$$\boldsymbol{a} = \frac{dv}{dt}\boldsymbol{T} + v(\kappa v\boldsymbol{N}) = \dot{v}\boldsymbol{T} + \kappa v^2\boldsymbol{N}.$$

■

Note

$\boldsymbol{a} \bullet \boldsymbol{B} = 0$; there is no acceleration in the binormal direction.

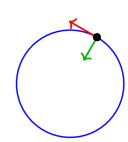


Figure 2.10: Uniform circular motion. Velocity $\boldsymbol{v}$ (red) is tangent; acceleration $\boldsymbol{a}$ (green) points toward the centre with $|\boldsymbol{a}| = v^2/R$.

Example 2.9. Circular Motion. Consider a particle moving in a circle of radius R with constant angular velocity ω:

$$\boldsymbol{r}(t) = \langle R\cos(\omega t), R\sin(\omega t)\rangle.$$

Differentiation yields:

$$\boldsymbol{v}(t) = \langle -R\omega\sin(\omega t), R\omega\cos(\omega t)\rangle, \quad v = R\omega.$$
$$\boldsymbol{a}(t) = \langle -R\omega^2\cos(\omega t), -R\omega^2\sin(\omega t)\rangle = -\omega^2\boldsymbol{r}(t).$$

Here, $\frac{dv}{dt} = 0$, so $a_T = 0$. The acceleration is purely normal (centripetal):

$$a_N = \|\boldsymbol{a}\| = R\omega^2 = \frac{(R\omega)^2}{R} = \frac{v^2}{R} = \kappa v^2,$$

consistent with $\kappa = 1/R$.

範例

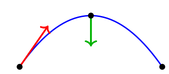


Figure 2.11: Projectile motion under gravity. The path $y = v_{0y}t - \frac{1}{2}gt^2$ traces a parabola. Initial velocity (red); constant acceleration $-g\hat{y}$ (green).

Projectile Motion

If we assume a constant gravitational acceleration $\boldsymbol{a}(t) = \langle 0, -g\rangle$, we can integrate to find the trajectory.

Example 2.10. Projectile with Initial Velocity. Suppose a projectile is launched from the origin with initial velocity $\boldsymbol{v}_0 = \langle v_{0x}, v_{0y}\rangle$. Integrating $\boldsymbol{a}(t) = \langle 0, -g\rangle$:

$$\boldsymbol{v}(t) = \int \boldsymbol{a}(t)\,dt = \langle C_1, -gt + C_2\rangle.$$

Applying $\boldsymbol{v}(0) = \boldsymbol{v}_0$ gives $\boldsymbol{v}(t) = \langle v_{0x}, v_{0y} - gt\rangle$. Integrating again for position:

$$\boldsymbol{r}(t) = \int \boldsymbol{v}(t)\,dt = \left\langle v_{0x}t, v_{0y}t - \frac{1}{2}gt^2\right\rangle,$$

assuming $\boldsymbol{r}(0) = \boldsymbol{0}$. This describes a parabola. The maximum height occurs when $v_y(t) = 0$, i.e., $t = v_{0y}/g$.

範例

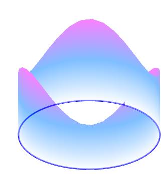


Figure 2.12: Scalar line integral $\int_C f\,ds$. The "curtain" has height $f(\gamma(t))$ and the integral computes its area.

2.6 *Scalar Line Integrals*

We now consider the integration of a scalar field $f : \mathbb{R}^n \to \mathbb{R}$ along a curve C. This generalises the concept of arclength to "weighted" arclength, useful for calculating mass, charge, or average values along a wire.

Definition 2.7. ***Scalar Line Integral.***
Let C be a piecewise smooth curve parametrised by $\gamma : [a, b] \to \mathbb{R}^n$, and let f be a continuous function defined on C. The **line integral of** f **along** C is:

$$\int_C f \, ds = \int_a^b f(\gamma(t)) \|\gamma'(t)\| \, dt.$$

定義

This line integral is independent of parametrisation and does not change if we reverse the orientation of the curve. If we reverse the curve $(-C)$, the scalar line integral remains unchanged:

$$\int_{-C} f \, ds = \int_C f \, ds.$$

This contrasts with vector line integrals (work), which change sign upon reversal.

Applications

Mass and Centre of Mass

If $\rho(x, y, z)$ represents the linear mass density (mass per unit length) of a wire C, the total mass M is:

$$M = \int_C \rho(x, y, z) \, ds.$$

The coordinates of the centre of mass $(\bar{x}, \bar{y}, \bar{z})$ are given by the moments:

$$\bar{x} = \frac{1}{M} \int_C x\rho \, ds, \quad \bar{y} = \frac{1}{M} \int_C y\rho \, ds, \quad \bar{z} = \frac{1}{M} \int_C z\rho \, ds.$$

Example 2.11. Centroid of a Semicircle. Find the centroid of a wire with uniform density $\rho = 1$ shaped like a semicircle of radius R in the upper half-plane.
Parametrisation: $\gamma(t) = \langle R\cos t, R\sin t\rangle$ for $t \in [0, \pi]$.

$$\|\gamma'(t)\| = R.$$

Total mass (length): $M = \int_0^\pi R \, dt = \pi R$. Centre of mass $\bar{y}$:

$$\bar{y} = \frac{1}{\pi R} \int_0^\pi (R\sin t) R \, dt = \frac{R}{\pi} \int_0^\pi \sin t \, dt = \frac{R}{\pi} [-\cos t]_0^\pi = \frac{2R}{\pi}.$$

By symmetry, $\bar{x} = 0$. Hence, the centroid is $(0, 2R/\pi)$.

範例

Averages

The average value of a function f over a curve C of length L is:

$$f_{\text{avg}} = \frac{1}{L}\int_C f\,ds.$$

Example 2.12. Average Height on a Helix. Consider one turn of the helix $\boldsymbol{r}(t) = \langle \cos t, \sin t, t\rangle$ for $t \in [0, 2\pi]$. The speed is $\|\boldsymbol{r}'\| = \sqrt{2}$. The length is $L = 2\pi\sqrt{2}$.
The average height (z-coordinate) is:

$$\bar{z} = \frac{1}{2\pi\sqrt{2}}\int_0^{2\pi} t(\sqrt{2})\,dt = \frac{1}{2\pi}\left[\frac{t^2}{2}\right]_0^{2\pi} = \frac{1}{2\pi}\frac{4\pi^2}{2} = \pi.$$

範例

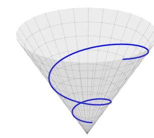


Figure 2.13: The exponential spiral $\langle e^{-t}\cos t, e^{-t}\sin t, e^{-t}\rangle$ lies on the cone $z = \sqrt{x^2+y^2}$.

2.7 Exercises

1. **Differentiation and Integration.** Perform the following calculations for vector-valued functions.
   (a) Compute $\frac{d}{dt}\langle t^2, e^t, \ln t\rangle$.
   (b) Compute $\frac{d}{dt}\langle \cosh(t^2), \sinh(\ln t)\rangle$.
   (c) Evaluate the indefinite integral $\int \langle 1, t, \sin t\rangle\,dt$.
2. **Kinematics from Acceleration.** Let a particle move with constant jerk (the derivative of acceleration). Specifically, let $\mathbf{r}'''(t) = \mathbf{c}$ for some constant vector $\mathbf{c}$. If the initial position, velocity, and acceleration are $\mathbf{r}_0, \mathbf{v}_0, \mathbf{a}_0$ respectively, find the general formula for the position $\mathbf{r}(t)$.
3. **Initial Value Problem.** Suppose the velocity of a particle is given by
$$\mathbf{v}(t) = \langle t, 3, t\cosh(t^2)\rangle$$
   and its initial position is $\mathbf{r}(0) = (1, 2, 3)$.
   (a) Find the acceleration $\mathbf{a}(t)$.
   (b) Find the position $\mathbf{r}(t)$ for $t \geq 0$.
4. **Parametric Tangents.** Let the path of a particle be described by $\mathbf{r}(t) = \langle 2t, \ln t, t^2+1\rangle$ for $t > 0$. Find the parametric equations of the tangent line to the path at the point $(2, 0, 2)$.
5. **Intersection vs Collision.** Two particles travel in the plane along the following paths for $t \geq 0$:
$$\mathbf{r}_1(t) = \langle -10+t, 1+t\rangle, \quad \mathbf{r}_2(t) = \langle 20-4t, 6+t\rangle.$$
   (a) Determine the point in the plane where the geometric paths intersect.
   (b) Determine if the particles collide (i.e., does $\mathbf{r}_1(t) = \mathbf{r}_2(t)$ for some time $t \geq 0$?).

Solve $\mathbf{r}_1(s) = \mathbf{r}_2(t)$ for $s, t \geq 0$.

6. **Helix Properties.** Consider the helix $\mathbf{r}(t) = \langle \cos(\pi t), \sin(\pi t), t/8 \rangle$.
   (a) How many complete revolutions does this helix make as it travels from $z = 0$ to $z = 1$?
   (b) Calculate the total arclength of the curve from $z = 0$ to $z = 1$.
7. **Reparametrisation.** Find the arclength function $s(t)$ for the helix $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, t \rangle$ starting from $t = 0$. Invert this function to find $t(s)$ and write the curve $\mathbf{r}(s)$ parametrised by arclength.
8. **Speed Limits.** A hover car travels along the path $\mathbf{r}(t) = \langle 2^{-t}, 3\sin t, 4t \rangle$ for $t \geq 0$ (units in miles and seconds). If the speed limit is 6 miles per second, determine if the car ever exceeds the limit. Provide a rigorous upper bound for the speed $\|\mathbf{r}'(t)\|$.
9. **Frenet Frame Calculation.** Let $\mathbf{r}(t) = \langle e^{-t}\cos t, e^{-t}\sin t, e^{-t} \rangle$. This curve is a spiral lying on a cone.
   (a) Calculate the unit tangent vector $\mathbf{T}(t)$.
   (b) Calculate the principal normal $\mathbf{N}(t)$.
   (c) Calculate the binormal $\mathbf{B}(t)$.
10. **Curvature and Torsion.** Using the results from the previous exercise, compute the curvature $\kappa(t)$ and torsion $\tau(t)$ of the exponential spiral.
11. **Maximising Curvature.** Find the point on the curve $y = 1/x$ (for $x > 0$) where the curvature κ is maximised.

    > *Remark.*
    >
    > Use the standard formula for the curvature of a graph $y = f(x)$: $\kappa(x) = \frac{|f''(x)|}{(1+[f'(x)]^2)^{3/2}}$.

12. **Planar Characterisation.** Prove that a curve with non-vanishing curvature is planar if and only if its torsion τ is identically zero.

    > *Remark.*
    >
    > One direction is easy. For the converse, consider the derivative of the binormal vector.

13. **Central Forces and Planarity.** A force is called *central* if $\mathbf{F}(\mathbf{r}) = f(\|\mathbf{r}\|)\mathbf{r}$, meaning it always points along the position vector.
    (a) Using Newton's Second Law $\mathbf{F} = m\mathbf{a}$, show that for a particle moving under a central force, the angular momentum $\mathbf{L} = \mathbf{r} \times m\mathbf{v}$ is constant.
    (b) Deduce that the trajectory of the particle must lie in a fixed plane.
14. $\star$ **Lancret's Theorem.** A curve $\mathbf{r}(s)$ is called a *generalised helix* if its tangent vector $\mathbf{T}(s)$ makes a constant angle with a fixed unit vector $\mathbf{u}$ (the axis). Prove that a curve is a generalised helix if and only if the ratio of curvature to torsion, κ/τ, is constant.

> *Remark.*
>
> Consider the projection $\mathbf{T} \bullet \mathbf{u} = \cos\alpha$. Differentiate this condition and express $\mathbf{u}$ in the Frenet basis.

15. ⋆ **The Darboux Vector.** In the theory of rigid body dynamics, the change of a rotating frame is described by an angular velocity vector. For the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$, we define the *Darboux vector* as:

$$\boldsymbol{\omega} = \tau\mathbf{T} + \kappa\mathbf{B}.$$

Prove the following kinematic relations, which show that the evolution of the frame corresponds to rotation about $\boldsymbol{\omega}$:

(a) $\mathbf{T}' = \boldsymbol{\omega} \times \mathbf{T}$

(b) $\mathbf{N}' = \boldsymbol{\omega} \times \mathbf{N}$

(c) $\mathbf{B}' = \boldsymbol{\omega} \times \mathbf{B}$

16. ⋆ **Implicit Curvature.** Let C be a plane curve defined implicitly by the level set $F(x, y) = 0$. By differentiating the relation $F(x, y(x)) = 0$ implicitly twice, show that the curvature is given by:

$$\kappa = \frac{|-F_{xx}F_y^2 + 2F_{xy}F_xF_y - F_{yy}F_x^2|}{(F_x^2 + F_y^2)^{3/2}}.$$

This formula allows the calculation of curvature without finding an explicit parametrisation.

17. ⋆ **The Evolute.** The *evolute* of a plane curve $\mathbf{r}(s)$ is the locus of the centres of its osculating circles. It is given by $\mathbf{e}(s) = \mathbf{r}(s) + \rho(s)\mathbf{N}(s)$, where $\rho(s) = 1/\kappa(s)$ is the radius of curvature. Prove that the tangent vector to the evolute, $\mathbf{e}'(s)$, is normal to the original curve $\mathbf{r}(s)$. Specifically, show that $\mathbf{e}'(s)$ is parallel to $\mathbf{N}(s)$.

Assume $\rho'(s) \neq 0$. Use the Frenet equations.

3
Review: Point Sets and Topology

Normally in one dimension, we rely heavily on open intervals (a, b) to define limits. In higher dimensions, the geometry becomes richer, and we must generalise concepts such as "closeness", "boundaries", and "connectedness" using the language of set theory.
This chapter formalises the classification of points and sets in $\mathbb{R}^n$, providing the necessary framework for multivariable analysis.

Standard single-variable calculus.

Review for readers of the analysis notes.

3.1 Neighbourhoods and Point Classification

The fundamental building block of topology in metric spaces is the open ball, which generalises the open interval. Recall Euclidean distance and write $\|\boldsymbol{x} - \boldsymbol{y}\|$.

> **Definition 3.1.** ***Open Ball (Neighbourhood).***
> Let $\boldsymbol{a} \in \mathbb{R}^n$ and $r > 0$. The **open ball** of radius r centred at $\boldsymbol{a}$, denoted $B_r(\boldsymbol{a})$ (or sometimes $O_r(\boldsymbol{a})$), is the set of all points strictly within distance r of $\boldsymbol{a}$:
>
> $$B_r(\boldsymbol{a}) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x} - \boldsymbol{a}\| < r\}.$$
>
> We often refer to $B_r(\boldsymbol{a})$ as the r-**neighbourhood** of $\boldsymbol{a}$. The **deleted neighbourhood**, denoted $\mathring{B}_r(\boldsymbol{a})$, is the open ball with the centre removed:
>
> $$\mathring{B}_r(\boldsymbol{a}) = B_r(\boldsymbol{a}) \setminus \{\boldsymbol{a}\} = \{\boldsymbol{x} \in \mathbb{R}^n \mid 0 < \|\boldsymbol{x} - \boldsymbol{a}\| < r\}.$$
>
> 定義

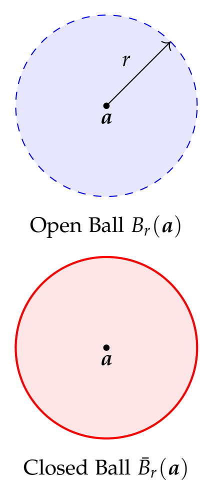


Figure 3.1: In $\mathbb{R}^2$, an open ball is a disc without its rim (dashed boundary). A closed ball includes the rim.

Using neighbourhoods, we classify points in $\mathbb{R}^n$ relative to a given set S.

> **Definition 3.2.** ***Topological Classification of Points.***
> Let $S \subseteq \mathbb{R}^n$ be a set and $\boldsymbol{x} \in \mathbb{R}^n$ be a point.
> - $\boldsymbol{x}$ is an **interior point** of S if there exists $r > 0$ such that $B_r(\boldsymbol{x}) \subseteq S$.
> - $\boldsymbol{x}$ is an **exterior point** of S if there exists $r > 0$ such that $B_r(\boldsymbol{x}) \cap S =$

$\emptyset$.

- $\boldsymbol{x}$ is a **boundary point** of S if every neighbourhood $B_r(\boldsymbol{x})$ contains at least one point in S and at least one point not in S.

定義

We "re"introduce specific notation for the collections of these points:

- The **interior** of S, denoted $\text{int}(S)$ or S°, is the set of all interior points.
- The **boundary** of S, denoted ∂S, is the set of all boundary points.

Example 3.1. Interior and Boundary of a Disc. Consider the set $S = \{\boldsymbol{x} \in \mathbb{R}^2 \mid \|\boldsymbol{x}\| \leq 1\}$.

- Any point with $\|\boldsymbol{x}\| < 1$ is an interior point. If $\|\boldsymbol{x}\| = 0.9$, we can choose $r = 0.09$, and the entire ball fits inside S.
- Any point with $\|\boldsymbol{x}\| > 1$ is an exterior point.
- Any point with $\|\boldsymbol{x}\| = 1$ is a boundary point. Any ball centred on the rim of the disc will capture points inside the disc and points outside it.

Thus, $\text{int}(S) = \{\boldsymbol{x} \mid \|\boldsymbol{x}\| < 1\}$ and $\partial S = \{\boldsymbol{x} \mid \|\boldsymbol{x}\| = 1\}$.

範例

Cluster Points and Limits

In the study of limits (e.g., $\lim_{\boldsymbol{x}\to\boldsymbol{a}} f(\boldsymbol{x})$), it is not necessary for $\boldsymbol{a}$ to belong to the domain of f, but $\boldsymbol{a}$ must be "arbitrarily close" to the domain. This motivates the concept of a cluster point.

Once again meaning basic calculus not the my study on analysis

Definition 3.3. ***Cluster Point.***
A point $\boldsymbol{x} \in \mathbb{R}^n$ is a **cluster point** (or accumulation point) of a set S if every deleted neighbourhood $\mathring{B}_r(\boldsymbol{x})$ contains at least one point of S. The set of all cluster points of S is called the **derived set**, denoted S'.

定義

Note

If $\boldsymbol{x} \in S$ but $\boldsymbol{x}$ is not a cluster point of S, then $\boldsymbol{x}$ is called an **isolated point**. For an isolated point, there exists a neighbourhood containing no other points of S other than $\boldsymbol{x}$ itself.

Cluster points are intrinsic to the behaviour of sequences. The following proposition links the topological definition to sequences, which is often more practical for proofs.

Proposition 3.1. ***Sequential Characterisation of Cluster Points.***
Let $\boldsymbol{x} \in \mathbb{R}^n$ and $S \subseteq \mathbb{R}^n$. The following are equivalent:

1. $\boldsymbol{x}$ is a cluster point of S.

2. Every neighbourhood $B_r(x)$ contains infinitely many points of S.
3. There exists a sequence of distinct points $\{x_k\} \subseteq S \setminus \{x\}$ such that $x_k \to x$ as $k \to \infty$.

命題

$(1 \implies 2)$

Assume x is a cluster point. Suppose, for contradiction, there exists a neighbourhood $B_r(x)$ containing only finitely many points of S distinct from x, say $\{s_1, \ldots, s_m\}$. Let $\delta = \min_{1 \le j \le m} \|s_j - x\|$. Then the ball $B_\delta(x)$ contains no points of S other than possibly x. This contradicts the definition of a cluster point. Thus, every neighbourhood must contain infinitely many points.

証明終

$(2 \implies 3)$

Construct the sequence inductively. Pick $x_1 \in B_1(x) \cap (S \setminus \{x\})$. Then pick $x_2 \in B_{1/2}(x) \cap (S \setminus \{x, x_1\})$. In general, choose $x_k \in B_{1/k}(x) \cap S$ distinct from all previous terms. By construction, $\|x_k - x\| < 1/k$, so the sequence converges to x.

証明終

$(3 \implies 1)$

If such a sequence exists, then for any $r > 0$, there exists K such that for all $k > K$, $x_k \in B_r(x)$. Since the points are distinct, at least one is not x, so $\mathring{B}_r(x) \cap S \neq \emptyset$.

証明終

Example 3.2. Cluster Points of a Discrete Set. Consider the set $S = \{1/n \mid n \in \mathbb{Z}^+\} \subset \mathbb{R}$.

- The point 0 is a cluster point of S because the sequence $1/n$ converges to 0. Note that $0 \notin S$.
- The point $1 \in S$ is an isolated point. We can choose $r = 1/2$; the interval $(0.5, 1.5)$ contains no other elements of S.
- The derived set is $S' = \{0\}$.

範例

Lemma 3.1. ***Transitivity of Accumulation.***
The cluster points of the derived set are cluster points of the original set. That is, $(S')' \subseteq S'$.

引理

Proof

Let $x \in (S')'$. By the sequential characterisation, there exists a sequence $\{y_k\} \subseteq S'$ converging to x. For any $r > 0$, there exists some y_k such that $\|y_k - x\| < r/2$. Since y_k is itself a cluster point of S, the neighbourhood $B_{r/2}(\boldsymbol{y}_k)$ contains infinitely many points of S. By the triangle inequality, $B_{r/2}(\boldsymbol{y}_k) \subset B_r(\boldsymbol{x})$. Thus, $B_r(\boldsymbol{x})$ contains points of S. Hence $\boldsymbol{x} \in S'$.

■

3.2 *Open and Closed Sets*

We now classify sets based on the nature of their boundaries.

Definition 3.4. ***Open and Closed Sets.***
A set $S \subseteq \mathbb{R}^n$ is:

- **Open** if $S = \text{int}(S)$. Equivalently, for every $\boldsymbol{x} \in S$, there exists a neighbourhood fully contained in S.
- **Closed** if its complement $S^c = \mathbb{R}^n \setminus S$ is open.

定義

The properties of open sets follow directly from the properties of unions and intersections.

- The union of any collection of open sets is open.
- The intersection of finitely many open sets is open.

By De Morgan's laws (Proposition 17.1.2 in the context of set theory), we deduce the dual properties for closed sets:

- The intersection of any collection of closed sets is closed.
- The union of finitely many closed sets is closed.

Note

Sets are not doors; a set can be neither open nor closed (e.g., the interval $(0, 1]$), or both open and closed (e.g., $\mathbb{R}^n$ and $\emptyset$).

Closure and Boundary Properties

The **closure** of a set S, denoted $\bar{S}$, is the union of the set and its cluster points:

$$\bar{S} = S \cup S'.$$

This represents the smallest closed set containing S. We have the following characterisations:

1. S is closed $\iff S' \subseteq S$. (A closed set contains all its cluster points).
2. $\bar{S} = S \cup \partial S$.
3. $\partial S = \bar{S} \cap \overline{S^c}$.

Theorem 3.1. ***The Boundary is Always Closed.***

For any set $S \subseteq \mathbb{R}^n$, the boundary ∂S is a closed set.

定理

Proof

We show that the complement $(\partial S)^c$ is open. Let $\boldsymbol{x} \notin \partial S$. By definition, it is not the case that *every* neighbourhood of $\boldsymbol{x}$ intersects both S and S^c. Thus, there exists some $r > 0$ such that $B_r(\boldsymbol{x})$ is disjoint from S (making $\boldsymbol{x}$ an exterior point) or $B_r(\boldsymbol{x})$ is contained in S (making $\boldsymbol{x}$ an interior point).

- If $B_r(\boldsymbol{x}) \cap S = \emptyset$, then every point $\boldsymbol{y} \in B_r(\boldsymbol{x})$ is also an exterior point. Thus $B_r(\boldsymbol{x}) \subseteq (\partial S)^c$.
- If $B_r(\boldsymbol{x}) \subseteq S$, then every point $\boldsymbol{y} \in B_r(\boldsymbol{x})$ is also an interior point. Thus $B_r(\boldsymbol{x}) \subseteq (\partial S)^c$.

In either case, $\boldsymbol{x}$ has a neighbourhood strictly contained in the complement of the boundary. Thus $(\partial S)^c$ is open, and ∂S is closed.

■

3.3 Compactness

Compactness is a generalisation of the property of being "finite" in a topological sense. It ensures that local properties can be extended globally, which is essential for theorems like the Extreme Value Theorem.

Definition 3.5. ***Covering and Compactness.***
Let $K \subseteq \mathbb{R}^n$. An **open covering** of K is a collection of open sets $\{U_\alpha\}_{\alpha \in I}$ such that $K \subseteq \bigcup_{\alpha \in I} U_\alpha$. The set K is **compact** if *every* open covering admits a **finite subcover**. That is, there exist finitely many indices $\alpha_1, \ldots, \alpha_k$ such that

$$K \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_k}.$$

定義

In Euclidean space $\mathbb{R}^n$, the Heine-Borel Theorem provides a concrete characterisation of compactness: A set $K \subseteq \mathbb{R}^n$ is compact if and only if it is both **closed** and **bounded**.

Proposition 3.2. ***Closed Subsets of Compact Sets.***
Let K be a compact set and let $F \subseteq K$ be a closed set. Then F is compact.

命題

Proof

Let $\{U_\alpha\}$ be an open covering of F. We must produce a finite subcover. Since F is closed, its complement F^c is open. Consider the collection of open sets $\{U_\alpha\} \cup \{F^c\}$. Since $F \subseteq \bigcup U_\alpha$, the union of this expanded collection covers the entirety of $\mathbb{R}^n$, and specifically covers K. Because K is compact, there exists a finite subcover of K from this collection:

$$K \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_m} \cup F^c.$$

Since $F \subseteq K$ and $F \cap F^c = \emptyset$, the sets $U_{\alpha_1}, \ldots, U_{\alpha_m}$ must cover F. Thus, we have found a finite subcover for F.

■

Example 3.3. Non-Compact Sets.

- The set $\mathbb{R}$ is not compact. The covering $\{(-n, n) \mid n \in \mathbb{Z}^+\}$ covers $\mathbb{R}$, but no finite sub-collection covers $\mathbb{R}$. (It is closed but not bounded).
- The interval $(0, 1)$ is not compact. The covering $\{(1/n, 1) \mid n \geq 2\}$ covers $(0, 1)$, but any finite subcover has a maximum n, leaving the interval $(0, 1/n]$ uncovered. (It is bounded but not closed).

範例

3.4 *Connectedness and Regions*

In calculus, we often require a domain to be "in one piece" to define integrals or establish the Intermediate Value Theorem.

Definition 3.6. ***Connected Set.***
A set S is **disconnected** if there exist two disjoint open sets U, V such that:

1. $S \cap U \neq \emptyset$ and $S \cap V \neq \emptyset$.
2. $S \subseteq U \cup V$.

If no such separation exists, S is **connected**.

定義

Definition 3.7. ***Region.***
An open, connected set in $\mathbb{R}^n$ is called a **region** (or domain).

定義

Verifying connectedness using the separation definition can be difficult. A more intuitive notion is **path-connectedness**: can we walk from any point to any other point without leaving the set?

Definition 3.8. ***Path-Connectedness.***
A set S is **path-connected** if for any $\boldsymbol{x}, \boldsymbol{y} \in S$, there exists a continuous function $\gamma : [0, 1] \to S$ such that $\gamma(0) = \boldsymbol{x}$ and $\gamma(1) = \boldsymbol{y}$.

定義

While path-connectedness implies connectedness, the converse is not always true (e.g., the Topologist's Sine Curve). However, for open sets in $\mathbb{R}^n$, the two concepts coincide.

Theorem 3.2. ***Regions are Path-Connected.***
Every region (connected open set) in $\mathbb{R}^n$ is path-connected.

定理

Proof

Let D be a region and fix a point $\boldsymbol{a} \in D$. Let U be the set of all points $\boldsymbol{x} \in D$ that can be connected to $\boldsymbol{a}$ by a path in D. We aim to show $U = D$.

U is open

Let $\boldsymbol{x} \in U$. Since D is open, there exists a ball $B_r(\boldsymbol{x}) \subseteq D$. Any point $\boldsymbol{y} \in B_r(\boldsymbol{x})$ can be connected to $\boldsymbol{x}$ by a straight line segment (which lies in the ball, and thus in D). Concatenating the path from $\boldsymbol{a}$ to $\boldsymbol{x}$ with the segment from $\boldsymbol{x}$ to $\boldsymbol{y}$ shows $\boldsymbol{y} \in U$. Thus $B_r(\boldsymbol{x}) \subseteq U$.

証明終

$D \setminus U$ is open

Let $\boldsymbol{z} \in D \setminus U$. Since D is open, there is a ball $B_r(\boldsymbol{z}) \subseteq D$. If any point $\boldsymbol{w} \in B_r(\boldsymbol{z})$ were in U, we could path-connect $\boldsymbol{a}$ to $\boldsymbol{w}$ and then $\boldsymbol{w}$ to $\boldsymbol{z}$, implying $\boldsymbol{z} \in U$, a contradiction. Thus $B_r(\boldsymbol{z})$ is disjoint from U, so $B_r(\boldsymbol{z}) \subseteq D \setminus U$.

証明終

Since D is connected, it cannot be the union of two disjoint non-empty open sets U and $D \setminus U$. Since $\boldsymbol{a} \in U$ (so $U \neq \emptyset$), it must be that $D \setminus U = \emptyset$. Therefore, $U = D$.

■

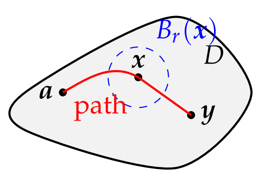


Figure 3.2: In an open connected set, the ability to connect $\boldsymbol{x}$ to a local neighbourhood implies global path-connectedness.

3.5 *Fundamental Theorems in* $\mathbb{R}^n$

The topological structure of $\mathbb{R}^n$ allows us to generalise several fundamental theorems from single-variable calculus. While properties relying on the order of real numbers do not extend directly to higher dimensions, concepts related to completeness, compactness, and convergence do.

If you've read my analysis notes these should NOT be new to you

Completeness and Concentration

The completeness of $\mathbb{R}$—the property that "there are no gaps"—extends to $\mathbb{R}^n$. This is formally captured by the Cauchy criterion.

Theorem 3.3. ***Cauchy Convergence Criterion.***
A sequence $\{\mathbf{x}_k\}$ in $\mathbb{R}^n$ converges if and only if it is a **Cauchy sequence**. That is, for every $\epsilon > 0$, there exists an integer N such that

$$\|\mathbf{x}_k - \mathbf{x}_m\| < \epsilon \quad \text{for all } k, m > N.$$

定理

Proof

This follows from the component-wise convergence in $\mathbb{R}^n$. A vector sequence is Cauchy if and only if each of its coordinate sequences is Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete, the coordinates converge, implying the vector sequence converges.

■

A powerful consequence of completeness is the ability to extract convergent subsequences from bounded sets. This is known as the Bolzano-Weierstrass Theorem.

Theorem 3.4. ***Bolzano-Weierstrass.***
Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence. Equivalently, every bounded infinite subset of $\mathbb{R}^n$ has at least one cluster point.

定理

Proof

Let $\{\mathbf{x}_k\}$ be a bounded sequence. The sequence of first components $\{x_k^{(1)}\}$ is bounded in $\mathbb{R}$, so by the one-dimensional Bolzano-Weierstrass theorem, it has a convergent subsequence. We pass to this subsequence. The sequence of second components is also bounded, so we extract a further subsequence where the second components converge. Repeating this n times yields a subsequence where every coordinate converges.

■

The Nested Closed Set Theorem

In $\mathbb{R}$, the Nested Interval Theorem guarantees that a sequence of nested closed intervals with lengths tending to zero intersects at a single point. The generalisation to $\mathbb{R}^n$ replaces intervals with closed sets and length with diameter.
Recall that the **diameter** of a set S is $\text{diam}(S) = \sup\{\|\mathbf{x} - \mathbf{y}\| : \mathbf{x}, \mathbf{y} \in S\}$.

Theorem 3.5. ***Nested Closed Set Theorem.***
Let $\{D_k\}_{k=1}^{\infty}$ be a sequence of non-empty closed sets in $\mathbb{R}^n$ satisfying:

1. **Nested:** $D_{k+1} \subseteq D_k$ for all k.
2. **Vanishing Diameter:** $\lim_{k\to\infty} \operatorname{diam}(D_k) = 0$.

Then the intersection $\bigcap_{k=1}^{\infty} D_k$ consists of exactly one point.

定理

Existence

Select a point $\mathbf{x}_k \in D_k$ for each k. Since $D_{k+1} \subseteq D_k$, for any $m > k$, both $\mathbf{x}_k$ and $\mathbf{x}_m$ lie in D_k. Thus, $\|\mathbf{x}_k - \mathbf{x}_m\| \le \operatorname{diam}(D_k)$. Since the diameter tends to zero, $\{\mathbf{x}_k\}$ is a Cauchy sequence and converges to some limit $\mathbf{x}$. Fix an index k_0. For all $k \ge k_0$, $\mathbf{x}_k \in D_k \subseteq D_{k_0}$. Since D_{k_0} is closed, the limit $\mathbf{x}$ must belong to D_{k_0}. This holds for any k_0, so $\mathbf{x} \in \bigcap D_k$.

証明終

Uniqueness

Suppose $\mathbf{x}$ and $\mathbf{y}$ are in the intersection. Then for all k, $\|\mathbf{x} - \mathbf{y}\| \le \operatorname{diam}(D_k)$. Taking the limit as $k \to \infty$ implies $\|\mathbf{x} - \mathbf{y}\| = 0$, so $\mathbf{x} = \mathbf{y}$.

証明終

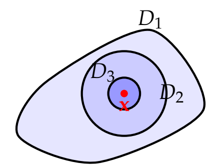


Figure 3.3: A sequence of nested closed sets with diameters tending to zero converges to a unique point $\mathbf{x}$.

The *Nested Closed Set Theorem* provides an elegant method to prove geometric connectivity properties.

Example 3.4. Connectivity of $\mathbb{R}^n$. A set $S \subseteq \mathbb{R}^n$ that is both open and closed must be either $\emptyset$ or $\mathbb{R}^n$.

範例

Proof

Assume S is a non-empty proper subset of $\mathbb{R}^n$ that is both open and closed. Then S^c is also non-empty, open, and closed. Pick $\mathbf{a} \in S$ and $\mathbf{b} \in S^c$. Consider the line segment L connecting $\mathbf{a}$ and $\mathbf{b}$. Let $M_0 = L$. We perform a bisection procedure: Divide M_0 into two equal sub-segments. At least one sub-segment must have one endpoint in S and the other in S^c; its midpoint lies in exactly one of S or S^c. Call this segment M_1. Repeat this process to generate a sequence of nested closed intervals $M_0 \supset M_1 \supset M_2 \ldots$ where each M_k has endpoints $\mathbf{u}_k \in S$ and $\mathbf{v}_k \in S^c$, and $\operatorname{diam}(M_k) \to 0$. By the *theorem 3.5*, there is a unique common point $\mathbf{p} = \bigcap M_k$. Since $\mathbf{u}_k \to \mathbf{p}$ and S is closed, $\mathbf{p} \in S$. Since $\mathbf{v}_k \to \mathbf{p}$ and S^c is closed, $\mathbf{p} \in S^c$. Thus $\mathbf{p} \in S \cap S^c$, which is impossible. Hence, no such set exists.

■

Characterisations of Compactness

We previously defined a compact set as one where every open cover admits a finite subcover. In $\mathbb{R}^n$, compactness equates to being closed and bounded (Heine-Borel). There are two other equivalent characterisations useful in proofs.

Proposition 3.3. ***Finite Intersection Property.***
A set K is compact if and only if every family of closed subsets $\{F_\alpha\}_{\alpha \in I}$ of K with the **finite intersection property** has a non-empty total intersection. A family has the finite intersection property if the intersection of any finite sub-collection is non-empty.

命題

Proof

This is the De Morgan dual of the open cover definition. Let $\{F_\alpha\}$ be closed subsets of K. Then $U_\alpha = K \setminus F_\alpha$ are open relative to K.

$$\begin{aligned} \bigcap F_\alpha = \emptyset &\iff K \cap \bigcap F_\alpha = \emptyset \\ &\iff K \subseteq \left(\bigcap F_\alpha\right)^c = \bigcup U_\alpha \\ &\iff \{U_\alpha\} \text{ covers } K. \end{aligned}$$

If K is compact, the cover $\{U_\alpha\}$ has a finite subcover $U_{\alpha_1}, \ldots, U_{\alpha_m}$. This is equivalent to $K \subseteq \bigcup_{i=1}^m U_{\alpha_i}$, which implies $\bigcap_{i=1}^m F_{\alpha_i} = \emptyset$. Thus, empty total intersection implies empty finite intersection. Conversely, if every finite intersection is non-empty, the total intersection must be non-empty.

■

Theorem 3.6. ***Sequential Compactness.***
For a set $S \subseteq \mathbb{R}^n$, the following are equivalent:

1. S is compact (Open Cover definition).
2. S is closed and bounded (Heine-Borel).
3. Every infinite subset of S has a cluster point in S (Bolzano-Weierstrass property).

定理

$(1 \implies 3)$

Assume S is compact and let $A \subseteq S$ be infinite. Suppose A has no cluster point in S. Then for every $\mathbf{x} \in S$, there is a neighbourhood $B_{r_x}(\mathbf{x})$ containing no point of A (except possibly $\mathbf{x}$ itself). The collection $\{B_{r_x}(\mathbf{x})\}$ covers S. By compactness, a finite subcover exists. Since each ball contains at most one point of A, A must be finite, a contradiction.

証明終

(3 $\implies$ 2)

If S is unbounded, we can pick a sequence with $\|\mathbf{x}_k\| \to \infty$ having no cluster point. Thus S must be bounded. If S is not closed, there is a cluster point $\mathbf{y} \notin S$. We can construct a sequence in S converging to $\mathbf{y}$, which creates an infinite subset with no cluster point *inside* S. Thus S must be closed.

証明終

(2 $\implies$ 1)

Assume $S \subseteq \mathbb{R}^n$ is closed and bounded. Since S is bounded, there exists $M > 0$ such that $S \subseteq Q := [-M, M]^n$. The set Q is a closed n-dimensional cube. By the proposition on closed subsets of compact sets, it is enough to show that Q is compact; then S, being closed in Q, will also be compact. Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of Q. Suppose, for contradiction, that no finite subcollection of $\{U_\alpha\}$ covers Q. We construct a sequence of nested closed cubes inside Q:

- Set $Q_0 := Q$.
- Given a closed cube Q_k, divide it into 2^n congruent closed subcubes by bisecting each edge.

If *every* one of these 2^n subcubes admitted a finite subcover by sets from $\{U_\alpha\}$, then taking the union of these finitely many finite subcovers would give a finite subcover of Q_k, and hence of Q. This contradicts our assumption that no finite subcollection covers Q. Therefore, at least one of the subcubes of Q_k does *not* admit a finite subcover; choose one such cube and call it Q_{k+1}. By construction we obtain a nested sequence of non-empty closed cubes $Q_0 \supseteq Q_1 \supseteq Q_2 \supseteq \cdots$ and at each step the side length (and hence the diameter) is divided by 2. Thus $\lim_{k\to\infty} \operatorname{diam}(Q_k) = 0$. Nested Closed Set Theorem now applies: there exists a unique point $\mathbf{x} \in \bigcap_{k=0}^{\infty} Q_k$. Since $\{U_\alpha\}_{\alpha \in I}$ covers Q, in particular $\mathbf{x} \in U_{\alpha_0}$ for some index α_0. Because U_{α_0} is open, there exists $r > 0$ such that $B_r(\mathbf{x}) \subseteq U_{\alpha_0}$. For k sufficiently large, we have $\operatorname{diam}(Q_k) < r$, and since $\mathbf{x} \in Q_k$ this implies $Q_k \subseteq B_r(\mathbf{x}) \subseteq U_{\alpha_0}$. But then the single set U_{α_0} is a finite subcover of Q_k, contradicting the way we chose Q_k (each Q_k was chosen so that no finite subcollection of $\{U_\alpha\}$ covers it). This contradiction shows that our assumption was false. Therefore every open cover of Q has a finite subcover; that is, Q is compact. Since S is a closed subset of the compact set Q, the earlier proposition implies S is compact as well.

証明終

Separation Theorems

Compactness allows us to separate disjoint sets by strictly positive distances, a property that fails for general closed sets (e.g., the hyperbola $xy = 1$ and the axis $y = 0$ are disjoint closed sets in $\mathbb{R}^2$ with distance zero).

Definition 3.9. ***Distance Between Sets.***
The distance between two non-empty sets $A, B \subseteq \mathbb{R}^n$ is defined as:

$$d(A, B) = \inf\{\|\mathbf{a} - \mathbf{b}\| : \mathbf{a} \in A, \mathbf{b} \in B\}.$$

定義

Proposition 3.4. ***Separation of Compact Sets.***
Let K_1 and K_2 be disjoint, non-empty, compact sets in $\mathbb{R}^n$. Then $d(K_1, K_2) > 0$. Moreover, there exist disjoint open sets U_1, U_2 such that $K_1 \subseteq U_1$ and $K_2 \subseteq U_2$.

命題

Positive Distance

Assume $d(K_1, K_2) = 0$. Then there exist sequences $\mathbf{x}_k \in K_1$ and $\mathbf{y}_k \in K_2$ such that $\|\mathbf{x}_k - \mathbf{y}_k\| \to 0$. By compactness, we can extract convergent subsequences $\mathbf{x}_{k_j} \to \mathbf{x} \in K_1$ and $\mathbf{y}_{k_j} \to \mathbf{y} \in K_2$. The condition $\|\mathbf{x}_k - \mathbf{y}_k\| \to 0$ implies $\|\mathbf{x} - \mathbf{y}\| = 0$, so $\mathbf{x} = \mathbf{y}$. Thus $\mathbf{x} \in K_1 \cap K_2$, contradicting disjointness. Hence $d(K_1, K_2) = \delta > 0$.

証明終

Open Separation

Let $\delta = d(K_1, K_2)$. Define

$$U_1 = \bigcup_{\mathbf{x} \in K_1} B_{\delta/3}(\mathbf{x}) \quad \text{and} \quad U_2 = \bigcup_{\mathbf{y} \in K_2} B_{\delta/3}(\mathbf{y}).$$

These are open sets containing K_1 and K_2 respectively. By the triangle inequality, if $U_1 \cap U_2 \neq \emptyset$, there exist $\mathbf{x} \in K_1, \mathbf{y} \in K_2$ such that $\|\mathbf{x} - \mathbf{y}\| < \frac{2\delta}{3} < \delta$, a contradiction.

証明終

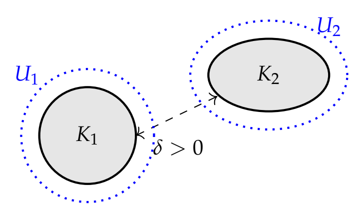


Figure 3.4: Two disjoint compact sets can always be separated by a non-zero distance δ, allowing them to be encased in disjoint open neighbourhoods.

3.6 *Exercises*

In the following exercises, unless otherwise specified, S denotes a subset of $\mathbb{R}^n$. The distance between a point $\mathbf{x}$ and a set S is defined as $d(\mathbf{x}, S) = \inf\{\|\mathbf{x} - \mathbf{s}\| : \mathbf{s} \in S\}$.

1. **Algebra of Open and Closed Sets.** Using the definitions provided in the text:

(a) Prove that the intersection of an arbitrary collection of closed sets is closed.
(b) Prove that the union of a finite collection of closed sets is closed.
(c) Is the intersection of infinitely many open sets necessarily open? Provide a counter-example, such as $\bigcap_{n=1}^{\infty}(-1/n, 1/n)$.
(d) Prove that the interior of S, denoted $\text{int}(S)$, is an open set, and moreover, it is the largest open set contained in S.

2. **Cluster Point Equivalence.** In the text, we provided a sequential characterisation of cluster points. Prove the equivalence of the following definitions using the structure of neighbourhoods:

Consider how one might construct a contradiction if a neighbourhood contained only finitely many points.

(a) $\mathbf{x}$ is a cluster point of S if every deleted neighbourhood $\mathring{B}_r(\mathbf{x})$ contains at least one point of S.
(b) $\mathbf{x}$ is a cluster point of S if every neighbourhood $B_r(\mathbf{x})$ contains infinitely many points of S.

3. **Boundary Identities.** Establish the following relationships between the closure, interior, and boundary of a set.

(a) Prove that $\overline{S} = S \cup \partial S$.
(b) Prove that $\partial S = \overline{S} \setminus \text{int}(S)$.
(c) Prove that if $A \cap B = \emptyset$, then $A \cap \text{int}(B) = \emptyset$.

4. **Distance Characterisation of Closure.** Prove that the closure of S consists precisely of those points at zero distance from S. That is:

$$\overline{S} = \{\mathbf{x} \in \mathbb{R}^n \mid d(\mathbf{x}, S) = 0\}.$$

5. **Derived Sets and Isolation.** Recall that S' denotes the set of cluster points of *SSets satisfying $S \subseteq S'$ are often called perfect sets.*. Prove that $S = S'$ if and only if S is closed and contains no isolated points.

6. **Convexity and Closure.** A set $S \subseteq \mathbb{R}^n$ is *convex* if for any $\mathbf{x}, \mathbf{y} \in S$ and any $t \in [0, 1]$, the point $(1-t)\mathbf{x} + t\mathbf{y}$ lies in S. Prove that if S is convex, then its closure $\overline{S}$ is also convex.

7. **Countable Generations.**

For (a), consider sets like $\{\mathbf{x} \mid d(\mathbf{x}, S) < 1/k\}$. For (b), consider $\{\mathbf{x} \mid d(\mathbf{x}, S^c) \geq 1/k\}$.

(a) Prove that every closed set in $\mathbb{R}^n$ can be expressed as the intersection of a countable number of open sets. (Sets of this type are called G_δ sets).
(b) Prove that every open set in $\mathbb{R}^n$ can be expressed as the union of a countable number of closed sets. (Sets of this type are called F_σ sets).

8. **The Graph of a Continuous Function.** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that its graph, defined as the set $\Gamma = \{(x, f(x)) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$, is a closed subset of the plane $\mathbb{R}^2$.

9. **Normality of $\mathbb{R}^n$.** The separation property discussed in the text for compact sets can be strengthened. Let A and B be disjoint *closed* sets in $\mathbb{R}^n$ (not necessarily bounded). Prove that there exist disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

Consider the cover formed by balls $B_{d(\mathbf{a},B)/2}(\mathbf{a})$ for each $\mathbf{a} \in A$.

10. **The Lebesgue Number Lemma.** This result is a crucial tool in algebraic topology and analysis. Let K be a compact subset of $\mathbb{R}^n$.
    (a) Prove that for any $\delta > 0$, there exist finitely many points $\mathbf{p}_1, \ldots, \mathbf{p}_k$ in K such that the union of balls $\bigcup_{i=1}^{k} B_\delta(\mathbf{p}_i)$ covers K.
    (b) Let $\mathcal{U} = \{U_\alpha\}$ be an open covering of K. Prove there exists a number $\lambda > 0$ (the Lebesgue number of the cover) such that for any $\mathbf{x} \in K$, the ball $B_\lambda(\mathbf{x})$ is contained entirely within at least one set $U_\alpha \in \mathcal{U}$.
    (c) Show that the statement in (b) is equivalent to the standard definition of compactness (every open cover admits a finite subcover).

11. **The Minkowski Sum.** Given two sets $A, B \subseteq \mathbb{R}^n$, their Minkowski sum is defined as $A + B = \{\mathbf{a} + \mathbf{b} \mid \mathbf{a} \in A, \mathbf{b} \in B\}$.
    (a) Prove that if K is compact and C is closed, then $K + C$ is closed.
    (b) Prove that the sum of two closed sets need not be closed.

    > *Remark.*
    > Consider $K = \mathbb{Z}$ and $C = \{m\alpha \mid m \in \mathbb{Z}\}$ in $\mathbb{R}$, where α is irrational. You may assume that $\{m\alpha - \lfloor m\alpha \rfloor\}$ is dense in $[0,1]$. Show that $K + C$ is dense in $\mathbb{R}$ but not equal to $\mathbb{R}$.

12. ⋆ **Geometric Application: The Centroid.** Use *theorem 3.5* to prove that the three medians of a triangle intersect at a single point.

Construct a sequence of nested triangles. Let Δ_0 be the original triangle. Let Δ_{k+1} be the triangle formed by the midpoints of the sides of Δ_k. Analyse the limit of these sets.

13. **Basic Connectedness Properties.**
    (a) Prove that a subset $S \subseteq \mathbb{R}$ is connected if and only if S is an interval (possibly unbounded).
    (b) Prove that if $\{S_\alpha\}$ is a collection of connected sets such that $\bigcap S_\alpha \neq \emptyset$, then their union $\bigcup S_\alpha$ is connected.
    (c) Prove that if A is connected and $A \subseteq B \subseteq \overline{A}$, then B is connected.

Recall that intervals are characterized by the property: $x, y \in I, x < z < y \implies z \in I$.

14. **Rational Paths.** Consider the set $S = \{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{Q} \text{ or } y \in \mathbb{Q}\}$. Prove that S is path-connected.

Try to construct a path consisting of horizontal and vertical line segments between any two points $\mathbf{p}_1, \mathbf{p}_2 \in S$.

15. **The Topologist's Sine Curve.** Connectedness does not always

imply path-connectedness. Consider the set:

$$S = \left\{ \left(x, \sin\frac{1}{x} \right) \;\middle|\; 0 < x \leq \frac{1}{\pi} \right\} \cup \{(0, y) \mid -1 \leq y \leq 1\}.$$

(a) Prove that S is connected.

> *Remark.*
>
> Use the result from Exercise 13(c).

(b) Prove that S is *not* path-connected.

> *Remark.*
>
> Suppose such a path $\gamma(t)$ exists starting at $(0, 0)$. Analyse the behaviour of γ as it enters the oscillating region.

4
Limits and Continuity of Functions of Several Variables

The study of calculus in higher dimensions begins with the fundamental concepts of limits and continuity. While the intuitive definitions mirror those of single-variable calculus, the geometry of $\mathbb{R}^n$ introduces new complexities. In one dimension, a point can only be approached from two directions (left and right). In $\mathbb{R}^n$, there are infinitely many paths approaching a point, requiring a more robust definition of the limit.

4.1 Limits of Functions of Several Variables

Double Limits

Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function of n variables, and let $\mathbf{a} \in \mathbb{R}^n$ be a cluster point of the domain D. We define the limit of f as $\mathbf{x}$ approaches $\mathbf{a}$ using the standard ε-δ formulation.

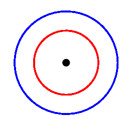


Figure 4.1: The ε-δ definition in $\mathbb{R}^2$: if $\mathbf{x}$ lies in the δ-disk (red), then $f(\mathbf{x})$ must lie within ε of L.

Definition 4.1. ***Limit of a Function.***
Let f be defined in a deleted neighbourhood of $\mathbf{a} \in \mathbb{R}^n$. We say that the **limit** of $f(\mathbf{x})$ as $\mathbf{x}$ approaches $\mathbf{a}$ is L, denoted by

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L,$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $\mathbf{x} \in D$ satisfying $0 < \|\mathbf{x} - \mathbf{a}\| < \delta$, we have

$$|f(\mathbf{x}) - L| < \varepsilon.$$

In terms of neighbourhoods, this condition is:

$$\mathbf{x} \in \mathring{B}_\delta(\mathbf{a}) \implies f(\mathbf{x}) \in B_\varepsilon(L).$$

This is often called the **double limit** (or n-fold limit) to distinguish it from iterated limits.

定義

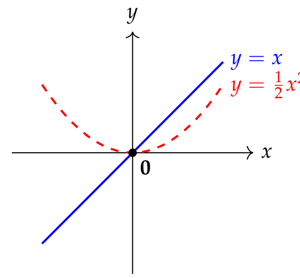
The crucial difference in multivariable calculus is that $\mathbf{x} \to \mathbf{a}$ implies approach along *any* path. If $f(\mathbf{x})$ approaches different values along different paths terminating at $\mathbf{a}$, the limit does not exist.

Example 4.1. Non-Existence of a Limit. Consider the function $f(x,y) = \frac{-xy}{x^2+y^2}$. We investigate the limit as $(x,y) \to (0,0)$.
Along the line $y = mx$ (where m is a constant slope), the function becomes:

$$\lim_{x\to 0} f(x,mx) = \lim_{x\to 0} \frac{-x(mx)}{x^2+(mx)^2} = \lim_{x\to 0} \frac{-mx^2}{x^2(1+m^2)} = \frac{-m}{1+m^2}.$$

The limit depends on the slope m. For $m = 0$ (approach along the x-axis), the limit is 0. For $m = 1$ (approach along $y = x$), the limit is $-1/2$. Since the value is not unique, the double limit does not exist.

範例



Path dependence at the origin

Figure 4.2: To demonstrate a limit does not exist, it suffices to find two paths (e.g., a line and a parabola) yielding different limiting values.

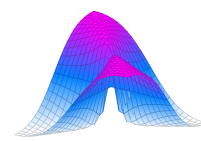


Figure 4.3: The surface $z = \frac{-xy}{x^2+y^2}$ near the origin. Different radial directions yield different limiting values, so the limit does not exist.

We can also define limits where variables tend to infinity. The definitions are analogous to the single-variable case but must account for the multidimensional nature of the domain.

Definition 4.2. ***Limits at Infinity.***
Let $f : \mathbb{R}^2 \to \mathbb{R}$.

- $\lim_{(x,y)\to(\infty,\infty)} f(x,y) = L$ if $\forall \varepsilon > 0, \exists M > 0$ such that $x > M$ and $y > M$ implies $|f(x,y) - L| < \varepsilon$.
- $\lim_{(x,y)\to(x_0,+\infty)} f(x,y) = L$ if $\forall \varepsilon > 0, \exists \delta, M > 0$ such that $0 < |x - x_0| < \delta$ and $y > M$ implies $|f(x,y) - L| < \varepsilon$.

定義

The standard properties of limits — uniqueness, arithmetic rules (sum, product, quotient), and the squeeze theorem — hold for functions of several variables.

Theorem 4.1. ***Uniqueness of Limits.***
If the limit of a function $f(\mathbf{x})$ as $\mathbf{x} \to \mathbf{a}$ exists, it is unique. That is, if $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L_1$ and $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L_2$, then $L_1 = L_2$.

定理

Proof

Suppose, for the sake of contradiction, that $L_1 \neq L_2$. Let $\varepsilon = \frac{|L_1 - L_2|}{2}$. Since $L_1 \neq L_2$, we have $\varepsilon > 0$. By the definition of the limit, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that:

- If $0 < \|\mathbf{x} - \mathbf{a}\| < \delta_1$, then $|f(\mathbf{x}) - L_1| < \varepsilon$.
- If $0 < \|\mathbf{x} - \mathbf{a}\| < \delta_2$, then $|f(\mathbf{x}) - L_2| < \varepsilon$.

Let $\delta = \min(\delta_1, \delta_2)$. For any $\mathbf{x}$ satisfying $0 < \|\mathbf{x} - \mathbf{a}\| < \delta$, both inequalities hold. Using the triangle inequality:

$$|L_1 - L_2| = |(L_1 - f(\mathbf{x})) + (f(\mathbf{x}) - L_2)| \leq |f(\mathbf{x}) - L_1| + |f(\mathbf{x}) - L_2|.$$

Substituting the bounds:

$$|L_1 - L_2| < \varepsilon + \varepsilon = 2\varepsilon = |L_1 - L_2|.$$

This implies $|L_1 - L_2| < |L_1 - L_2|$, a contradiction. Thus, $L_1 = L_2$.

■

Theorem 4.2. ***Squeeze Theorem.***
Let f, g, h be functions defined on a deleted neighbourhood of $\mathbf{a}$ such that:

$$g(\mathbf{x}) \leq f(\mathbf{x}) \leq h(\mathbf{x}) \quad \text{for all } \mathbf{x} \neq \mathbf{a}.$$

If $\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) = L$ and $\lim_{\mathbf{x}\to\mathbf{a}} h(\mathbf{x}) = L$, then $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$.

定理

Proof

Let $\varepsilon > 0$. Since $g(\mathbf{x}) \to L$ and $h(\mathbf{x}) \to L$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that:

- If $0 < \|\mathbf{x} - \mathbf{a}\| < \delta_1$, then $|g(\mathbf{x}) - L| < \varepsilon \implies L - \varepsilon < g(\mathbf{x})$.
- If $0 < \|\mathbf{x} - \mathbf{a}\| < \delta_2$, then $|h(\mathbf{x}) - L| < \varepsilon \implies h(\mathbf{x}) < L + \varepsilon$.

Let $\delta = \min(\delta_1, \delta_2)$. For $0 < \|\mathbf{x} - \mathbf{a}\| < \delta$, the ordering implies:

$$L - \varepsilon < g(\mathbf{x}) \leq f(\mathbf{x}) \leq h(\mathbf{x}) < L + \varepsilon.$$

Subtracting L yields $-\varepsilon < f(\mathbf{x}) - L < \varepsilon$, or $|f(\mathbf{x}) - L| < \varepsilon$. Thus, the limit of $f(\mathbf{x})$ is L.

■

Theorem 4.3. ***Arithmetic of Limits.***
If $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = A$ and $\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) = B$, then:

1. $\lim_{\mathbf{x}\to\mathbf{a}} (f(\mathbf{x}) \pm g(\mathbf{x})) = A \pm B$.
2. $\lim_{\mathbf{x}\to\mathbf{a}} (f(\mathbf{x})g(\mathbf{x})) = AB$.
3. If $B \neq 0$, $\lim_{\mathbf{x}\to\mathbf{a}} \frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{A}{B}$.

The proofs follow directly from the properties of absolute values and are analogous to the single-variable cases.

定理

Methods for Computing Limits

Computing double limits often requires transforming the expression to bound the error term or reducing it to a single-variable limit.

Polar Coordinates

For limits at the origin $(0,0)$, the substitution $x = r\cos\theta, y = r\sin\theta$ is powerful. If $|f(r\cos\theta, r\sin\theta) - L| \leq g(r)$ where $g(r) \to 0$ independent of θ, then the limit is L.

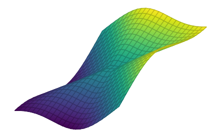


Figure 4.4: The surface $z = \frac{x^3+y^3}{x^2+y^2}$. In polar form, $z = r(\cos^3\theta + \sin^3\theta) \to 0$ as $r \to 0$, independent of θ.

Example 4.2. Polar Coordinate Calculation. Find

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}.$$

Let $x = r\cos\theta$ and $y = r\sin\theta$.

$$\frac{x^3+y^3}{x^2+y^2} = \frac{r^3(\cos^3\theta + \sin^3\theta)}{r^2} = r(\cos^3\theta + \sin^3\theta).$$

Since $|\cos^3\theta + \sin^3\theta| \leq |\cos\theta|^3 + |\sin\theta|^3 \leq 2$, we have:

$$\left|\frac{x^3+y^3}{x^2+y^2}\right| \leq 2r.$$

As $(x,y) \to (0,0)$, $r \to 0$. Therefore the limit is 0.

範例

Squeeze Theorem and Inequalities

Basic inequalities such as $|xy| \leq \frac{1}{2}(x^2+y^2)$ or $|\sin z| \leq |z|$ are often used to bound functions.

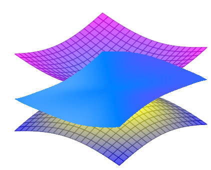


Figure 4.5: The squeeze theorem: $|f| \leq 2r$ (cone) forces $f \to 0$ as $r \to 0$.

Example 4.3. Squeeze Theorem Application. Find

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^3+y^3)}{x^2+y^2}.$$

Using the inequality $|\sin z| \leq |z|$, we have:

$$\left|\frac{\sin(x^3+y^3)}{x^2+y^2}\right| \leq \frac{|x^3+y^3|}{x^2+y^2} \leq \frac{|x|^3+|y|^3}{x^2+y^2}.$$

Note that $|x| \leq \sqrt{x^2+y^2} = r$ and $|y| \leq r$. Thus $|x|^3 + |y|^3 \leq 2r^3$.

$$\frac{|x|^3+|y|^3}{x^2+y^2} \leq \frac{2r^3}{r^2} = 2r = 2\sqrt{x^2+y^2}.$$

As $(x,y) \to (0,0)$, the upper bound approaches 0. The limit is 0.

範例

Example 4.4. Piecewise Function and Continuity. Let

$$f(x,y) = \begin{cases} \frac{\sin xy}{x} & x \neq 0 \\ y & x = 0 \end{cases}.$$

Find $\lim_{(x,y)\to(0,0)} f(x,y)$ and discuss continuity at the origin.

For $x \neq 0$, we have $|\frac{\sin xy}{x}| = |y||\frac{\sin xy}{xy}|$. Using the inequality $|\sin u| \leq |u|$ gives $|\frac{\sin xy}{x}| \leq \frac{|xy|}{|x|} = |y|$. For $x = 0$, $f(0,y) = y$, so $|f(0,y)| = |y|$. In all cases, $|f(x,y)| \leq |y|$. Since $\lim_{(x,y)\to(0,0)} |y| = 0$, the limit is 0 by the squeeze theorem. Since $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$, the function is continuous at the origin.

範例

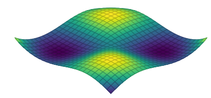


Figure 4.6: The surface $z = (x^2+y^2)^{xy}$ near the origin. Despite the indeterminate form 0^0, the limit is 1.

Example 4.5. Exponential Limit. Find

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)^{xy}.$$

We examine the logarithm $L = \ln\left((x^2+y^2)^{xy}\right) = xy\ln(x^2+y^2)$. Using polar coordinates, $x^2+y^2 = r^2$ and $|xy| \leq r^2$.

$$|xy\ln(x^2+y^2)| \leq r^2|\ln(r^2)| = 2r^2|\ln r|.$$

Using L'Hôpital's rule (single variable), $\lim_{r\to 0^+} r^2 \ln r = 0$. Thus the log-limit is 0, and the original limit is $e^0 = 1$.

範例

Example 4.6. Limit at Infinity. Prove

$$\lim_{(x,y)\to(+\infty,+\infty)} \left(\frac{xy}{x^2+y^2}\right)^{x^2} = 0.$$

For $x, y > 0$, we have $x^2+y^2 \geq 2xy$, so $0 < \frac{xy}{x^2+y^2} \leq \frac{1}{2}$. Therefore,

$$0 < \left(\frac{xy}{x^2+y^2}\right)^{x^2} \leq \left(\frac{1}{2}\right)^{x^2}.$$

As $x \to +\infty$, $(1/2)^{x^2} \to 0$. By the squeeze theorem, the limit is 0.

範例

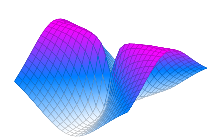


Figure 4.7: The surface $z = \frac{x^2-y^2}{x^2+y^2}$. The iterated limits differ: $\lim_{x\to 0}\lim_{y\to 0} = 1$ but $\lim_{y\to 0}\lim_{x\to 0} = -1$.

4.2 *Iterated Limits*

While the double limit considers approach from all directions simultaneously, **iterated limits** involve taking limits with respect to one variable at a time.

Definition 4.3. ***Iterated Limits.***
The iterated limits of $f(x,y)$ at (x_0, y_0) are:

$$L_{12} = \lim_{x\to x_0}\left(\lim_{y\to y_0} f(x,y)\right) \quad \text{and} \quad L_{21} = \lim_{y\to y_0}\left(\lim_{x\to x_0} f(x,y)\right).$$

定義

The existence of a double limit imposes strict conditions on iterated limits, but the converse is not true.

Proposition 4.1. ***Relationship between Double and Iterated Limits.***
Let $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = A$.

1. If $\lim_{x\to x_0} f(x,y)$ exists for each $y \neq y_0$, then $\lim_{y\to y_0}\lim_{x\to x_0} f(x,y) = A$.
2. If $\lim_{y\to y_0} f(x,y)$ exists for each $x \neq x_0$, then $\lim_{x\to x_0}\lim_{y\to y_0} f(x,y) = A$.

命題

In other words, if the double limit exists, any iterated limit that can be formed must equal the double limit.

Note

Counter-examples:

- **Iterated exist, Double does not:** $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ at $(0,0)$.

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0}\frac{x^2}{x^2} = 1.$$

$$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{y\to 0}\frac{-y^2}{y^2} = -1.$$

  Since $1 \neq -1$, the double limit cannot exist.

- **Double exists, Iterated do not:** $f(x,y) = x\sin(1/y)$ at $(0,0)$ (with $f = 0$ on axes). $|f(x,y)| \leq |x| \to 0$, so the double limit is 0. However, for fixed $x \neq 0$, $\lim_{y\to 0} x\sin(1/y)$ does not exist.

Interchanging the Order of Limits

A fundamental question in analysis is when limit operations commute. *proposition* 4.1 gives a sufficient condition if the double limit is known to exist. If the double limit is unknown, we require **uniform convergence** of the inner limit.

Proposition 4.2. ***Moore-Osgood Theorem for Iterated Limits.***
Let $f(x,y)$ be defined on a deleted neighbourhood of (x_0, y_0). Suppose:

1. For each x, the limit $\lim_{y \to y_0} f(x, y) = g(x)$ exists.
2. The limit $\lim_{x \to x_0} f(x, y) = h(y)$ exists **uniformly** with respect to y. That is, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < |x - x_0| < \delta$ implies $|f(x, y) - h(y)| < \varepsilon$ for all y.

Then both iterated limits exist and are equal:

$$\lim_{x \to x_0} g(x) = \lim_{y \to y_0} h(y).$$

命题

Proof

We prove this directly using the ε characterisation. Let $\lim_{x \to x_0} f(x, y) = h(y)$ uniformly. By the Cauchy criterion for uniform convergence, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x', x'' \in \mathring{B}_\delta(x_0)$, we have

$$|f(x', y) - f(x'', y)| < \frac{\varepsilon}{2} \quad \text{for all } y.$$

Taking the limit as $y \to y_0$ (which implies $f(x, y) \to g(x)$), we obtain:

$$|g(x') - g(x'')| \leq \frac{\varepsilon}{2} < \varepsilon.$$

Thus, $g(x)$ satisfies the Cauchy criterion as $x \to x_0$, so $\lim_{x \to x_0} g(x)$ exists. Let this limit be A. We now show $\lim_{y \to y_0} h(y) = A$. Fix $\varepsilon > 0$.

1. Since $g(x) \to A$, choose δ_1 such that $0 < |x - x_0| < \delta_1 \implies |g(x) - A| < \varepsilon/3$.
2. Since $f(x, y) \to h(y)$ uniformly, choose δ_2 such that $0 < |x - x_0| < \delta_2 \implies |f(x, y) - h(y)| < \varepsilon/3$ for all y.

Let $\delta = \min(\delta_1, \delta_2)$. Fix an $x^* \in \mathring{B}_\delta(x_0)$. Since $\lim_{y \to y_0} f(x^*, y) = g(x^*)$, there exists $\eta > 0$ such that $0 < |y - y_0| < \eta \implies |f(x^*, y) - g(x^*)| < \varepsilon/3$. Then, for y in this range:

$$\begin{aligned} |h(y) - A| &\leq |h(y) - f(x^*, y)| + |f(x^*, y) - g(x^*)| + |g(x^*) - A| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Thus, $\lim_{y \to y_0} h(y) = A$.

■

Sequential Characterisation

Just as in $\mathbb{R}$, limits of functions in $\mathbb{R}^n$ can be characterised by sequences.

Theorem 4.4. ***Heine's Principle for Multivariate Limits.***
Let $f : D \to \mathbb{R}$ and $\mathbf{a}$ be a cluster point of D. Then $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$ if and only if for every sequence $\{\mathbf{x}_k\} \subset D \setminus \{\mathbf{a}\}$ converging to $\mathbf{a}$, the sequence of values $\{f(\mathbf{x}_k)\}$ converges to L.

定理

$(\implies)$

Assume the limit is L. For any $\varepsilon > 0$, there exists $\delta > 0$ such that $\mathbf{x} \in \mathring{B}_\delta(\mathbf{a}) \implies |f(\mathbf{x}) - L| < \varepsilon$. Since $\mathbf{x}_k \to \mathbf{a}$, there exists K such that for $k > K$, $\|\mathbf{x}_k - \mathbf{a}\| < \delta$. Thus $|f(\mathbf{x}_k) - L| < \varepsilon$, so $f(\mathbf{x}_k) \to L$.

証明終

$(\impliedby)$

We proceed by contrapositive. Suppose $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) \neq L$. Then there exists $\varepsilon_0 > 0$ such that for every $\delta = 1/k$, there is a point $\mathbf{x}_k \in \mathring{B}_{1/k}(\mathbf{a})$ with $|f(\mathbf{x}_k) - L| \geq \varepsilon_0$. The sequence $\{\mathbf{x}_k\}$ converges to $\mathbf{a}$ (by construction), but $\{f(\mathbf{x}_k)\}$ does not converge to L.

証明終

This principle is particularly useful for proving that a limit does *not* exist: simply find two sequences converging to $\mathbf{a}$ that yield different limits for $f(\mathbf{x})$.

Theorem 4.5. ***Cauchy Convergence Criterion.***
A function $f(\mathbf{x})$ has a limit at $\mathbf{a}$ if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\mathbf{x}', \mathbf{x}'' \in \mathring{B}_\delta(\mathbf{a})$:

$$|f(\mathbf{x}') - f(\mathbf{x}'')| < \varepsilon.$$

定理

$(\Rightarrow)$

Assume that $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$ exists. Let $\varepsilon > 0$ be arbitrary. By the ε–δ definition of the limit, there exists $\delta > 0$ such that

$$0 < \|\mathbf{x} - \mathbf{a}\| < \delta \implies |f(\mathbf{x}) - L| < \frac{\varepsilon}{2}.$$

Now take any $\mathbf{x}', \mathbf{x}'' \in \mathring{B}_\delta(\mathbf{a})$, so that

$$0 < \|\mathbf{x}' - \mathbf{a}\| < \delta, \qquad 0 < \|\mathbf{x}'' - \mathbf{a}\| < \delta.$$

Then

$$|f(\mathbf{x}') - f(\mathbf{x}'')| \leq |f(\mathbf{x}') - L| + |L - f(\mathbf{x}'')| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This is exactly the desired Cauchy-type condition.

証明終

($\Leftarrow$)
Assume now that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\mathbf{x}', \mathbf{x}'' \in \mathring{B}_\delta(\mathbf{a}) \implies |f(\mathbf{x}') - f(\mathbf{x}'')| < \varepsilon.$$

We must show that $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})$ exists. We use the sequential characterisation of limits (Heine's Principle) proved earlier: it is enough to prove that for every sequence $\{\mathbf{x}_k\} \subset D \setminus \{\mathbf{a}\}$ with $\mathbf{x}_k \to \mathbf{a}$, the sequence $\{f(\mathbf{x}_k)\}$ converges, and that the limit does not depend on the particular sequence.

Step 1: Each such sequence $\{f(\mathbf{x}_k)\}$ is Cauchy (hence convergent). Let $\{\mathbf{x}_k\}$ be any sequence in $D \setminus \{\mathbf{a}\}$ with $\mathbf{x}_k \to \mathbf{a}$. Fix $\varepsilon > 0$, and choose $\delta > 0$ as in the hypothesis. Since $\mathbf{x}_k \to \mathbf{a}$, there exists K such that

$$k \geq K \implies \|\mathbf{x}_k - \mathbf{a}\| < \delta.$$

Thus for all $m, n \geq K$, we have $\mathbf{x}_m, \mathbf{x}_n \in \mathring{B}_\delta(\mathbf{a})$ and hence

$$|f(\mathbf{x}_m) - f(\mathbf{x}_n)| < \varepsilon.$$

So $\{f(\mathbf{x}_k)\}$ is a Cauchy sequence in $\mathbb{R}$ and therefore converges (since $\mathbb{R}$ is complete). Denote this limit by $L(\{\mathbf{x}_k\})$ for now.

Step 2: The limit does not depend on the choice of sequence. Let $\{\mathbf{x}_k\}$ and $\{\mathbf{y}_k\}$ be two sequences in $D \setminus \{\mathbf{a}\}$ with $\mathbf{x}_k \to \mathbf{a}$ and $\mathbf{y}_k \to \mathbf{a}$. Consider the interleaved sequence

$$\mathbf{z}_1 = \mathbf{x}_1, \quad \mathbf{z}_2 = \mathbf{y}_1, \quad \mathbf{z}_3 = \mathbf{x}_2, \quad \mathbf{z}_4 = \mathbf{y}_2, \quad \ldots$$

Then $\{\mathbf{z}_k\} \subset D \setminus \{\mathbf{a}\}$ and $\mathbf{z}_k \to \mathbf{a}$. By Step 1, $\{f(\mathbf{z}_k)\}$ is Cauchy in $\mathbb{R}$ and thus converges to some value L. But $\{f(\mathbf{x}_k)\}$ and $\{f(\mathbf{y}_k)\}$ are subsequences of the convergent sequence $\{f(\mathbf{z}_k)\}$. Therefore they both converge to the same limit L. Hence for *every* sequence $\{\mathbf{x}_k\} \subset D \setminus \{\mathbf{a}\}$ with $\mathbf{x}_k \to \mathbf{a}$, the sequence $\{f(\mathbf{x}_k)\}$ converges to the same real number L. By Heine's Principle for multivariate limits, this implies

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L.$$

証明終

This allows us to prove the existence of a limit without knowing its value, leveraging the completeness of $\mathbb{R}$.

4.3 *Continuity of Functions*

We now turn to the concept of continuity. Intuitively, a function is continuous if small changes in the input result in small changes in the output. In $\mathbb{R}^n$, this definition remains formally identical to the single-variable case, yet the implications of the multidimensional domain are far richer.

Definition and Basic Properties

Definition 4.4. ***Continuity at a Point.***
Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and let $\mathbf{a} \in D$ be a cluster point of D. The function f is **continuous** at $\mathbf{a}$ if:

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a}).$$

In the language of neighbourhoods, f is continuous at $\mathbf{a}$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$\mathbf{x} \in D \cap B_\delta(\mathbf{a}) \implies f(\mathbf{x}) \in B_\varepsilon(f(\mathbf{a})).$$

If f is continuous at every point in a set $S \subseteq D$, we say f is **continuous on** S.

定義

Note

If $\mathbf{a}$ is an isolated point of D, the limit condition is vacuously satisfied (there are no sequences in $D \setminus \{\mathbf{a}\}$ converging to $\mathbf{a}$), and f is automatically continuous there. However, in calculus, we almost exclusively deal with regions where every point is a cluster point.

The algebraic properties of limits transfer directly to continuous functions.

Proposition 4.3. ***Algebra of Continuous Functions.***
Let $f, g : D \to \mathbb{R}$ be continuous at $\mathbf{a}$, and let $c \in \mathbb{R}$. Then the following functions are continuous at $\mathbf{a}$:

1. **Scalar multiplication:** cf.
2. **Sum and Difference:** $f \pm g$.
3. **Product:** fg.
4. **Quotient:** f/g (provided $g(\mathbf{a}) \neq 0$).

Furthermore, if $f : D \to E \subseteq \mathbb{R}^m$ is continuous at $\mathbf{a}$, and $g : E \to \mathbb{R}^k$ is continuous at $f(\mathbf{a})$, then the composition $g \circ f$ is continuous at $\mathbf{a}$.

命題

Partial Continuity vs. Joint Continuity

A common misconception is that a function $f(x, y)$ is continuous if it is continuous in x (for fixed y) and continuous in y (for fixed x). This condition, known as *separate* or *partial continuity*, is **not** sufficient for *joint continuity* (continuity as a function of the vector $\mathbf{x}$).

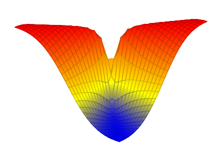


Figure 4.8: The function $z = \frac{xy}{x^2+y^2}$ is separately continuous (continuous along each axis) but discontinuous at the origin.

Example 4.7. Separate Continuity does not imply Joint Continuity. Consider the function discussed in the previous section:

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

- Fix $y = 0$: $f(x,0) = 0$ for all x. This is a constant function, hence continuous in x.
- Fix $x = 0$: $f(0,y) = 0$ for all y. This is continuous in y.

Thus, f is separately continuous at the origin. However, approaching along the line $y = x$ yields a limit of $1/2 \neq f(0,0)$. The function is discontinuous at the origin.

範例

For separate continuity to imply joint continuity, stronger conditions are required, such as monotonicity or uniform continuity in one variable.

Proposition 4.4. ***Sufficient Condition for Joint Continuity.*** Let $f(x,y)$ be defined on a region D. If f is continuous in y for each fixed x, and continuous in x **uniformly** with respect to y, then f is jointly continuous.

命題

Proof

Let $(x_0,y_0) \in D$. We wish to bound $|f(x,y) - f(x_0,y_0)|$. By the triangle inequality:

$$|f(x,y) - f(x_0,y_0)| \leq |f(x,y) - f(x_0,y)| + |f(x_0,y) - f(x_0,y_0)|.$$

Let $\varepsilon > 0$.

1. **Uniform continuity in x:** There exists $\delta_1 > 0$ (independent of y) such that $|x - x_0| < \delta_1$ implies $|f(x,y) - f(x_0,y)| < \varepsilon/2$.
2. **Continuity in y:** For the fixed x_0, there exists $\delta_2 > 0$ such that $|y - y_0| < \delta_2$ implies $|f(x_0,y) - f(x_0,y_0)| < \varepsilon/2$.

Choosing $\delta = \min(\delta_1,\delta_2)$, if $(x,y) \in B_\delta(x_0,y_0)$, both terms are bounded by $\varepsilon/2$. Thus the sum is bounded by ε.

■

Topological Characterisation

In the topology review, we defined open and closed sets. Continuity provides the natural link between topological structures of the domain and codomain. In advanced analysis, the following proposition often serves as the *definition* of continuity.

Theorem 4.6. ***Topological Continuity.***
Let $f : \mathbb{R}^n \to \mathbb{R}^m$. The following are equivalent:

1. f is continuous on $\mathbb{R}^n$.
2. For every open set $V \subseteq \mathbb{R}^m$, the preimage $f^{-1}(V)$ is open in $\mathbb{R}^n$.
3. For every closed set $C \subseteq \mathbb{R}^m$, the preimage $f^{-1}(C)$ is closed in $\mathbb{R}^n$.
4. For every subset $E \subseteq \mathbb{R}^n$, $f(\overline{E}) \subseteq \overline{f(E)}$.

定理

We prove the cycle $(1) \implies (2) \implies (3) \implies (4) \implies (1)$.

(1) $\implies$ (2)

Let V be open. If $f^{-1}(V)$ is empty, it is open. Suppose $\mathbf{x}_0 \in f^{-1}(V)$. Then $f(\mathbf{x}_0) \in V$. Since V is open, there is an ε-ball $B_\varepsilon(f(\mathbf{x}_0)) \subseteq V$. By continuity, there exists $\delta > 0$ such that $f(B_\delta(\mathbf{x}_0)) \subseteq B_\varepsilon(f(\mathbf{x}_0)) \subseteq V$. Thus $B_\delta(\mathbf{x}_0) \subseteq f^{-1}(V)$, proving $f^{-1}(V)$ is open.

証明終

(2) $\implies$ (3)

Let $C \subseteq \mathbb{R}^m$ be closed. Then its complement $V = \mathbb{R}^m \setminus C$ is open. By (2), the preimage $f^{-1}(V)$ is open. Since preimages preserve set operations, we have $f^{-1}(V) = f^{-1}(\mathbb{R}^m \setminus C) = \mathbb{R}^n \setminus f^{-1}(C)$. Since the complement of $f^{-1}(C)$ is open, $f^{-1}(C)$ must be closed.

証明終

(3) $\implies$ (4)

Note that $f(E) \subseteq \overline{f(E)}$. Taking preimages, $E \subseteq f^{-1}(\overline{f(E)})$. The set $\overline{f(E)}$ is closed, so by (3), $f^{-1}(\overline{f(E)})$ is closed. Since the closure $\overline{E}$ is the smallest closed set containing E, we must have $\overline{E} \subseteq f^{-1}(\overline{f(E)})$. Applying f to both sides yields $f(\overline{E}) \subseteq \overline{f(E)}$.

証明終

(4) $\implies$ (1)

We prove the contrapositive. Suppose f is discontinuous at $\mathbf{a}$. By Heine's principle, there exists a sequence $\mathbf{x}_k \to \mathbf{a}$ such that $f(\mathbf{x}_k)$ does not converge to $f(\mathbf{a})$. This implies there is an ε_0 such that $|f(\mathbf{x}_k) - f(\mathbf{a})| \geq \varepsilon_0$ for infinitely many k (passing to a subsequence if necessary). Let $E = \{\mathbf{x}_k\}$. Then $\mathbf{a} \in \overline{E}$. However, $f(\mathbf{a})$ is separated from the set $f(E)$ by ε_0, so $f(\mathbf{a}) \notin \overline{f(E)}$. This contradicts (4).

証明終

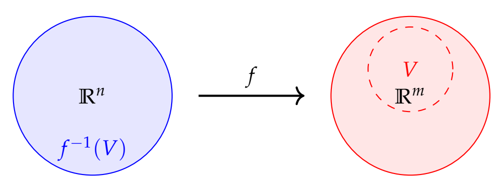


Figure 4.9: Topological continuity: The preimage of an open set V is always open.

Continuity on Compact and Connected Sets

The topological definitions allow us to swiftly extend the Extreme Value Theorem and Intermediate Value Theorem to higher dimensions.

Theorem 4.7. ***Preservation of Compactness.***
Let $K \subseteq \mathbb{R}^n$ be a compact set and $f : K \to \mathbb{R}^m$ be continuous. Then the image $f(K)$ is compact.

定理

Proof

Let $\{V_\alpha\}$ be an open cover of $f(K)$. Since f is continuous, each $f^{-1}(V_\alpha)$ is open in K. The collection $\{f^{-1}(V_\alpha)\}$ covers K. Since K is compact, there exists a finite subcover corresponding to indices $\alpha_1, \ldots, \alpha_k$. The sets $V_{\alpha_1}, \ldots, V_{\alpha_k}$ then cover $f(K)$. Thus $f(K)$ is compact.

■

Corollary 4.1. Extreme Value Theorem

推論

If $f : K \to \mathbb{R}$ is continuous on a compact set K, then f is bounded and attains its maximum and minimum values on K.

Proof

By the theorem, $f(K)$ is a compact subset of $\mathbb{R}$. By the Heine-Borel theorem in $\mathbb{R}$, $f(K)$ is closed and bounded. A closed bounded set in $\mathbb{R}$ contains its supremum and infimum.

■

Theorem 4.8. ***Uniform Continuity (Heine-Cantor).***
If $f : K \to \mathbb{R}^m$ is continuous on a compact set K, then f is **uniformly continuous** on K. That is, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\mathbf{x}, \mathbf{y} \in K$:

$$\|\mathbf{x} - \mathbf{y}\| < \delta \implies \|f(\mathbf{x}) - f(\mathbf{y})\| < \varepsilon.$$

定理

The proof mirrors the single-variable covering argument and is omitted here.

Theorem 4.9. ***Preservation of Connectedness.***
Let $E \subseteq \mathbb{R}^n$ be a connected set and $f : E \to \mathbb{R}^m$ be continuous. Then $f(E)$ is connected.

定理

Proof

Suppose $f(E)$ is disconnected. Then there exist disjoint open sets U, V in $\mathbb{R}^m$ separating $f(E)$. By continuity, $f^{-1}(U)$ and $f^{-1}(V)$ are open in E. They are disjoint and cover E, implying E is disconnected, a contradiction.

∎

Corollary 4.2. *Intermediate Value Theorem.* Let $f : E \subseteq \mathbb{R}^n \to \mathbb{R}$ be continuous on a connected set E. If $\mathbf{a}, \mathbf{b} \in E$ and $f(\mathbf{a}) < c < f(\mathbf{b})$, there exists $\mathbf{x} \in E$ such that $f(\mathbf{x}) = c$.

推論

Proof

The image $f(E)$ is a connected subset of $\mathbb{R}$. The only connected sets in $\mathbb{R}$ are intervals. Since $f(\mathbf{a}), f(\mathbf{b}) \in f(E)$, the interval $[f(\mathbf{a}), f(\mathbf{b})]$ is contained in $f(E)$. Thus $c \in f(E)$.

∎

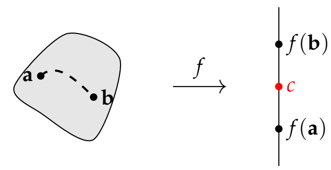


Figure 4.10: Since f maps the connected domain to a connected interval, every intermediate value c is attained.

Contraction Mapping Principle

We conclude this section with a result for vector-valued functions that underpins the solution of differential equations.

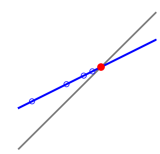


Figure 4.11: Contraction mapping iteration. The sequence $x_{k+1} = f(x_k)$ spirals toward the fixed point x^* where $y = x$ meets $y = f(x)$.

Definition 4.5. ***Contraction Mapping.***
A function $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a **contraction mapping** if there exists a constant $L \in (0, 1)$ such that for all $\mathbf{x}, \mathbf{y} \in \Omega$:

$$\|f(\mathbf{x}) - f(\mathbf{y})\| \leq L\|\mathbf{x} - \mathbf{y}\|.$$

定義

Note that any contraction mapping is uniformly continuous (it is Lipschitz with constant $L < 1$).

Proposition 4.5. ***Banach Fixed Point Theorem.***
Let Ω be a closed subset of $\mathbb{R}^n$ and $f : \Omega \to \Omega$ be a contraction mapping. Then f has a unique **fixed point** $\mathbf{x}^* \in \Omega$ (i.e., $f(\mathbf{x}^*) = \mathbf{x}^*$).

命題

Proof

Pick any $\mathbf{x}_0 \in \Omega$. Define the sequence $\mathbf{x}_{k+1} = f(\mathbf{x}_k)$. Observe that $\|\mathbf{x}_{k+1} - \mathbf{x}_k\| = \|f(\mathbf{x}_k) - f(\mathbf{x}_{k-1})\| \leq L\|\mathbf{x}_k - \mathbf{x}_{k-1}\|$. By induction, $\|\mathbf{x}_{k+1} - \mathbf{x}_k\| \leq L^k\|\mathbf{x}_1 - \mathbf{x}_0\|$. To show that $\{\mathbf{x}_k\}$ is a Cauchy sequence, let $m > n \geq 0$:

$$\|\mathbf{x}_m - \mathbf{x}_n\| \leq \sum_{j=n}^{m-1} \|\mathbf{x}_{j+1} - \mathbf{x}_j\| \leq \sum_{j=n}^{m-1} L^j \|\mathbf{x}_1 - \mathbf{x}_0\|$$
$$= \|\mathbf{x}_1 - \mathbf{x}_0\| L^n \sum_{j=0}^{m-n-1} L^j < \|\mathbf{x}_1 - \mathbf{x}_0\| \frac{L^n}{1 - L}.$$

Since $0 < L < 1$, $L^n \to 0$ as $n \to \infty$, so the sequence $\{\mathbf{x}_k\}$ is Cauchy. Since Ω is closed (and $\mathbb{R}^n$ is complete), the sequence converges to some $\mathbf{x}^* \in \Omega$. Continuity of f implies

$\mathbf{x}^* = \lim \mathbf{x}_{k+1} = \lim f(\mathbf{x}_k) = f(\mathbf{x}^*)$. Uniqueness follows easily: if $f(\mathbf{y}) = \mathbf{y}$, then $\|\mathbf{x}^* - \mathbf{y}\| = \|f(\mathbf{x}^*) - f(\mathbf{y})\| \leq L\|\mathbf{x}^* - \mathbf{y}\|$, which implies $(1 - L)\|\mathbf{x}^* - \mathbf{y}\| \leq 0$. Since $1 - L > 0$, we have $\mathbf{x}^* = \mathbf{y}$.

■

4.4 *Further Properties of Continuity*

We now consider the global behaviour of specific classes of functions, particularly polynomials and linear maps, and introduce the stronger concept of uniform continuity.

Polynomials and Rational Functions

The algebraic limit laws imply that continuity is preserved under elementary operations.

Proposition 4.6. ***Continuity of Polynomials.***
Any polynomial function $P : \mathbb{R}^n \to \mathbb{R}$ is continuous on $\mathbb{R}^n$. Any rational function $R(\mathbf{x}) = \frac{P(\mathbf{x})}{Q(\mathbf{x})}$ is continuous on its natural domain $D = \{\mathbf{x} \in \mathbb{R}^n \mid Q(\mathbf{x}) \neq 0\}$.

命题

Proof

The coordinate projection functions $\pi_i(\mathbf{x}) = x_i$ are clearly continuous (given ε, take $\delta = \varepsilon$). A monomial $M(\mathbf{x}) = c x_1^{k_1} \ldots x_n^{k_n}$ is a product of constants and coordinate functions, hence continuous by the product rule for limits. A polynomial is a finite sum of monomials, hence continuous by the sum rule. A rational function is a quotient of polynomials, hence continuous where the denominator is non-zero.

■

Uniform Continuity

Continuity is a local property: for a given ε, the required δ depends on the point $\mathbf{x}$. If a single δ works for the entire domain, the function is *uniformly continuous.*

Definition 4.6. ***Uniform Continuity.***
Let $D \subseteq \mathbb{R}^n$. A function $f : D \to \mathbb{R}^m$ is **uniformly continuous** on D if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\mathbf{x}, \mathbf{y} \in D$:

$$\|\mathbf{x} - \mathbf{y}\| < \delta \implies \|f(\mathbf{x}) - f(\mathbf{y})\| < \varepsilon.$$

定義

We previously stated that continuous functions on compact sets are uniformly continuous. Here, we investigate a class of functions that are uniformly continuous on *all* of $\mathbb{R}^n$: linear maps.
To analyse linear maps, we introduce the **operator norm**.

Definition 4.7. ***Operator Norm.***
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. The operator norm of T, denoted $\|T\|$, is defined as:

$$\|T\| = \sup_{\mathbf{x}\neq\mathbf{0}} \frac{\|T\mathbf{x}\|}{\|\mathbf{x}\|} = \sup_{\|\mathbf{x}\|=1} \|T\mathbf{x}\|.$$

定義

The supremum exists because the unit sphere is compact and the map $\mathbf{x} \mapsto \|T\mathbf{x}\|$ is continuous. This definition implies the inequality $\|T\mathbf{x}\| \leq \|T\|\|\mathbf{x}\|$ for all $\mathbf{x}$.

Theorem 4.10. ***Linear Maps are Uniformly Continuous.***
Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is uniformly continuous on $\mathbb{R}^n$.

定理

Proof

Let $\varepsilon > 0$. If T is the zero map, the result is trivial. Assume $\|T\| > 0$. Choose $\delta = \frac{\varepsilon}{\|T\|}$. Then for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ with $\|\mathbf{x} - \mathbf{y}\| < \delta$:

$$\|T\mathbf{x} - T\mathbf{y}\| = \|T(\mathbf{x} - \mathbf{y})\| \leq \|T\|\|\mathbf{x} - \mathbf{y}\| < \|T\| \cdot \frac{\varepsilon}{\|T\|} = \varepsilon.$$

Thus, T is uniformly continuous.

■

4.5 *Series and Matrix Functions*

Finally the concepts of convergence extend naturally from sequences of points to series of vectors and matrices. This generalisation is crucial for the study of differential equations, where solutions are often expressed as matrix exponentials.

Series of Vectors

Definition 4.8. ***Convergent Series.***
A series of vectors $\sum_{k=1}^{\infty} \mathbf{a}_k$ in $\mathbb{R}^n$ converges to a vector $\mathbf{s}$ if the sequence of partial sums $\mathbf{s}_m = \sum_{k=1}^{m} \mathbf{a}_k$ converges to $\mathbf{s}$ as $m \to \infty$.

定義

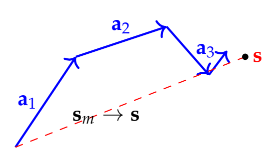


Figure 4.12: Geometric interpretation of a convergent vector series. The partial sums trace a path converging to the limit point $\mathbf{s}$.

Proposition 4.7. ***Absolute Convergence Implies Convergence.***
If the series of scalars $\sum_{k=1}^{\infty} \|\mathbf{a}_k\|$ converges, then the series of vectors $\sum_{k=1}^{\infty} \mathbf{a}_k$ converges.

命題

Proof

Let $S_m = \sum_{k=1}^{m} \|\mathbf{a}_k\|$. Since this scalar series converges, $\{S_m\}$ is a Cauchy sequence. For the vector partial sums $\mathbf{s}_m$, consider $p > q$:

$$\|\mathbf{s}_p - \mathbf{s}_q\| = \left\| \sum_{k=q+1}^{p} \mathbf{a}_k \right\| \leq \sum_{k=q+1}^{p} \|\mathbf{a}_k\| = |S_p - S_q|.$$

Since $\{S_m\}$ is Cauchy, for any $\varepsilon > 0$, there exists N such that for $p, q > N$, $|S_p - S_q| < \varepsilon$. It follows that $\|\mathbf{s}_p - \mathbf{s}_q\| < \varepsilon$. Thus $\{\mathbf{s}_m\}$ is a Cauchy sequence in $\mathbb{R}^n$. By the completeness of $\mathbb{R}^n$, the series converges.

■

Matrix Series and the Neumann Series

The space of $n \times n$ matrices, denoted $M_n(\mathbb{R})$, can be identified with $\mathbb{R}^{n^2}$. Thus, notions of convergence apply to matrices entry-wise. However, it is often more powerful to use the operator norm. The operator norm satisfies the sub-multiplicative property:

$$\|AB\| \leq \|A\|\|B\|.$$

From this, it follows that $\|A^k\| \leq \|A\|^k$.

Proposition 4.8. ***Geometric Series of Matrices.***
Let $A \in M_n(\mathbb{R})$. If $\|A\| < 1$, then the series $\sum_{k=0}^{\infty} A^k$ converges to $(I - A)^{-1}$.

命題

Proof

Since $\|A\| < 1$, the scalar series $\sum \|A\|^k = \sum \|A^k\|$ converges (it is a geometric series). Thus, the matrix series $S = \sum_{k=0}^{\infty} A^k$ converges absolutely. To show the sum is the inverse of $I - A$, consider the partial sum $S_m = \sum_{k=0}^{m} A^k$. Observe that:

$$S_m(I - A) = (I + A + \cdots + A^m)(I - A) = I - A^{m+1}.$$

Since $\|A\| < 1$, $\lim_{m \to \infty} A^{m+1} = O$ (the zero matrix). Taking the limit as $m \to \infty$:

$$S(I - A) = I.$$

Similarly, $(I - A)S = I$. Thus $S = (I - A)^{-1}$.

■

This result has profound topological implications for the general linear group $GL_n(\mathbb{R})$, the set of all invertible $n \times n$ matrices.

Theorem 4.11. ***Openness of the General Linear Group.***
The set of invertible matrices $GL_n(\mathbb{R})$ is an open subset of $M_n(\mathbb{R})$.
定理

Proof

Let $B \in GL_n(\mathbb{R})$. We wish to show that any matrix sufficiently close to B is also invertible. Let $H \in M_n(\mathbb{R})$ be a perturbation matrix such that $\|H\| < \frac{1}{\|B^{-1}\|}$. We can factor the perturbed matrix as:

$$B + H = B(I + B^{-1}H).$$

Let $A = -B^{-1}H$. Then $\|A\| \leq \|B^{-1}\|\|H\| < 1$. By the geometric series proposition, $I - A = I + B^{-1}H$ is invertible. Since B is invertible and the product of invertible matrices is invertible, $B + H$ is invertible. Thus, the ball of radius $1/\|B^{-1}\|$ centred at B is contained in $GL_n(\mathbb{R})$, proving the set is open.

■

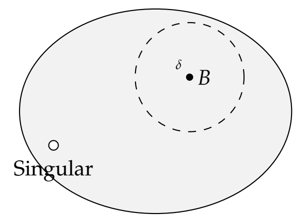


Figure 4.13: The set of invertible matrices is open; if you perturb an invertible matrix B by a sufficiently small amount, it remains invertible.

4.6 Exercises

1. **Classification of Sets.** For each of the following subsets, determine whether it is open, closed, both, or neither, and provide a justification.
   (a) $\{x \in \mathbb{R} \mid 0 < x \leq 1\}$ as a subset of $\mathbb{R}$.
   (b) $\{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| < 1\}$ as a subset of $\mathbb{R}^2$.
   (c) The interval $(0, 1]$ as a subset of $\mathbb{R}$.
   (d) $\{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| \leq 1\}$ as a subset of $\mathbb{R}^2$.
   (e) $\{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$ as a subset of $\mathbb{R}$.
   (f) The "punctured" unit ball $\{\mathbf{x} \in \mathbb{R}^3 \mid \|\mathbf{x}\| \leq 1, \mathbf{x} \neq \mathbf{0}\}$ as a subset of $\mathbb{R}^3$.
   (g) The empty set $\emptyset$ as a subset of $\mathbb{R}$.
   (h) The subspace $\mathbb{R} \times \{0\} \subset \mathbb{R}^2$ (the x-axis).
   (i) The rational numbers $\mathbb{Q}$ as a subset of $\mathbb{R}$.
   (j) The set $\{\mathbf{x} \in \mathbb{R}^2 \mid xy \neq 0\}$.
2. **Set Operations.** Let U be an open subset and F a closed subset of $\mathbb{R}^n$.
   (a) Prove that any union of open sets is open, but an infinite intersection of open sets is not necessarily open.
   (b) Prove that a finite intersection of open sets is open.
   (c) Show that $\text{int}(S)$ is the largest open set contained in S.
   (d) Show that $\overline{S}$ is the smallest closed set containing S.

(e) Verify the boundary identity: $\partial S = \overline{S} \setminus \operatorname{int}(S)$.
(f) Show that $\partial S = \overline{S} \cap \overline{S^c}$.

3. **Natural Domains.** Determine the natural domain $D \subseteq \mathbb{R}^n$ for the following expressions and determine if D is open, closed, or neither.

   (a) $f(x,y) = \sin(1/xy)$
   (b) $f(x,y) = \ln\sqrt{x^2 - y}$
   (c) $f(x) = \ln(\ln x)$
   (d) $f(x,y) = \arcsin\left(\frac{3}{x^2+y^2}\right)$
   (e) $f(x,y,z) = \frac{1}{xyz}$

4. **Boundedness.** Prove that a set $S \subseteq \mathbb{R}^n$ is bounded if and only if it is contained in a ball $B_R(\mathbf{x}_0)$ for some $\mathbf{x}_0 \in \mathbb{R}^n$. (i.e., the centre of the bounding ball need not be the origin).

5. **Computation of Limits.** Evaluate the following limits or prove that they do not exist.

   (a) $\lim_{(x,y)\to(0,a)} \dfrac{\sin xy}{x}$
   (b) $\lim_{(x,y)\to(0,0)} (x+y)\ln(x^2+y^2)$
   (c) $\lim_{(x,y)\to(0,0)} \left(1+\dfrac{1}{x}\right)^{\frac{x^2}{x+y}}$
   (d) $\lim_{(x,y)\to(0,0)} \dfrac{x+y}{x^2-xy+y^2}$
   (e) $\lim_{\mathbf{x}\to(1,2)} \dfrac{x^2}{x+y}$
   (f) $\lim_{\mathbf{x}\to\mathbf{0}} \dfrac{\sqrt{|x|}y}{x^2+y^2}$
   (g) $\lim_{\mathbf{x}\to\mathbf{0}} \dfrac{\sqrt{|xy|}}{\sqrt{x^2+y^2}}$

6. **Non-Existence.** Prove that $\lim_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^3+y^3}$ does not exist.

   > *Remark.*
   >
   > Consider paths of the form $y = mx$ versus non-linear paths, or examine the behaviour near $y = -x$.

7. **Iterated Limits.** Discuss the existence of the double limit and both iterated limits at the origin for the following functions:

   (a) $f(x,y) = \dfrac{x^2y^2}{x^2y^2+(x-y)^2}$
   (b) $f(x,y) = (x+y)\sin\dfrac{1}{x}\sin\dfrac{1}{y}$

(c) $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} + y\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

8. **Difference Quotients.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable. Define $g(x,y) = \frac{f(x)-f(y)}{x-y}$ for $x \neq y$. Determine $\lim\limits_{(x,y)\to(t,t)} g(x,y)$.

9. **Limit Theorems.**

   (a) Prove the Local Boundedness Theorem: If $\lim\limits_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$, then f is bounded on some neighbourhood of $\mathbf{a}$.

   (b) Prove the Local Sign Preservation Theorem: If $\lim\limits_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L > 0$, then $f(\mathbf{x}) > 0$ on some deleted neighbourhood of $\mathbf{a}$.

10. **Dominated Convergence.** Suppose $\lim\limits_{y\to y_0} \varphi(y) = A$ and $\lim\limits_{x\to x_0} \psi(x) = 0$. If $|f(x,y) - \varphi(y)| < \psi(x)$ holds in a neighbourhood of (x_0, y_0), prove that $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = A$.

11. **Continuity Extension.** For the following functions, determine if a value can be assigned at the origin to make the function continuous on $\mathbb{R}^2$.

    (a) $f(x,y) = \frac{1}{x^2+y^2+1}$

    (b) $f(x,y) = \frac{\sqrt{x^2+y^2}}{|x|+|y|^{1/3}}$

    (c) $f(x,y) = (x^2+y^2)\ln(x^2+2y^2)$

    (d) $f(x,y) = (x^2+y^2)\ln|x+y|$

12. **Open Maps.** If $f : \mathbb{R} \to \mathbb{R}$ has the property that for every open set $U \subset \mathbb{R}$, the image $f(U)$ is open, is f necessarily continuous? Provide a proof or a counter-example.

13. **Limits and Subsequences.** Prove that a sequence $\mathbf{a}_k \in \mathbb{R}^n$ converges to $\mathbf{a}$ if and only if every subsequence converges to $\mathbf{a}$.

14. **Triangle Inequality for Series.** Suppose $\sum_{i=1}^{\infty} \mathbf{x}_i$ is a convergent series in $\mathbb{R}^n$. Prove that $\|\sum_{i=1}^{\infty} \mathbf{x}_i\| \leq \sum_{i=1}^{\infty} \|\mathbf{x}_i\|$.

15. **The Matrix Exponential.** Let A be an $n \times n$ matrix. Define $e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$.

    (a) Show that the series converges for all A, and find a bound for $\|e^A\|$ in terms of $\|A\|$.

    (b) Compute e^A explicitly for: (i) $A = \text{diag}(a,b)$, (ii) $A = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$, (iii) $A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$.

    (c) Prove or disprove:
    - $e^{A+B} = e^A e^B$ for all A, B.
    - $e^{A+B} = e^A e^B$ if $AB = BA$.

- $e^{2A} = (e^A)^2$ for all A.

16. **Approximating the Identity.** Let $U = \{A \in M_2(\mathbb{R}) \mid \det(I - A) \neq 0\}$.
    (a) Show that U is open.
    (b) Let $f : U \to M_2(\mathbb{R})$ be defined by $f(A) = (A^2 - I)(A - I)^{-1}$. Does $\lim_{A \to I} f(A)$ exist?
    (c) Let $B = \text{diag}(1, -1)$. Let $V = \{A \in M_2(\mathbb{R}) \mid \det(A - B) \neq 0\}$. Define $g(A) = (A^2 - B^2)(A - B)^{-1}$. Does $\lim_{A \to B} g(A)$ exist?

17. **Inversion via Series.**
    (a) Let $B = \begin{bmatrix} 1 & \epsilon & \epsilon \\ 0 & 1 & \epsilon \\ 0 & 0 & 1 \end{bmatrix}$. Find B^{-1} by writing $B = I - N$ and using the geometric series $\sum N^k$.
    (b) Compute the inverse of $C = \begin{bmatrix} 1 & -\epsilon \\ \epsilon & 1 \end{bmatrix}$ for $|\epsilon| < 1$ using a similar method.

18. **Matrix Power Convergence.** Let $A = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$. For what values of $a \in \mathbb{R}$ does the sequence A^k converge? Generalise to $n \times n$ matrices where every entry is a.

19. **Derivative of the Inverse.** Let A be an invertible matrix. Discuss the existence of the limit $\lim_{B \to A} (A - B)^{-1}(A^2 - B^2)$. Does it exist for $A = I$? For $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$?

20. **Uniform Convergence of Compositions.** Let $f(x, y)$ be continuous on $[a, b] \times [c, d]$. Let $\{\varphi_n(x)\}$ be a sequence of functions converging uniformly on $[a, b]$ such that $c \leq \varphi_n(x) \leq d$. Prove that the sequence $F_n(x) = f(x, \varphi_n(x))$ converges uniformly on $[a, b]$.

21. **Uniform Continuity on $\mathbb{R}^2$.**
    (a) Prove that $f(x, y) = \sqrt{x^2 + y^2}$ is uniformly continuous on $\mathbb{R}^2$.
    (b) Suppose $f(x, y)$ is continuous on $\mathbb{R}^2$ and $\lim_{\|\mathbf{x}\| \to \infty} f(\mathbf{x})$ exists. Prove that f is bounded and uniformly continuous on $\mathbb{R}^2$.
    (c) Let $A = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}$. This defines a linear map $L(\mathbf{x}) = A\mathbf{x}$. Find the optimal δ in terms of ϵ for the uniform continuity condition of L.

22. **Dini's Theorem Generalisation.** Let $K \subset \mathbb{R}^n$ be compact. Let $\{f_k\}$ be a sequence of continuous functions on K converging pointwise to 0. Suppose the sequence is monotone, i.e., $f_1(\mathbf{x}) \geq f_2(\mathbf{x}) \geq$

$\cdots \geq 0$. Prove that $\{f_k\}$ converges uniformly to 0 on K.

23. **Modulus of Continuity.** The oscillation of f at $\mathbf{a}$, denoted $\omega_f(\mathbf{a})$, is defined as $\lim_{\delta \to 0^+} \sup_{\mathbf{x},\mathbf{y} \in B_\delta(\mathbf{a})} |f(\mathbf{x}) - f(\mathbf{y})|$. Show that this is equivalent to $\lim_{\delta \to 0^+} \sup_{\mathbf{x} \in B_\delta(\mathbf{a})} |f(\mathbf{x}) - f(\mathbf{a})|$ if f is continuous at $\mathbf{a}$. Discuss the relationship between $\omega_f(\mathbf{a}) = 0$ and continuity.

24. **Compact Images.** Let $A \subset \mathbb{R}^n$ be a non-compact set. Prove there exists a continuous unbounded function on A.

25. **Roots of Polynomials (Complex Analysis Preview).** Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ be a polynomial with complex coefficients.
    (a) Show that there exists $R > 0$ such that $|p(z)| > |p(0)|$ for all $|z| > R$.
    (b) Use the Extreme Value Theorem (on the disk $|z| \leq R$) to prove that $|p(z)|$ attains a minimum on $\mathbb{C}$.
    (c) (Harder) Using the fact that a minimum of $|p|$ must be a root (Fundamental Theorem of Algebra proof logic), construct an argument for the existence of a root for $p(z) = z^5 + 4z^3 + 3iz - 3$.

26. $\star$ **Separate vs Joint Continuity.** Construct a function $f : \mathbb{R}^2 \to \mathbb{R}$ that is continuous on every line passing through the origin (i.e., for any fixed θ, $g(r) = f(r\cos\theta, r\sin\theta)$ is continuous), yet f is discontinuous at the origin.

    > *Remark.*
    >
    > Consider functions that vanish on lines but have spikes on parabolic paths, such as $\frac{x^2y}{x^4+y^2}$.

27. $\star$ **Pathological Continuity.** Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 1/q$ if $x = p/q$ in lowest terms ($q > 0$), and $f(x) = 0$ if x is irrational (Thomae's function).
    (a) Prove that f is discontinuous at every rational number.
    (b) Prove that f is continuous at every irrational number.
    (c) Can a function be continuous *only* on the rational numbers?

5 Differentiation

In single-variable calculus, the derivative $f'(a)$ serves two primary roles: it describes the instantaneous rate of change (slope) of the function at a point, and it constructs the best linear approximation to the function near that point. Specifically, for a small change in the input Δx, the resulting change in output is approximated by $\Delta y \approx f'(a)\Delta x$.

This characterisation of the derivative as a coefficient of linear approximation generalises naturally to higher dimensions. For a function $f : \mathbb{R}^n \to \mathbb{R}^m$, the scalar $f'(a)$ is replaced by a matrix (often called the Jacobian), and the increments $\Delta \mathbf{x}$ and $\Delta \mathbf{y}$ become vectors. The entries of this derivative matrix are constructed from **partial derivatives**, which measure rates of change along the coordinate axes.

However, to rigorously define these concepts, we first consider the rate of change along an arbitrary line in the domain. This leads to the concept of the **directional derivative**.

5.1 Directional Derivatives

Consider a scalar field $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ and a point $\mathbf{p}_0 = (x_0, y_0)$ in the interior of D. To measure how f changes as we move away from $\mathbf{p}_0$, we must specify a direction. Let $\mathbf{u} = \langle a, b \rangle$ be a unit vector representing this direction.

Geometrically, we restrict the domain of f to the line passing through $\mathbf{p}_0$ parallel to $\mathbf{u}$. This path can be parameterised by:

$$\mathbf{r}(t) = \mathbf{p}_0 + t\mathbf{u} = (x_0 + ta, y_0 + tb).$$

Lifting this path to the graph of the function yields a curve on the surface $z = f(x, y)$ given by $\gamma(t) = (\mathbf{r}(t), f(\mathbf{r}(t)))$. The rate of change of f with respect to distance along this path is the slope of the tangent to $\gamma(t)$ at $t = 0$.

Definition 5.1. ***Directional Derivative.***
Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ be defined on an open set containing $\mathbf{p}_0$, and let $\mathbf{u} \in \mathbb{R}^n$ be a unit vector ($\|\mathbf{u}\| = 1$). The **directional derivative** of f at $\mathbf{p}_0$ in the direction $\mathbf{u}$ is defined as:

$$D_{\mathbf{u}} f(\mathbf{p}_0) = \lim_{t\to 0} \frac{f(\mathbf{p}_0 + t\mathbf{u}) - f(\mathbf{p}_0)}{t}.$$

Provided the limit exists, this may be expressed in terms of single-variable calculus as:

$$D_{\mathbf{u}} f(\mathbf{p}_0) = \frac{d}{dt}\left[f(\mathbf{p}_0 + t\mathbf{u})\right]\Big|_{t=0}.$$

定義

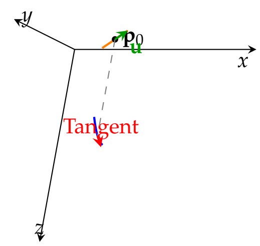


Figure 5.1: The directional derivative corresponds to the slope of the curve (blue) generated by slicing the surface $z = f(x, y)$ with the vertical plane defined by $\mathbf{p}_0$ and $\mathbf{u}$.

Note

Some authors define the directional derivative for arbitrary non-zero vectors $\mathbf{v}$. In that convention, $D_{\mathbf{v}} f$ scales with the length of $\mathbf{v}$. By restricting $\mathbf{u}$ to be a unit vector, our definition represents the rate of change *per unit distance*.

This definition reduces the multivariable problem to a single-variable derivative calculation, as demonstrated in the following example.

Example 5.1. Calculating a Directional Derivative. Let $f(x, y) = 25xy$. We wish to calculate the rate of change of f at the point $\mathbf{p}_0 = (1, 2)$ in the direction of the vector $\mathbf{v} = \langle 3, 4\rangle$.
First, we must normalise $\mathbf{v}$ to obtain a unit vector $\mathbf{u}$.

$$\|\mathbf{v}\| = \sqrt{3^2 + 4^2} = 5 \implies \mathbf{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle.$$

We parameterise the path:

$$\mathbf{p}_0 + t\mathbf{u} = \left(1 + \frac{3t}{5}, 2 + \frac{4t}{5}\right).$$

Substituting this into f:

$$g(t) = f\left(1 + \frac{3t}{5}, 2 + \frac{4t}{5}\right) = 25\left(1 + \frac{3t}{5}\right)\left(2 + \frac{4t}{5}\right).$$

Expanding the terms:

$$g(t) = 25\left(2 + \frac{4t}{5} + \frac{6t}{5} + \frac{12t^2}{25}\right) = 50 + 20t + 30t + 12t^2 = 50 + 50t + 12t^2.$$

Differentiating with respect to t:

$$g'(t) = 50 + 24t.$$

Evaluating at $t = 0$:

$$D_{\mathbf{u}} f(1, 2) = g'(0) = 50.$$

範例

Compare this result with the rates of change restricted to the Cartesian coordinate axes. These are the directional derivatives in the directions $\mathbf{e}_1 = \langle 1, 0\rangle$ and $\mathbf{e}_2 = \langle 0, 1\rangle$, commonly known as **partial derivatives**.

Example 5.2. Coordinate Direction Derivatives. Using the same function $f(x, y) = 25xy$ at $\mathbf{p}_0 = (1, 2)$:
Case 1: Direction $\mathbf{u} = \langle 1, 0\rangle$.

$$f(1+t, 2) = 25(1+t)(2) = 50 + 50t.$$

$$D_{\langle 1,0\rangle} f(1, 2) = \frac{d}{dt}[50 + 50t]\bigg|_{t=0} = 50.$$

Case 2: Direction $\mathbf{u} = \langle 0, 1\rangle$.

$$f(1, 2+t) = 25(1)(2+t) = 50 + 25t.$$

$$D_{\langle 0,1\rangle} f(1, 2) = \frac{d}{dt}[50 + 25t]\bigg|_{t=0} = 25.$$

Observe the relationship between these results and the previous example where $\mathbf{u} = \langle 3/5, 4/5\rangle$:

$$\frac{3}{5}(50) + \frac{4}{5}(25) = 30 + 20 = 50.$$

This suggests a fundamental linearity property:

$$D_{\langle a,b\rangle} f(\mathbf{p}_0) = aD_{\langle 1,0\rangle} f(\mathbf{p}_0) + bD_{\langle 0,1\rangle} f(\mathbf{p}_0).$$

範例

This observation—that the directional derivative is a linear combination of the derivatives along the coordinate axes—is not a coincidence. It forms the basis for the definition of the total derivative and the gradient vector in subsequent sections. While the definition extends to $\mathbb{R}^n$ without modification, the geometric intuition of slicing the graph with a plane is most vivid in two dimensions.

5.2 *Partial Differentiation in* $\mathbb{R}^2$

We continue the discussion of the previous section concerning the rates of change of functions of two variables. While the directional derivative allows us to analyse change in any direction $\mathbf{u}$, the most natural directions to consider are those aligned with the coordinate axes.

Definition and Basic Computations

Definition 5.2. ***Partial Derivative.***
Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a function and let $(x_0, y_0) \in D$. The **partial derivative** of f with respect to x at (x_0, y_0) is the directional derivative in the direction of the standard basis vector $\mathbf{e}_1 = \langle 1, 0 \rangle$:

$$\frac{\partial f}{\partial x}(x_0, y_0) = D_{\langle 1,0 \rangle} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}.$$

Similarly, the partial derivative with respect to y is the directional derivative in the direction $\mathbf{e}_2 = \langle 0, 1 \rangle$:

$$\frac{\partial f}{\partial y}(x_0, y_0) = D_{\langle 0,1 \rangle} f(x_0, y_0) = \lim_{k \to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k}.$$

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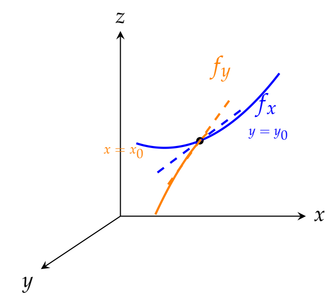

Figure 5.2: Partial derivatives as slopes of traces: f_x is the slope of the curve obtained by fixing $y = y_0$ (blue), while f_y is the slope when $x = x_0$ is fixed (orange).

If the partial derivatives exist at every point in a region, they define new functions $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. The subscript notation is used for brevity:

$$f_x = \frac{\partial f}{\partial x}, \quad f_y = \frac{\partial f}{\partial y}.$$

Higher-order derivatives are defined by successive differentiation:

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \quad \text{etc.}$$

Note

For most functions encountered in physical applications, the order of differentiation does not matter for mixed partials (i.e., $f_{xy} = f_{yx}$). This equality holds provided the mixed partial derivatives are continuous (Clairaut's Theorem), a fact we assume for the "smooth" functions in this chapter unless stated otherwise.

Operationally, calculating a partial derivative with respect to x is equivalent to treating y as a constant and differentiating with respect to x using single-variable rules.

Note

Conceptual Warning: In applied fields like thermodynamics or economics, the phrase "holding other variables constant" must be treated with care. A partial derivative $\partial f / \partial x$ is ambiguous unless the full set of independent variables is specified. For instance, the change in gas pressure with respect to temperature depends crucially on whether volume or entropy is held fixed.

Proposition 5.1. ***Properties of Partial Derivatives.***
Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be differentiable functions and $c \in \mathbb{R}$.
1. **Linearity:** $(cf + g)_x = cf_x + g_x$.

2. **Product Rule:** $(fg)_x = f_x g + f g_x$.
3. **Quotient Rule:** $(f/g)_x = \frac{f_x g - f g_x}{g^2}$ (where $g \neq 0$).
4. **Chain Rule (Scalar):** If $h : \mathbb{R} \to \mathbb{R}$ is differentiable, then

$$\frac{\partial}{\partial x}[h(f(x,y))] = h'(f(x,y))\frac{\partial f}{\partial x}.$$

Analogous rules hold for differentiation with respect to y.

命題

Proof

These follow immediately from the limit definition, which reduces to the single-variable derivative definition when one variable is fixed. For example, property (4) is simply the single-variable chain rule applied to the function $x \mapsto h(f(x, y_0))$.

■

Example 5.3. Asymmetry of Variables. Power functions and exponential functions behave differently depending on which variable is the base.

$$\frac{\partial}{\partial x}(x^y) = yx^{y-1} \quad \text{(Power rule: } y \text{ is constant)},$$
$$\frac{\partial}{\partial y}(x^y) = x^y \ln(x) \quad \text{(Exponential rule: } x \text{ is constant base)}.$$

Similarly:

$$\frac{\partial}{\partial x}(y^x) = y^x \ln(y),$$
$$\frac{\partial}{\partial y}(y^x) = xy^{x-1}.$$

範例

Example 5.4. Higher Order Derivatives. Let $f(x,y) = xy^2$. We calculate the second-order derivatives:

$$f_x = y^2, \qquad f_y = 2xy.$$
$$f_{xx} = \frac{\partial}{\partial x}(y^2) = 0, \qquad f_{yy} = \frac{\partial}{\partial y}(2xy) = 2x.$$
$$f_{xy} = \frac{\partial}{\partial y}(y^2) = 2y, \qquad f_{yx} = \frac{\partial}{\partial x}(2xy) = 2y.$$

Observe that $f_{xy} = f_{yx}$, consistent with the continuity of the derivatives.

範例

Directional Derivatives and the Gradient

In the previous section, we observed that for $f(x,y) = 25xy$, the directional derivative satisfied the linear relationship $D_{\mathbf{u}}f = af_x + bf_y$, where $\mathbf{u} = \langle a, b \rangle$. One might be tempted to conjecture that this formula holds for all functions. However, the existence of partial derivatives alone is insufficient to guarantee this structure.

Example 5.5. Failure of Linearity for Non-Smooth Functions. Consider the function

$$f(x,y) = \begin{cases} 1 & \text{if } y = 0 \text{ or } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

At the origin $(0,0)$:
- Along the x-axis ($y = 0$), $f(x,0) = 1$, so $f_x(0,0) = 0$.
- Along the y-axis ($x = 0$), $f(0,y) = 1$, so $f_y(0,0) = 0$.

However, in any direction $\mathbf{u} = \langle a, b \rangle$ where $a \neq 0$ and $b \neq 0$, $f(ta, tb) = 0$ for $t \neq 0$. The function jumps from 1 (at the origin) to 0 immediately. The limit defining $D_{\mathbf{u}}f$ diverges (or is technically $-\infty$ if approached carefully, but simply put, the function is discontinuous). Even if we modified f to be continuous but "kinked", linearity could fail.

範例

To resolve this, we require a stronger condition than simply the existence of partial derivatives.

Definition 5.3. ***Continuously Differentiable.***
A function f is **continuously differentiable**, denoted $f \in C^1$, at a point $\mathbf{p}$ if the partial derivatives f_x and f_y exist and are **continuous** functions in a neighbourhood of $\mathbf{p}$.

定義

For C^1 functions, the linear approximation property holds, leading to the following fundamental theorem. We introduce the **gradient** vector to capture the partial derivatives.

Definition 5.4. ***The Gradient.***
Let f be a function for which the partial derivatives exist. The **gradient** of f, denoted ∇f (or sometimes grad f), is the vector field:

$$\nabla f(x,y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = f_x \mathbf{i} + f_y \mathbf{j}.$$

定義

Theorem 5.1. ***Gradient Formula for Directional Derivatives.***
If f is continuously differentiable at $\mathbf{p}_0$, then for any unit vector $\mathbf{u} = \langle a, b \rangle$, the directional derivative is given by the dot product:

$$D_{\mathbf{u}} f(\mathbf{p}_0) = \nabla f(\mathbf{p}_0) \cdot \mathbf{u}.$$

定理

This formula provides an immediate geometric interpretation of the gradient. Let θ be the angle between ∇f and the direction vector $\mathbf{u}$. Then:

$$D_{\mathbf{u}} f = \|\nabla f\| \|\mathbf{u}\| \cos\theta = \|\nabla f\| \cos\theta.$$

Proposition 5.2. ***Geometric Properties of the Gradient.***
Let f be differentiable at $\mathbf{p}$.

1. **Maximum Increase:** f increases most rapidly in the direction of ∇f ($\theta = 0$). The rate of change is $\|\nabla f\|$.
2. **Maximum Decrease:** f decreases most rapidly in the direction of $-\nabla f$ ($\theta = \pi$). The rate of change is $-\|\nabla f\|$.
3. **Zero Change:** The directional derivative is zero when $\mathbf{u}$ is orthogonal to ∇f ($\theta = \pi/2$). This direction is tangent to the level curve passing through $\mathbf{p}$.

命題

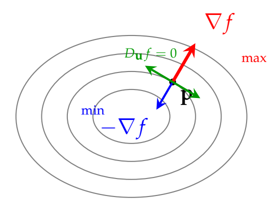

Figure 5.3: The gradient ∇f points in the direction of steepest ascent; $-\nabla f$ is steepest descent. Directions orthogonal to ∇f (green) are tangent to level curves, where $D_{\mathbf{u}} f = 0$.

Example 5.6. Analysis of a Scalar Field. Let $f(x, y) = x^2 + y^2$. We calculate the gradient:

$$\nabla f = \langle 2x, 2y \rangle.$$

Consider the point $\mathbf{p} = (2, 3)$.

- The gradient is $\nabla f(2, 3) = \langle 4, 6 \rangle$.
- The direction of maximum increase is parallel to $\langle 4, 6 \rangle$, or normalised $\mathbf{u} = \frac{1}{\sqrt{13}} \langle 2, 3 \rangle$.
- The rate of maximum increase is $\|\langle 4, 6 \rangle\| = \sqrt{16 + 36} = \sqrt{52} = 2\sqrt{13}$.
- The function is constant in the direction orthogonal to ∇f. If $\mathbf{u} = \langle a, b \rangle$ is orthogonal to $\langle 4, 6 \rangle$, then $4a + 6b = 0$, implying $b = -2/3a$. Normalizing gives directions $\pm\frac{1}{\sqrt{13}} \langle 3, -2 \rangle$.

範例

Definition 5.5. ***Critical Point.***
A point (x_0, y_0) is a **critical point** of f if $\nabla f(x_0, y_0) = \mathbf{0}$ or if the gradient does not exist. At such points, the tangent plane (if it exists) is

horizontal.

定義

Gradient Vector Fields and Level Curves

The assignment of the vector $\nabla f(\mathbf{p})$ to every point $\mathbf{p}$ in the domain creates a **vector field**. This field visualises the "flow" of steepest ascent.

Example 5.7. Coordinate Frames via Gradients. The gradient can be used to derive natural basis vectors for curvilinear coordinate systems. Consider polar coordinates:

$$r(x,y) = \sqrt{x^2+y^2}, \quad \theta(x,y) = \tan^{-1}(y/x).$$

The gradient of the radial coordinate is:

$$\nabla r = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle = \left\langle \frac{x}{r}, \frac{y}{r} \right\rangle = \langle \cos\theta, \sin\theta \rangle.$$

This is exactly the unit vector $\hat{\mathbf{r}}$ pointing radially outward. For the angular coordinate:

$$\nabla\theta = \left\langle \frac{-y/x^2}{1+(y/x)^2}, \frac{1/x}{1+(y/x)^2} \right\rangle = \left\langle \frac{-y}{r^2}, \frac{x}{r^2} \right\rangle = \frac{1}{r}\langle -\sin\theta, \cos\theta \rangle.$$

Here, $\nabla\theta$ points in the tangential direction ˆ, but has magnitude $1/r$. This reflects that a small displacement affects the angle θ more significantly near the origin than far away.

範例

A **contour plot** displays the level curves $f(x,y) = k$ for various constants k. The fundamental geometric relationship between the function and its contours is orthogonality.

Theorem 5.2. ***Orthogonality of Gradient to Level Curves.***
At any point (x_0, y_0) where $\nabla f \neq \mathbf{0}$, the gradient vector is perpendicular to the level curve $f(x,y) = k$ passing through that point.

定理

This property is extensively used in physics. For example, in electrostatics, the electric field $\mathbf{E} = -\nabla V$ is perpendicular to the equipotential lines (level curves of the voltage V).

Example 5.8. Finding Normals to Curves. To find a normal vector to the hyperbola $xy = 1$ at the point $(2, 1/2)$, we define the level function $F(x,y) = xy$. The curve is the level set $F(x,y) = 1$. The gradient is:

$$\nabla F(x,y) = \langle y, x \rangle.$$

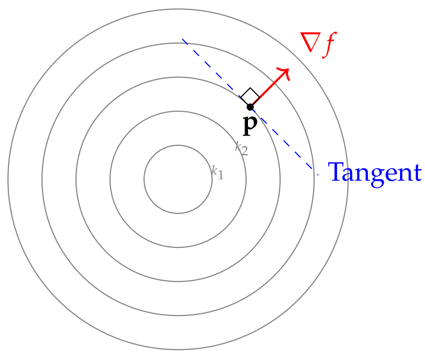

Figure 5.4: The gradient vector ∇f at $\mathbf{p}$ is orthogonal to the tangent of the level curve $f(x,y) = k$. It points in the direction of increasing k.

At $(2, 1/2)$, the normal vector is $\nabla F(2, 1/2) = \langle 1/2, 2\rangle$. Any nonzero scalar multiple, such as $\langle 1, 4\rangle$, is also a valid normal vector.

範例

5.3 *Partial Differentiation in* $\mathbb{R}^3$ *and* $\mathbb{R}^n$

The concepts of partial differentiation and gradients extend naturally to functions of three or more variables. The logic remains identical: we isolate variation in a single coordinate direction while holding all others constant.

Definitions and Basic Properties

Definition 5.6. ***Partial Derivative in*** $\mathbb{R}^n$**.**
Using *definition 5.2*, let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ be a scalar field, and let $\mathbf{p}_0 = (x_1, \ldots, x_n) \in D$. The **partial derivative** of f with respect to the j-th variable x_j at $\mathbf{p}_0$ is defined as the directional derivative along the j-th standard basis vector $\mathbf{e}_j$:

$$\frac{\partial f}{\partial x_j}(\mathbf{p}_0) = \lim_{t\to 0} \frac{f(\mathbf{p}_0 + t\mathbf{e}_j) - f(\mathbf{p}_0)}{t}.$$

Common notations include:

$$\frac{\partial f}{\partial x_j} = \partial_j f = f_{x_j}.$$

In $\mathbb{R}^3$, with variables (x, y, z), we write f_x, f_y, f_z.

定義

Operationally, calculating $\frac{\partial f}{\partial x_j}$ is equivalent to differentiating f with respect to x_j while treating all other variables $\{x_k\}_{k\neq j}$ as constants. The properties of linearity, product, and quotient rules extend directly from the two-variable case.

Proposition 5.3. ***Fundamental Properties.***
Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be differentiable. Then for any scalar $c \in \mathbb{R}$:

1. $(cf)_j = cf_j$.
2. $(f+g)_j = f_j + g_j$.
3. $(fg)_j = f_j g + f g_j$.
4. **Chain Rule (Scalar):** If $h : \mathbb{R} \to \mathbb{R}$ is differentiable, then
$$\frac{\partial}{\partial x_j}[h(f(\mathbf{x}))] = h'(f(\mathbf{x}))\frac{\partial f}{\partial x_j}.$$
5. **Independence of Coordinates:** $\frac{\partial x_i}{\partial x_j} = \delta_{ij}$, where δ_{ij} is the Kronecker delta.

命題

Proof
For property (5), let $f(\mathbf{x}) = x_i$. Then:
$$\frac{\partial x_i}{\partial x_j} = \lim_{t \to 0} \frac{(\mathbf{x} + t\mathbf{e}_j)_i - (\mathbf{x})_i}{t} = \lim_{t \to 0} \frac{x_i + t\delta_{ij} - x_i}{t} = \delta_{ij}.$$
The other properties follow from single-variable calculus applied to the restricted function.

■

Example 5.9. Derivative Calculation. Let $g(x, y, z) = xy^2z^3 + \sin(xyz)$.
$$g_x = y^2z^3 + yz\cos(xyz),$$
$$g_y = 2xyz^3 + xz\cos(xyz),$$
$$g_z = 3xy^2z^2 + xy\cos(xyz).$$

範例

Example 5.10. Radial Derivatives. Let $r = \|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2}$. For any $j \in \{1, \ldots, n\}$:
$$\frac{\partial r}{\partial x_j} = \frac{\partial}{\partial x_j}(x_1^2 + \cdots + x_n^2)^{1/2} = \frac{1}{2}(x_1^2 + \cdots + x_n^2)^{-1/2} \cdot 2x_j = \frac{x_j}{r}.$$
This identity $\partial_j r = x_j/r$ is ubiquitous in physics, particularly in potentials dependent only on distance.

範例

Example 5.11. Laplace's Equation in $\mathbb{R}^3$. We verify that the potential $u = 1/r$ satisfies Laplace's equation $\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0$

for $r \neq 0$. First derivatives:

$$u_{x_j} = \frac{\partial}{\partial x_j}(r^{-1}) = -r^{-2}\frac{\partial r}{\partial x_j} = -r^{-2}\frac{x_j}{r} = -\frac{x_j}{r^3}.$$

Second derivatives:

$$\begin{aligned} u_{x_j x_j} &= \frac{\partial}{\partial x_j}\left(-\frac{x_j}{r^3}\right) \\ &= -\frac{1 \cdot r^3 - x_j(3r^2\partial_j r)}{r^6} \quad \text{(Quotient Rule)} \\ &= -\frac{r^3 - 3x_j r^2(x_j/r)}{r^6} = -\frac{r^3 - 3x_j^2 r}{r^6} = -\frac{1}{r^3} + \frac{3x_j^2}{r^5}. \end{aligned}$$

Summing over $j = 1, 2, 3$:

$$\sum_{j=1}^{3} u_{x_j x_j} = \sum_{j=1}^{3}\left(-\frac{1}{r^3} + \frac{3x_j^2}{r^5}\right) = -\frac{3}{r^3} + \frac{3}{r^5}\sum_{j=1}^{3} x_j^2.$$

Since $\sum x_j^2 = r^2$, the second term becomes $3r^2/r^5 = 3/r^3$. Thus the sum is zero.

範例

Gradient and Directional Derivatives

The definition of continuously differentiable functions (C^1) extends directly: a function is C^1 if all partial derivatives exist and are continuous, as in *definition* 5.3. For such functions, the gradient vector completely characterises the local linear behaviour.

Definition 5.7. ***Gradient in*** $\mathbb{R}^n$**.**
Using *definition* 5.4, the gradient of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is the vector field:

$$\nabla f(\mathbf{x}) = \left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\mathbf{e}_j.$$

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Proposition 5.4. ***Directional Derivative Formula.***
By *theorem* 5.1, if f is continuously differentiable at $\mathbf{p}$, then the directional derivative in the direction of a unit vector $\mathbf{u}$ is:

$$D_{\mathbf{u}} f(\mathbf{p}) = \nabla f(\mathbf{p}) \cdot \mathbf{u} = \|\nabla f(\mathbf{p})\| \cos\theta,$$

where θ is the angle between ∇f and $\mathbf{u}$.

命題

This implies the standard geometric interpretation: ∇f points in the direction of steepest ascent, and its magnitude is the rate of that ascent.

Example 5.12. Maximizing Rate of Change. Let $f(x,y,z) = 5\sqrt{x^2+y^2+z^2} = 5r$. Find the maximum rate of change at $\mathbf{p} = (3,6,-2)$. First, calculate the gradient:

$$\nabla f = 5\nabla r = 5\left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right\rangle = \frac{5}{r}\mathbf{x}.$$

At $\mathbf{p}$, $r = \sqrt{9+36+4} = \sqrt{49} = 7$.

$$\nabla f(3,6,-2) = \frac{5}{7}\langle 3,6,-2\rangle.$$

The maximum rate of change is the magnitude:

$$\|\nabla f\| = \frac{5}{7}\|\langle 3,6,-2\rangle\| = \frac{5}{7}(7) = 5.$$

The direction is the normalised gradient vector $\mathbf{u} = \frac{1}{7}\langle 3,6,-2\rangle$.

範例

Gradients and Level Surfaces

In $\mathbb{R}^3$, the solution set to $f(x,y,z) = k$ defines a **level surface** (e.g., spheres for $x^2+y^2+z^2 = k$). The gradient provides a powerful tool for analysing the geometry of these surfaces.

Theorem 5.3. ***Normal Vector to Level Surfaces.***
If f is continuously differentiable and $\nabla f(\mathbf{p}) \neq \mathbf{0}$, then $\nabla f(\mathbf{p})$ is a normal vector to the tangent plane of the level surface $f(x,y,z) = k$ at $\mathbf{p}$.

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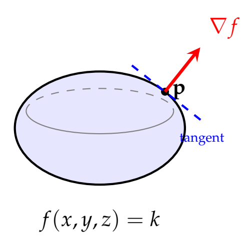

Figure 5.5: The gradient ∇f at a point $\mathbf{p}$ on the level surface $f = k$ is normal to the tangent plane. The surface shown represents a 2D cross-section of a 3D level surface.

This allows us to write the equation of the tangent plane to a surface $f(x,y,z) = k$ at $\mathbf{p} = (x_0, y_0, z_0)$ immediately as:

$$f_x(\mathbf{p})(x-x_0) + f_y(\mathbf{p})(y-y_0) + f_z(\mathbf{p})(z-z_0) = 0.$$

Example 5.13. Normal to an Ellipsoid. Consider the ellipsoid defined by $F(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. The gradient is:

$$\nabla F = \left\langle \frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2} \right\rangle.$$

At any point $\mathbf{p}$ on the surface, $\nabla F(\mathbf{p})$ is normal to the surface. This derivation is significantly more efficient than parameterising the surface and computing cross products of tangent vectors.

範例

Curvilinear Coordinates in $\mathbb{R}^3$

The gradient also allows us to derive the basis vectors for non-Cartesian coordinate systems. If a coordinate system is defined by scalar fields u_1, u_2, u_3 (e.g., r, θ, z), the unit vectors pointing in the direction of increasing coordinates are given by normalising the gradients of these fields.

Example 5.14. Spherical Basis Vectors. Recall the spherical coordinates ρ, ϕ, θ defined by:

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi.$$

Inverting these gives $\rho = (x^2 + y^2 + z^2)^{1/2}$, etc. The unit vector $\hat{\boldsymbol{\rho}}$ points in the direction of increasing ρ.

$$\nabla\rho = \left\langle \frac{x}{\rho}, \frac{y}{\rho}, \frac{z}{\rho} \right\rangle = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle.$$

Since $\|\nabla\rho\| = 1$, we have $\hat{\boldsymbol{\rho}} = \nabla\rho$. For ϕ and θ, the gradients $\nabla\phi$ and $\nabla\theta$ are orthogonal to $\hat{\boldsymbol{\rho}}$ (and each other), but must be normalised to form the unit vectors $\hat{\boldsymbol{\phi}}$ and $\hat{\boldsymbol{\theta}}$.

範例

This method generalises to any orthogonal coordinate system, providing a robust algebraic path to differential geometry in $\mathbb{R}^3$.

5.4 *The General Derivative*

Thus far, we have analysed the rate of change of a function restricted to specific lines, yielding partial and directional derivatives. While these provide valuable local information, they do not fully capture the local behaviour of the function. In single-variable calculus, differentiability at a point a implies that the graph of the function is well-approximated by a tangent line. The slope of this line is the derivative $f'(a)$.

In higher dimensions, the natural generalisation of the tangent line is the tangent space (a plane or hyperplane), and the generalisation of the derivative is a **linear transformation** that best approximates the function locally. This concept is known as the **total derivative** (or Fréchet derivative).

Linear Approximation and Definition

Let $U \subseteq \mathbb{R}^n$ be an open set and let $\mathbf{f} : U \to \mathbb{R}^m$ be a vector-valued function. We seek a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ such that $\mathbf{f}(\mathbf{a} + \mathbf{h}) \approx \mathbf{f}(\mathbf{a}) + L(\mathbf{h})$ for small perturbations $\mathbf{h}$.

Definition 5.8. ***Differentiability and the Total Derivative.***
The function $\mathbf{f} : U \to \mathbb{R}^m$ is **differentiable** at a point $\mathbf{a} \in U$ if there exists a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ such that:

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|\mathbf{f}(\mathbf{a}+\mathbf{h}) - \mathbf{f}(\mathbf{a}) - L(\mathbf{h})\|}{\|\mathbf{h}\|} = 0.$$

If such a linear map exists, it is unique. We call L the **total differential** (or simply the derivative) of $\mathbf{f}$ at $\mathbf{a}$, denoted by $d\mathbf{f}_{\mathbf{a}}$ or $D\mathbf{f}(\mathbf{a})$.

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The condition requires that the error in the linear approximation vanishes *faster* than the perturbation $\mathbf{h}$ itself (i.e., the error is $o(\|\mathbf{h}\|)$).

The Jacobian Matrix

Since $D\mathbf{f}(\mathbf{a})$ is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$, it can be represented by an $m \times n$ matrix with respect to the standard bases. This matrix is the **Jacobian matrix**.

Theorem 5.4. ***The Jacobian Matrix.***
If $\mathbf{f}$ is differentiable at $\mathbf{a}$, then its total derivative $D\mathbf{f}(\mathbf{a})$ is represented by the matrix of partial derivatives:

$$\mathbf{J}_{\mathbf{f}}(\mathbf{a}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{a}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{a}) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{a}) \end{bmatrix}.$$

Consequently, for any vector $\mathbf{h} \in \mathbb{R}^n$, the linear approximation is given by matrix-vector multiplication:

$$d\mathbf{f}_{\mathbf{a}}(\mathbf{h}) = \mathbf{J}_{\mathbf{f}}(\mathbf{a})\mathbf{h}.$$

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Proof

Let L be the total derivative. To find the j-th column of the matrix representing L, we evaluate $L(\mathbf{e}_j)$. By the definition of the limit:

$$\lim_{t\to 0} \frac{\mathbf{f}(\mathbf{a}+t\mathbf{e}_j) - \mathbf{f}(\mathbf{a}) - L(t\mathbf{e}_j)}{|t|} = 0.$$

This implies $L(\mathbf{e}_j) = \lim_{t\to 0} \frac{\mathbf{f}(\mathbf{a}+t\mathbf{e}_j)-\mathbf{f}(\mathbf{a})}{t}$, which is precisely the partial derivative vector $\frac{\partial \mathbf{f}}{\partial x_j}(\mathbf{a})$. These vectors form the columns of the Jacobian.

■

This general definition unifies the specific derivatives encountered previously:

- **Scalar Fields ($m = 1$):** The Jacobian is the row vector $\begin{bmatrix}\partial_1 f & \cdots & \partial_n f\end{bmatrix}$, which corresponds to the transpose of the gradient $(\nabla f)^T$. Formally, the gradient $\nabla f(\mathbf{a})$ is the transpose of the derivative $Df(\mathbf{a})$ when we identify covectors with vectors via the dot product.
- **Paths ($n = 1$):** The Jacobian is the column vector $\mathbf{f}'(t)$, representing velocity.
- **Coordinate Transformations ($n = m$):** The Jacobian is a square matrix, whose determinant measures local volume scaling.

Example 5.15. Linearisation of a Vector Field. Let $\mathbf{f}(x, y) = (xy, x^2, x + 3y)$. To linearise $\mathbf{f}$ at $\mathbf{a} = (x, y)$, we compute the Jacobian:

$$\mathbf{J_f}(x, y) = \begin{bmatrix} \frac{\partial}{\partial x}(xy) & \frac{\partial}{\partial y}(xy) \\ \frac{\partial}{\partial x}(x^2) & \frac{\partial}{\partial y}(x^2) \\ \frac{\partial}{\partial x}(x+3y) & \frac{\partial}{\partial y}(x+3y) \end{bmatrix} = \begin{bmatrix} y & x \\ 2x & 0 \\ 1 & 3 \end{bmatrix}.$$

The best linear approximation of the change $\Delta\mathbf{f}$ for a small increment $\mathbf{h} = (h, k)$ is:

$$\mathbf{f}(x+h, y+k) - \mathbf{f}(x, y) \approx \begin{bmatrix} y & x \\ 2x & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} yh + xk \\ 2xh \\ h + 3k \end{bmatrix}.$$

範例

Differentiability Classes and Continuity

The relationship between the existence of partial derivatives, continuity, and differentiability is subtle in higher dimensions.

Proposition 5.5. ***Implications of Differentiability.***
Let $\mathbf{f} : U \to \mathbb{R}^m$ be a function defined on an open set U.

1. If $\mathbf{f}$ is differentiable at $\mathbf{a}$, then $\mathbf{f}$ is continuous at $\mathbf{a}$.
2. If $\mathbf{f}$ is differentiable at $\mathbf{a}$, then all directional derivatives $D_\mathbf{u}\mathbf{f}(\mathbf{a})$ exist and $D_\mathbf{u}\mathbf{f}(\mathbf{a}) = D\mathbf{f}(\mathbf{a})\mathbf{u}$.

命題

Proof

For (1), note that $\mathbf{f}(\mathbf{a}+\mathbf{h}) - \mathbf{f}(\mathbf{a}) \approx L(\mathbf{h})$. Since L is linear, it is continuous and vanishes as $\mathbf{h} \to \mathbf{0}$. The error term also vanishes, so $\lim_{\mathbf{h}\to\mathbf{0}} \mathbf{f}(\mathbf{a}+\mathbf{h}) = \mathbf{f}(\mathbf{a})$.

■

Crucially, the converse is false. The existence of partial derivatives—or even all directional derivatives—does *not* guarantee differentiability, nor even continuity.

Example 5.16. Existence of Directional Derivatives without Continuity. Consider the function

$$f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

Along any line $y = mx$, the limit as $x \to 0$ is 0, so all directional derivatives exist. However, along the parabola $y = x^2$, the function approaches $1/2$. Thus, f is not continuous at the origin, and therefore cannot be differentiable there.

範例

To avoid such pathologies, we rely on a sufficient condition involving the continuity of the partial derivatives.

Theorem 5.5. ***Sufficient Condition for Differentiability.***
If the partial derivatives of $\mathbf{f}$ exist and are **continuous** on an open neighbourhood of $\mathbf{a}$, then $\mathbf{f}$ is differentiable at $\mathbf{a}$. Functions satisfying this property are said to be of class C^1 (continuously differentiable).

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The General Chain Rule

The primary advantage of the total derivative formulation is the elegance of the Chain Rule. It reduces the differentiation of composite functions to the multiplication of linear maps (or matrices).

Theorem 5.6. ***The General Chain Rule.***
Let $\mathbf{f} : U \subseteq \mathbb{R}^n \to \mathbb{R}^p$ be differentiable at $\mathbf{a}$, and $\mathbf{g} : V \subseteq \mathbb{R}^p \to \mathbb{R}^m$ be differentiable at $\mathbf{b} = \mathbf{f}(\mathbf{a})$. Then the composition $\mathbf{h} = \mathbf{g} \circ \mathbf{f}$ is differentiable at $\mathbf{a}$, and its derivative is the product of the derivatives:

$$D(\mathbf{g} \circ \mathbf{f})(\mathbf{a}) = D\mathbf{g}(\mathbf{f}(\mathbf{a})) \circ D\mathbf{f}(\mathbf{a}).$$

In terms of Jacobian matrices:

$$\mathbf{J}_{\mathbf{g}\circ\mathbf{f}}(\mathbf{a}) = \mathbf{J}_{\mathbf{g}}(\mathbf{f}(\mathbf{a}))\mathbf{J}_{\mathbf{f}}(\mathbf{a}).$$

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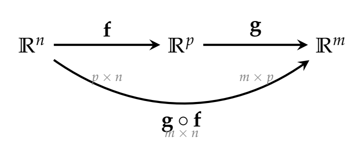


Figure 5.6: The chain rule: the Jacobian of $\mathbf{g} \circ \mathbf{f}$ is the matrix product $\mathbf{J}_{\mathbf{g}}\mathbf{J}_{\mathbf{f}}$, with dimensions $(m \times p)(p \times n) = m \times n$.

This single theorem encapsulates all specific chain rule formulas from introductory calculus. For example, if $z = g(u,v)$ with $u = u(x,y)$

and $v = v(x, y)$, the Jacobian product yields:

$$\begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}.$$

Matrix multiplication immediately recovers the familiar scalar formulas:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}.$$

Derivatives of Matrix-Valued Functions

The definition of the total derivative is coordinate-free, allowing us to differentiate functions acting on abstract vector spaces, such as spaces of matrices.

Example 5.17. Derivative of the Squaring Map. Let $S : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ be the map $S(A) = A^2$. We view $M_n(\mathbb{R})$ as a vector space isomorphic to $\mathbb{R}^{n^2}$. To find the derivative $DS(A)$, we examine the difference $S(A + H) - S(A)$ for a small matrix increment H:

$$S(A+H) - S(A) = (A+H)^2 - A^2 = A^2 + AH + HA + H^2 - A^2 = AH + HA + H^2.$$

The term linear in H is $AH + HA$. The remainder is H^2. Since $\|H^2\| \leq \|H\|^2$, the limit of the remainder divided by $\|H\|$ is 0. Thus, the total derivative is the linear map $L(H) = AH + HA$. Note that this is *not* $2AH$ unless A and H commute.

範例

Example 5.18. Derivative of the Inverse Map. Let $Inv : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ be defined by $Inv(A) = A^{-1}$. Using the identity $(A + H)^{-1} - A^{-1} \approx -A^{-1}HA^{-1}$ (derived from the geometric series expansion of $(A(I + A^{-1}H))^{-1}$), we find the derivative is the linear map:

$$D(Inv)(A)[H] = -A^{-1}HA^{-1}.$$

This generalises the single-variable rule $(1/x)' = -1/x^2$.

範例

Higher Order Derivatives and Clairaut's Theorem

Just as the first derivative approximates a function linearly, second derivatives provide information about curvature. The second partial derivatives can be organised into the **Hessian matrix**. A fundamental question is whether the order of differentiation matters.

Theorem 5.7. ***Clairaut's Theorem (Equality of Mixed Partials).***
Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$. If the mixed partial derivatives f_{xy} and f_{yx}

exist and are **continuous** on D, then:

$$f_{xy}(x,y) = f_{yx}(x,y)$$

for all $(x,y) \in D$.

定理

Example 5.19. Failure of Equality for Mixed Partials. Consider the function

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}.$$

Direct calculation shows $f_x(0,y) = -y$ and $f_y(x,0) = x$. Consequently:

$$f_{xy}(0,0) = \frac{\partial}{\partial y}(f_x)\Big|_{(0,0)} = \frac{d}{dy}(-y)\Big|_{y=0} = -1.$$

$$f_{yx}(0,0) = \frac{\partial}{\partial x}(f_y)\Big|_{(0,0)} = \frac{d}{dx}(x)\Big|_{x=0} = 1.$$

The mixed partials are unequal because they are not continuous at the origin.

範例

5.5 *Exercises*

Throughout these exercises, by a **region** we mean an open, connected subset of $\mathbb{R}^n$.

1. **Tangent Lines.** Find the equation of the line tangent to the graph of $f(x)$ at $(a, f(a))$ for the following functions:
   (a) $f(x) = \sin x, \quad a = 0$
   (b) $f(x) = \cos x, \quad a = \pi/3$
   (c) $f(x) = \cos x, \quad a = 0$
   (d) $f(x) = 1/x, \quad a = 1/2$
2. **Exponential Tangents.** For what value of a is the tangent to the graph of $f(x) = e^{-x}$ at (a, e^{-a}) a line of the form $y = mx$ (i.e., passing through the origin)?
3. **Chain Rule Practice.** Find $f'(x)$ for the following functions:
   (a) $f(x) = \sin^3(x^2 + \cos x)$
   (b) $f(x) = \cos^2((x + \sin x)^2)$
   (c) $f(x) = (\cos x)^4 \sin x$
   (d) $f(x) = (x + \sin^4 x)^3$
   (e) $f(x) = \sin x \sin(x^2 + \sin x)$
   (f) $f(x) = \sin\left(\frac{1}{x}\right)$
4. **Order of Approximation.** Using the definition of the derivative, check whether the following functions are differentiable at 0. If

differentiable, determine if the error term $f(0+h) - f(0) - f'(0)h$ is comparable to h^2 (i.e., is it $O(h^2)$?).

(a) $f(x) = x^{3/2}$

(b) $f(x) = \begin{cases} x \ln|x| & x \neq 0 \\ 0 & x = 0 \end{cases}$

(c) $f(x) = \begin{cases} x / \ln|x| & x \neq 0 \\ 0 & x = 0 \end{cases}$

5. **Partial Derivatives.** Calculate the partial derivatives $D_1 f$ and $D_2 f$ for the following functions at the specified points:

   (a) $f(x,y) = \sqrt{x^2+y}$ at $(1,3)$
   (b) $f(x,y) = x^2 y + y^4$ at $(2,1)$
   (c) $f(x,y) = \cos(xy) + y\cos y$ at $(0,\pi)$
   (d) $f(x,y) = \frac{xy^2}{y - \sqrt{x+y^2}}$ at $(3,1)$

6. **Vector-Valued Partials.** Calculate the partial derivatives $\frac{\partial \mathbf{f}}{\partial x}$ and $\frac{\partial \mathbf{f}}{\partial y}$ for the following $\mathbb{R}^m$-valued functions:

   (a) $\mathbf{f}(x,y) = \begin{bmatrix} \sqrt{x^2+y^2} \\ xy \end{bmatrix}$ (Assume $(x,y) \neq (0,0)$ when differentiating $\sqrt{x^2+y^2}$.)
   (b) $\mathbf{f}(x,y) = \begin{bmatrix} \sin^2(xy) \\ e^{xy} \end{bmatrix}$

7. **Jacobian Matrix Form.** Write the answers to the previous exercise in the form of the Jacobian matrix.

8. **Component Derivatives.**

   (a) Given a function $\mathbf{f}(x,y) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix}$ with Jacobian matrix

   $$\begin{bmatrix} 2x\cos(x^2+y) & \cos(x^2+y) \\ ye^{xy} & xe^{xy} \end{bmatrix},$$

   identify $D_1 f_1$, $D_2 f_1$, and $D_2 f_2$.

   (b) What are the dimensions of the Jacobian matrix of a function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ defined by $\mathbf{f}(x_1, \ldots, x_n) = (y_1, \ldots, y_m)^T$?

9. **Derivative Forms.** Assume the following functions are differentiable. Describe the form (dimensions) of their derivatives.

   (a) $f : \mathbb{R}^n \to \mathbb{R}^m$
   (b) $f : \mathbb{R}^3 \to \mathbb{R}$
   (c) $f : \mathbb{R} \to \mathbb{R}^4$

10. **Total Differential Calculation.** Find the total differential of $u = \ln(1 + x^2 + y^2)$ at the point $(x,y) = (1,2)$.

11. Jacobian Computation. Find the Jacobian matrices of the following mappings:

(a) $f(x, y) = \sin(xy)$
(b) $f(x, y) = e^{x^2+y^3}$
(c) $f(x, y) = \begin{bmatrix} x + y \\ x^2 - y^2 \end{bmatrix}$
(d) $f(r, \theta) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$

12. Linear Approximation. Let $\mathbf{f}(x, y) = \begin{bmatrix} x^2 - y^2 \\ 2xy \end{bmatrix}$. Let $\mathbf{p} = (1, 2)$ and $\mathbf{v} = (0.01, -0.01)$. Compute the value of the difference $\mathbf{f}(\mathbf{p} + t\mathbf{v}) - \mathbf{f}(\mathbf{p}) - t[D\mathbf{f}(\mathbf{p})]\mathbf{v}$ for $t = 1, 0.1, 0.01$. Does the difference scale like t^k for some integer k?

13. Error Propagation. The diameter of a cylinder is measured as $D_0 = 10.44$ and its height as $H_0 = 18.36$, with errors $|\Delta D| \leq 0.02$ and $|\Delta H| \leq 0.01$. Estimate the absolute error ΔV and relative error $\Delta V/V$ for the volume $V = \frac{1}{4}\pi D^2 H$.

14. Affine Maps. Let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$.

(a) Prove that if $\mathbf{f}$ is affine (i.e., $\mathbf{f}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$), then for any $\mathbf{a}, \mathbf{v} \in \mathbb{R}^n$:

$$\mathbf{f}(\mathbf{a} + \mathbf{v}) = \mathbf{f}(\mathbf{a}) + [D\mathbf{f}(\mathbf{a})]\mathbf{v}.$$

(b) Prove that if $\mathbf{f}$ is not affine, this equality does not hold for all $\mathbf{a}, \mathbf{v}$.

15. Non-Differentiability of Norm. Show that if $f(x) = |x|$, then

$$\lim_{h\to 0}(f(0 + h) - f(0) - mh) = 0$$

is never true for any number m. Thus, there is no linear map approximating $|x|$ at the origin.

16. Bounded Difference Quotients. Let $U \subseteq \mathbb{R}^n$ be open, $\mathbf{a} \in U$, and $g : U \to \mathbb{R}$ be differentiable at $\mathbf{a}$. Prove that the quantity $\frac{|g(\mathbf{a}+\mathbf{h})-g(\mathbf{a})|}{\|\mathbf{h}\|}$ is bounded as $\mathbf{h} \to \mathbf{0}$.

17. Derivative of Matrix Functions.

(a) Define differentiability for a mapping $F : \text{Mat}(n, m) \to \text{Mat}(k, l)$.

(b) Consider $F : \text{Mat}(n, m) \to \text{Mat}(n, n)$ given by $F(A) = AA^T$. Show that F is differentiable and compute $[DF(A)]$.

18. Matrix Squaring Map. Let $S : \mathbb{R}^4 \to \mathbb{R}^4$ correspond to the map $A \mapsto A^2$ for 2×2 matrices, where $A = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ is identified with

(x, y, z, w).

(a) Write the explicit formula for $S(x, y, z, w)$.

(b) Find the Jacobian matrix of S.

(c) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. If $H = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$ is a small increment, estimate $(A + H)^2$ using the derivative computed above.

19. Geometric Interpretation of Gradient. Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ be differentiable at $\mathbf{a}$. Show that if $\mathbf{v}$ is a unit vector making an angle θ with $\nabla f(\mathbf{a})$, then

$$[Df(\mathbf{a})]\mathbf{v} = \|\nabla f(\mathbf{a})\| \cos \theta.$$

Explain why this justifies the claim that $\nabla f(\mathbf{a})$ points in the direction of steepest ascent.

20. Continuity vs Differentiability.

(a) Is the mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ given by $f(\mathbf{x}) = \|\mathbf{x}\|\mathbf{x}$ differentiable at the origin? If so, what is its derivative?

(b) Give an example of a function $f(x, y)$ that has partial derivatives at a point but is not continuous there.

(c) Give an example of a function that is continuous at a point but has no partial derivatives there.

21. Derivative of the Inverse Matrix.

(a) Let A be a 2×2 invertible matrix. Compute the derivative of the function $f(A) = A^{-1}$ using the formula $A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

(b) Show that your result is consistent with the general formula $Df(A)[H] = -A^{-1}HA^{-1}$.

22. Derivative of the Determinant. Considering $\det : \mathrm{Mat}(2, 2) \to \mathbb{R}$, show that at the identity matrix I, the derivative acts on an increment H by the trace:

$$[D \det(I)]H = \mathrm{tr}(H).$$

23. Lipschitz Condition. Let $f(x, y)$ be defined on $I = [a, b] \times [c, d]$ and assume f_y is continuous on I. Prove that f satisfies a uniform Lipschitz condition in y: there exists $L > 0$ such that $|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|$ for all $(x, y_1), (x, y_2) \in I$.

24. $\star$ **Constant Function Criteria.**

(a) If f_x and f_y exist on a region D and are identically zero, prove that f is constant on D.

(b) Let $z = f(x, y)$ be differentiable on an open rectangle D. If $dz \equiv 0$, must f be constant?

25. $\star$ **Dependency of Functions.** Let $u(x, y)$ and $v(x, y)$ be continuously differentiable functions on a region Ω. Suppose they satisfy the system:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \quad u^2 + v^2 = C.$$

Prove that u and v must be constant on Ω.

26. $\star$ **Continuity of Partials vs Differentiability.** Let

$$f(x, y) = \begin{cases} xy \sin \frac{1}{x^2+y^2} & x^2 + y^2 \neq 0 \\ 0 & x^2 + y^2 = 0 \end{cases}$$

(a) Prove that $f_x(0, 0)$ and $f_y(0, 0)$ exist.

(b) Prove that f is differentiable at $(0, 0)$.

(c) Prove that the partial derivatives f_x and f_y are **not** continuous at $(0, 0)$.

This demonstrates that continuity of partials is a sufficient, but not necessary, condition for differentiability.

6

The Chain Rule and Applications

Symbolically, if $\mathbf{f}$ and $\mathbf{g}$ are differentiable appropriately, then $D(\mathbf{f} \circ \mathbf{g}) = D\mathbf{f} \circ D\mathbf{g}$. This chapter records concrete forms of the general chain rule for scalar fields, paths, and coordinate transformations, with applications to geometry, differential equations, and implicit functions.

6.1 Differentiation Along Paths

We first consider the case of a scalar field $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ evaluated along a differentiable path $\mathbf{r} : I \subseteq \mathbb{R} \to D$. This composition $g(t) = f(\mathbf{r}(t))$ represents the value of f observed by a particle moving through the domain.

Theorem 6.1. ***Chain Rule for Paths.***
Let f be continuously differentiable on an open set $D \subseteq \mathbb{R}^n$, and let $\mathbf{r} : I \to D$ be a differentiable path. Then the composite function $g(t) = f(\mathbf{r}(t))$ is differentiable, and its derivative is given by:

$$\frac{d}{dt} f(\mathbf{r}(t)) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t).$$

In coordinates, if $\mathbf{r}(t) = (x_1(t), \ldots, x_n(t))$, this becomes:

$$\frac{dg}{dt} = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \frac{dx_j}{dt}.$$

定理

Proof

This is *The General Chain Rule*. The Jacobian of f is the row vector $(\nabla f)^T$, and the Jacobian of $\mathbf{r}$ is the column vector $\mathbf{r}'(t)$. The product of a $1 \times n$ matrix and an $n \times 1$ matrix is the scalar dot product. ■

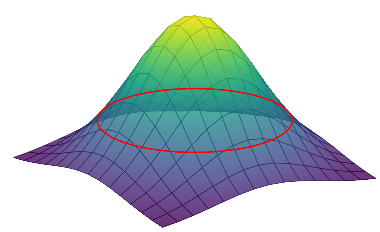


Figure 6.1: A path (red) on the surface $z = e^{-(x^2+y^2)/2}$. Along this circular path, $\nabla f \cdot \mathbf{r}' = 0$ since the path stays on a level set.

This formula relates the temporal rate of change to the spatial geometry. The term $\mathbf{r}'(t)$ represents the velocity of the trajectory, while ∇f

encodes the spatial variation of the field.

Example 6.1. Differentiation of a Power-Exponential Function.
Consider the function $u = x^y$ for $x > 0$. Let $x = \phi(t)$ and $y = \psi(t)$ be differentiable functions. By the chain rule:

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\phi'(t) + \frac{\partial u}{\partial y}\psi'(t).$$

Computing the partial derivatives:

$$\frac{\partial}{\partial x}(x^y) = yx^{y-1}, \quad \frac{\partial}{\partial y}(x^y) = x^y \ln x.$$

Thus:

$$\frac{du}{dt} = yx^{y-1}\phi'(t) + x^y(\ln x)\psi'(t).$$

Substituting ϕ and ψ back yields:

$$\frac{du}{dt} = \phi(t)^{\psi(t)}\left(\frac{\psi(t)}{\phi(t)}\phi'(t) + \ln(\phi(t))\psi'(t)\right).$$

範例

Geometric Interpretation: Orthogonality

The chain rule provides a rigorous proof for the geometric orthogonality between gradients and level sets, using *theorem* 6.1 and *theorem* 5.2.

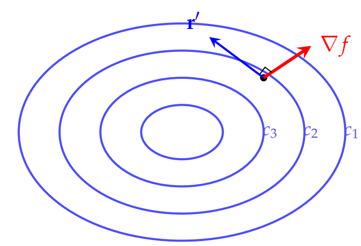


Figure 6.2: The gradient ∇f is orthogonal to level curves. The tangent $\mathbf{r}'$ lies along the curve where $f = c$.

Theorem 6.2. ***Gradient Orthogonality.***
Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function and $c \in \mathbb{R}$. If $\mathbf{r}(t)$ is any differentiable path lying entirely within the level set $S = \{\mathbf{x} : f(\mathbf{x}) = c\}$, then:

$$\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 0.$$

Consequently, $\nabla f(\mathbf{p})$ is orthogonal to every tangent vector to S at $\mathbf{p}$.

定理

Proof

This is *theorem* 5.2, applied to a level set.
Since the path lies in S, we have $f(\mathbf{r}(t)) = c$ for all t. Differentiating both sides with respect to t:

$$\frac{d}{dt}[f(\mathbf{r}(t))] = \frac{d}{dt}[c] = 0.$$

Applying the chain rule to the left-hand side yields $\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 0$.

■

Example 6.2. Normal to a Sphere. Consider the sphere defined by $f(x,y,z) = x^2 + y^2 + z^2 = R^2$. The gradient is $\nabla f = \langle 2x, 2y, 2z \rangle = 2\mathbf{x}$.

Let $\mathbf{r}(t)$ be any curve on the sphere. The velocity $\mathbf{r}'(t)$ is a tangent vector to the sphere. The theorem implies $2\mathbf{x} \cdot \mathbf{r}'(t) = 0$, which confirms that the position vector (radius) is orthogonal to the tangent plane at every point on a sphere.

範例

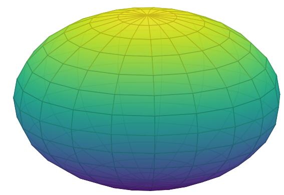


Figure 6.3: The unit sphere $x^2 + y^2 + z^2 = 1$. At each point $\mathbf{p}$, the gradient $\nabla f = 2\mathbf{p}$ points radially outward, orthogonal to all tangent vectors.

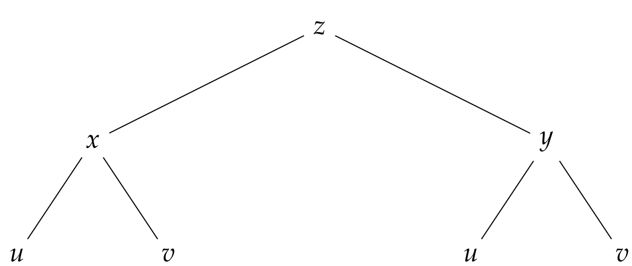
6.2 *Change of Variables*

When the intermediate variables are themselves functions of multiple independent variables, the chain rule expresses the partial derivatives of the composite in terms of the partial derivatives of the constituents.

Remark.

In this section, we frequently use u, v as independent variables (parameters), while z or w typically denote dependent variables. In other contexts, such as the Laplace equation, u often denotes the dependent variable itself; we rely on context to distinguish these roles.

Let $z = f(x,y)$, where $x = x(u,v)$ and $y = y(u,v)$. We view z as a function of u and v. The dependence structure can be visualised as:



The derivative with respect to u involves summing the contributions from all paths leading from z to u:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}.$$

Similarly for v:

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$

Example 6.3. Polar Coordinates. Let $f(x,y)$ be a differentiable function. Consider the polar transformation $x = r\cos\theta$, $y = r\sin\theta$.

We wish to express the partial derivatives $\partial f/\partial r$ and $\partial f/\partial\theta$ in terms of Cartesian derivatives.

$$\begin{aligned}\frac{\partial f}{\partial r} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} \\ &= f_x(\cos\theta) + f_y(\sin\theta).\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial \theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} \\ &= f_x(-r\sin\theta) + f_y(r\cos\theta).\end{aligned}$$

These relations can be inverted to express the Cartesian operators ∂_x, ∂_y in terms of polar operators, a technique essential for solving partial differential equations on circular domains.

範例

The Laplacian in Polar Coordinates

A classic application of the chain rule is transforming the Laplacian operator $\Delta = \nabla^2 = \partial_{xx} + \partial_{yy}$ into polar coordinates. This is non-trivial because it involves second derivatives.

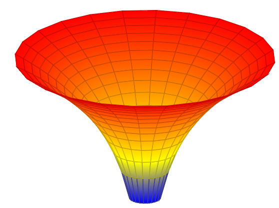


Figure 6.4: The harmonic function $u = \ln r = \ln\sqrt{x^2+y^2}$ satisfies $\Delta u = 0$ for $r > 0$. Its radial symmetry makes it natural to express in polar coordinates.

Theorem 6.3. ***Polar Laplacian.***
If $u(x,y) = U(r,\theta)$ is twice continuously differentiable, then:

$$\Delta u = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}.$$

定理

Proof

We differentiate the first-order result from the previous example. Recall:

$$\frac{\partial}{\partial r} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}.$$

To find $\partial_{rr}U$, we apply the operator $\frac{\partial}{\partial r}$ to $U_r = \cos\theta\, u_x + \sin\theta\, u_y$:

$$\frac{\partial^2 U}{\partial r^2} = \frac{\partial}{\partial r}(u_x\cos\theta + u_y\sin\theta).$$

Since θ is independent of r in the partial derivative, $\cos\theta$ and $\sin\theta$ are constant coefficients here.

$$\frac{\partial^2 U}{\partial r^2} = \cos\theta\frac{\partial}{\partial r}(u_x) + \sin\theta\frac{\partial}{\partial r}(u_y).$$

Now we apply the chain rule expansion of ∂_r to u_x and u_y:

$$\frac{\partial}{\partial r}(u_x) = u_{xx}\cos\theta + u_{xy}\sin\theta,$$
$$\frac{\partial}{\partial r}(u_y) = u_{yx}\cos\theta + u_{yy}\sin\theta.$$

Substituting these back:

$$U_{rr} = \cos^2\theta\, u_{xx} + 2\sin\theta\cos\theta\, u_{xy} + \sin^2\theta\, u_{yy}.$$

This accounts for the radial second derivative. The angular term $U_{\theta\theta}$ is more involved because the coefficients $\cos\theta$ and $\sin\theta$ must also be differentiated with respect to θ. Recall $U_\theta = -r\sin\theta\, u_x + r\cos\theta\, u_y$.

$$U_{\theta\theta} = \frac{\partial}{\partial\theta}(-r\sin\theta\, u_x + r\cos\theta\, u_y).$$

Using the product rule (differentiating the trig terms and the u terms):

$$\begin{aligned} U_{\theta\theta} &= -r\cos\theta\, u_x - r\sin\theta\frac{\partial u_x}{\partial\theta} - r\sin\theta\, u_y + r\cos\theta\frac{\partial u_y}{\partial\theta} \\ &= -r(\cos\theta u_x + \sin\theta u_y) + r\left(-\sin\theta\frac{\partial u_x}{\partial\theta} + \cos\theta\frac{\partial u_y}{\partial\theta}\right). \end{aligned}$$

Note that $\cos\theta u_x + \sin\theta u_y = U_r$. Thus the first term is $-rU_r$. For the derivatives $\partial_\theta u_x$ and $\partial_\theta u_y$:

$$\frac{\partial u_x}{\partial\theta} = u_{xx}(-r\sin\theta) + u_{xy}(r\cos\theta),$$
$$\frac{\partial u_y}{\partial\theta} = u_{yx}(-r\sin\theta) + u_{yy}(r\cos\theta).$$

Substituting these into the expression for $U_{\theta\theta}$ and simplifying yields:

$$U_{\theta\theta} = -rU_r + r^2(\sin^2\theta u_{xx} - 2\sin\theta\cos\theta u_{xy} + \cos^2\theta u_{yy}).$$

Dividing by r^2:

$$\frac{1}{r^2}U_{\theta\theta} = -\frac{1}{r}U_r + (\sin^2\theta u_{xx} - 2\sin\theta\cos\theta u_{xy} + \cos^2\theta u_{yy}).$$

Adding U_{rr} and $\frac{1}{r^2}U_{\theta\theta} + \frac{1}{r}U_r$:

$$U_{rr} + \frac{1}{r^2}U_{\theta\theta} + \frac{1}{r}U_r = (\cos^2\theta + \sin^2\theta)u_{xx} + 0\cdot u_{xy} + (\sin^2\theta + \cos^2\theta)u_{yy} = u_{xx} + u_{yy}.$$

Thus, $\Delta u = U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta}$.

■

6.3 *Implicit Differentiation*

The chain rule provides a systematic method for computing derivatives of functions defined implicitly by equations of the form $F(x, y, z) = 0$.
If the equation $F(x, y, z) = 0$ defines z locally as a function of x and y, say $z = g(x, y)$, then $F(x, y, g(x, y)) \equiv 0$ in that neighbourhood. Differentiating this identity with respect to x (holding y constant) gives:

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0.$$

Since x and y are independent variables, $\partial x/\partial x = 1$ and $\partial y/\partial x = 0$. Thus:

$$F_x + F_z\frac{\partial z}{\partial x} = 0 \implies \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad (F_z \neq 0).$$

Similarly, $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$.

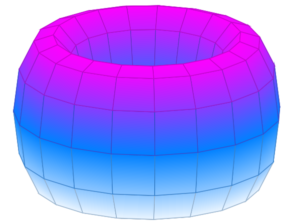


Figure 6.5: A torus defined implicitly by $(x^2 + y^2 + z^2 + R^2 - r^2)^2 = 4R^2(x^2 + y^2)$. At each point, z can be expressed locally as a function of x and y via implicit differentiation.

Example 6.4. Implicit Surface Derivatives. Consider the surface defined by $x^2 + y^2 + z^2 = 3xyz$. Let $F(x, y, z) = x^2 + y^2 + z^2 - 3xyz = 0$. We compute the partial derivatives of F:

$$F_x = 2x - 3yz, \quad F_y = 2y - 3xz, \quad F_z = 2z - 3xy.$$

Assuming $2z - 3xy \neq 0$, the partial derivatives of $z(x, y)$ are:

$$\frac{\partial z}{\partial x} = -\frac{2x - 3yz}{2z - 3xy} = \frac{3yz - 2x}{2z - 3xy}.$$

$$\frac{\partial z}{\partial y} = -\frac{2y - 3xz}{2z - 3xy} = \frac{3xz - 2y}{2z - 3xy}.$$

This method avoids the algebraic complexity of solving for z explicitly, which is often impossible.

範例

6.4 *Differentiation of Determinants*

A sophisticated application of the chain rule arises in linear algebra: computing the derivative of a determinant where the entries are functions of a parameter t.

Theorem 6.4. ***Derivative of a Determinant.***
Let $A(t) = (a_{ij}(t))$ be an $n \times n$ matrix of differentiable functions. Let $\Delta(t) = \det(A(t))$. Then:

$$\frac{d\Delta}{dt} = \sum_{i=1}^{n} \det(A_i(t)),$$

where $A_i(t)$ is the matrix obtained from $A(t)$ by replacing the i-th row with the derivatives of that row, $(a'_{i1}(t), \dots, a'_{in}(t))$.

定理

Proof

The determinant Δ is a polynomial function of the n^2 variables a_{ij}. By the chain rule:

$$\frac{d\Delta}{dt} = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial \Delta}{\partial a_{ij}} \frac{da_{ij}}{dt}.$$

From the cofactor expansion $\Delta = \sum_{j=1}^{n} a_{ij} C_{ij}$, the partial derivative with respect to an entry a_{ij} is simply its cofactor C_{ij} (since C_{ij} does not depend on a_{ij}). Thus:

$$\frac{d\Delta}{dt} = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} C_{ij} a'_{ij} \right).$$

The inner sum $\sum_{j=1}^{n} a'_{ij} C_{ij}$ is precisely the cofactor expansion of the determinant of the matrix where the i-th row is replaced by derivatives. Summing over all rows i gives the result.

■

6.5 *Homogeneous Functions and Euler's Theorem*

The chain rule yields a beautiful characterisation of homogeneous functions, which appear frequently in physics and economics.

Definition 6.1. ***Homogeneous Function.***
A function $f : D \subseteq \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}$ is **homogeneous of degree** k if for all $\mathbf{x} \in D$ and $t > 0$ such that $t\mathbf{x} \in D$:

$$f(t\mathbf{x}) = t^k f(\mathbf{x}).$$

定義

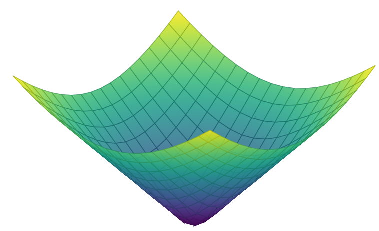


Figure 6.6: The cone $z = \sqrt{x^2 + y^2}$ is homogeneous of degree 1: scaling (x, y) by t scales z by t. Euler's theorem gives $xz_x + yz_y = z$.

Theorem 6.5. ***Euler's Homogeneous Function Theorem.***
Let f be continuously differentiable. Then f is homogeneous of degree k if and only if it satisfies the partial differential equation:

$$\mathbf{x} \cdot \nabla f(\mathbf{x}) = k f(\mathbf{x}).$$

In coordinates: $\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = k f(\mathbf{x})$.

定理

$(\implies)$

Assume $f(t\mathbf{x}) = t^k f(\mathbf{x})$. Differentiate both sides with respect to t:

$$\frac{d}{dt}[f(tx_1, \ldots, tx_n)] = \frac{d}{dt}[t^k f(\mathbf{x})].$$

Using the chain rule on the left (letting $u_i = tx_i$, so $du_i/dt = x_i$) and the power rule on the right:

$$\sum_{i=1}^{n} \frac{\partial f}{\partial u_i}(t\mathbf{x}) \cdot x_i = kt^{k-1} f(\mathbf{x}).$$

Setting $t = 1$, we obtain $\sum x_i f_{x_i}(\mathbf{x}) = kf(\mathbf{x})$.

証明終

($\Longleftarrow$)

Conversely, assume $\mathbf{x} \cdot \nabla f = kf$. Fix $\mathbf{x}$ and define $g(t) = f(t\mathbf{x})$. Then:

$$g'(t) = \nabla f(t\mathbf{x}) \cdot \mathbf{x} = \frac{1}{t}(t\mathbf{x}) \cdot \nabla f(t\mathbf{x}).$$

By hypothesis, $(t\mathbf{x}) \cdot \nabla f(t\mathbf{x}) = kf(t\mathbf{x}) = kg(t)$. Thus $g'(t) = \frac{k}{t} g(t)$. This is a separable differential equation $g'/g = k/t$, with solution $g(t) = Ct^k$. At $t = 1$, $g(1) = f(\mathbf{x}) = C$. Hence $f(t\mathbf{x}) = t^k f(\mathbf{x})$.

証明終

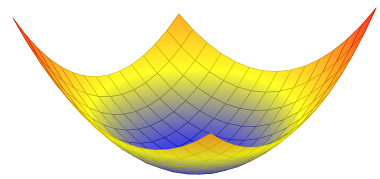


Figure 6.7: A positive-definite quadratic form $z = x^2 + \frac{1}{2}xy + y^2$, homogeneous of degree 2. Rays from the origin scale parabolically with distance.

Example 6.5. Verification for a Quadratic Form. Let $f(x, y) = Ax^2 + Bxy + Cy^2$. This is homogeneous of degree 2.
Euler's Theorem predicts $xf_x + yf_y = 2f$. Calculate partials:

$$f_x = 2Ax + By, \quad f_y = Bx + 2Cy.$$

Compute the sum:

$$\begin{aligned} x(2Ax + By) + y(Bx + 2Cy) &= 2Ax^2 + Bxy + Bxy + 2Cy^2 \\ &= 2(Ax^2 + Bxy + Cy^2) = 2f(x, y). \end{aligned}$$

範例

6.6 *Tangent Spaces and the Normal Vector Field*

In the study of the differential geometry of surfaces, the tangent space at a point plays the role of the best linear approximation to the surface, analogous to the tangent line for curves. This plane contains all possible velocity vectors of curves traversing the surface through that point. Orthogonal to this tangent space is the normal line, determined by a normal vector.
We analyse these structures through three complementary perspectives: as level sets of scalar fields, as images of parametrisations, and

as graphs of functions. While these viewpoints are mathematically equivalent, each offers distinct computational advantages.

Tangent Planes to Level Surfaces

Let S be a surface defined implicitly as the level set of a differentiable function $F : \mathbb{R}^3 \to \mathbb{R}$, so that $S = \{(x, y, z) \mid F(x, y, z) = k\}$.
In the previous section, we established that for any differentiable curve $\mathbf{r}(t)$ lying on S, the velocity vector $\mathbf{r}'(t)$ satisfies $\nabla F(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 0$. This implies that the gradient vector $\nabla F(\mathbf{p})$ is orthogonal to every tangent vector to the surface at $\mathbf{p}$.

> **Definition 6.2.** ***Normal and Tangent Plane (Implicit).***
> Let $\mathbf{p}_0 = (x_0, y_0, z_0)$ be a point on the level surface $F(x, y, z) = k$. If $\nabla F(\mathbf{p}_0) \neq \mathbf{0}$, we define:
> 1. The **normal vector** to S at $\mathbf{p}_0$ is $\mathbf{n} = \nabla F(\mathbf{p}_0)$.
> 2. The **tangent plane** to S at $\mathbf{p}_0$ is the plane passing through $\mathbf{p}_0$ with normal $\mathbf{n}$. Its equation is:
>
> $$\nabla F(\mathbf{p}_0) \cdot (\mathbf{x} - \mathbf{p}_0) = 0,$$
>
> or explicitly:
>
> $$F_x(\mathbf{p}_0)(x - x_0) + F_y(\mathbf{p}_0)(y - y_0) + F_z(\mathbf{p}_0)(z - z_0) = 0.$$
>
> 定義

The choice of the function F defines an orientation. Replacing F with $-F$ reverses the direction of the normal vector field. For example, on the unit sphere $x^2 + y^2 + z^2 = 1$, ∇F points radially outward, while $\nabla(-F)$ points radially inward.

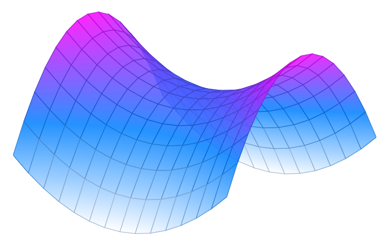


Figure 6.8: A hyperbolic paraboloid (saddle surface) $z = x^2 - 2y^2$. Such quadric surfaces arise from implicit equations like $x^2 - 2y^2 + z^2 = c$.

> **Example 6.6.** Tangent Plane to a Quadric Surface. Consider the surface defined by the equation $x^2 - 2y^2 + z^2 + yz = 2$. We wish to find the tangent plane and normal line at the point $\mathbf{p} = (2, 1, -1)$.
> Let $F(x, y, z) = x^2 - 2y^2 + z^2 + yz$. The gradient is:
>
> $$\nabla F = \langle 2x, -4y + z, 2z + y \rangle.$$
>
> Evaluating at $\mathbf{p} = (2, 1, -1)$:
>
> $$\mathbf{n} = \nabla F(2, 1, -1) = \langle 4, -4 - 1, -2 + 1 \rangle = \langle 4, -5, -1 \rangle.$$
>
> The equation of the tangent plane is:
>
> $$4(x - 2) - 5(y - 1) - 1(z + 1) = 0 \implies 4x - 5y - z = 4.$$
>
> The normal line is given parametrically by $\mathbf{l}(t) = \mathbf{p} + t\mathbf{n} = (2 + 4t, 1 - 5t, -1 - t)$.
>
> 範例

Tangent Planes to Parametrised Surfaces

Consider a surface S defined as the image of a differentiable map $\mathbf{r} : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$, written as $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$. Fixing one parameter creates **coordinate curves** on the surface. For a point $\mathbf{p}_0 = \mathbf{r}(u_0, v_0)$:

- The u-curve is $\boldsymbol{\alpha}(u) = \mathbf{r}(u, v_0)$. Its tangent vector is $\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}$.
- The v-curve is $\boldsymbol{\beta}(v) = \mathbf{r}(u_0, v)$. Its tangent vector is $\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}$.

These two vectors, $\mathbf{r}_u$ and $\mathbf{r}_v$, are tangent to the surface at $\mathbf{p}_0$. If they are linearly independent (i.e., non-collinear), they span the tangent plane.

Definition 6.3. *Normal Vector (Parametric).*
The **normal vector** induced by the parametrisation $\mathbf{r}(u,v)$ at a point (u_0, v_0) is the cross product of the partial derivatives:

$$\mathbf{N}(u_0, v_0) = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}.$$

A surface is called **smooth** (or regular) at (u_0, v_0) if $\mathbf{N}(u_0, v_0) \neq \mathbf{0}$.

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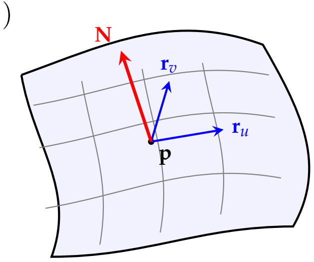

Figure 6.9: The tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ span the tangent plane. Their cross product $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$ is the surface normal.

The order of parameters matters: switching u and v results in $\mathbf{r}_v \times \mathbf{r}_u = -(\mathbf{r}_u \times \mathbf{r}_v) = -\mathbf{N}$, reversing the orientation.

Example 6.7. Normal to a Parametrised Surface. Let S be the surface parametrised by $\mathbf{r}(u,v) = \langle u^2, 2u\sin v, u\cos v\rangle$. We find the tangent plane at the parameter point $(u,v) = (1,0)$. First, identify the point on the surface:

$$\mathbf{p} = \mathbf{r}(1,0) = \langle 1^2, 2(1)\sin(0), 1\cos(0)\rangle = \langle 1,0,1\rangle.$$

Compute the partial derivative vectors:

$$\mathbf{r}_u = \langle 2u, 2\sin v, \cos v\rangle,$$
$$\mathbf{r}_v = \langle 0, 2u\cos v, -u\sin v\rangle.$$

Evaluate at $(1,0)$:

$$\mathbf{r}_u(1,0) = \langle 2,0,1\rangle, \quad \mathbf{r}_v(1,0) = \langle 0,2,0\rangle.$$

The normal vector is the cross product:

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 1 \\ 0 & 2 & 0 \end{vmatrix} = \langle -2, 0, 4\rangle.$$

The equation of the tangent plane at $(1,0,1)$ is:

$$-2(x-1) + 0(y-0) + 4(z-1) = 0 \iff -2x + 4z = 2 \iff x - 2z = -1.$$

範例

Tangent Planes to Graphs

A graph $z = f(x, y)$ is a specific type of surface that can be treated using either of the frameworks above.

1. **As a Level Surface:** Define $F(x, y, z) = z - f(x, y) = 0$. The gradient is:
$$\nabla F = \langle -f_x, -f_y, 1 \rangle.$$
2. **As a Parametrisation:** Define $\mathbf{r}(x, y) = \langle x, y, f(x, y) \rangle$. The tangent vectors are:
$$\mathbf{r}_x = \langle 1, 0, f_x \rangle, \quad \mathbf{r}_y = \langle 0, 1, f_y \rangle.$$
The normal is:
$$\mathbf{N} = \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = \langle -f_x, -f_y, 1 \rangle.$$

Both methods yield the same upward-pointing normal vector.

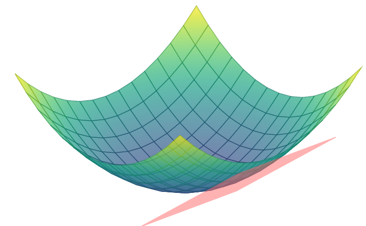

Figure 6.10: The paraboloid $z = x^2 + y^2$ with tangent plane at $(1, 0, 1)$. The plane $z = 2x - 1$ is the linearisation of f at that point.

Proposition 6.1. ***Tangent Plane Equation for a Graph.***
The tangent plane to the graph $z = f(x, y)$ at (a, b) is given by:
$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

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This equation is the **linearisation** of f at (a, b). It represents the first-order Taylor expansion of the function of two variables.

Example 6.8. Linear Approximation via Algebra. Consider the graph of $f(x, y) = 4x^2 - y^2 + 2y$. We find the tangent plane at $(-1, 2)$. The function value is $f(-1, 2) = 4(1) - 4 + 4 = 4$. We can compute partial derivatives directly, or we can use an algebraic "completion of the expansion" method around the point $(-1, 2)$. Let $x = -1 + \Delta x$ and $y = 2 + \Delta y$.
$$\begin{aligned} z &= 4(-1 + \Delta x)^2 - (2 + \Delta y)^2 + 2(2 + \Delta y) \\ &= 4(1 - 2\Delta x + \Delta x^2) - (4 + 4\Delta y + \Delta y^2) + (4 + 2\Delta y) \\ &= 4 - 8\Delta x + 4\Delta x^2 - 4 - 4\Delta y - \Delta y^2 + 4 + 2\Delta y \\ &= 4 - 8\Delta x - 2\Delta y + (4\Delta x^2 - \Delta y^2). \end{aligned}$$
The linear part determines the tangent plane:
$$L(x, y) = 4 - 8(x + 1) - 2(y - 2).$$
This matches the standard formula where $f_x(-1, 2) = -8$ and $f_y(-1, 2) = -2$. The quadratic terms $(4\Delta x^2 - \Delta y^2)$ represent the deviation of the surface from its tangent plane, providing information about the local curvature (Hessian) of the surface.

範例

6.7 Exercises

1. **Partial Derivatives via Substitution.**
   (a) Let $u = e^x + \sin y + t$, where $x = st$ and $y = s + t$. Find $\frac{\partial u}{\partial t}$.
   (b) Let $u = e^x \sin y$, where $x = 2st$ and $y = t + s^2$. Find $\frac{\partial u}{\partial s}$ and $\frac{\partial u}{\partial t}$.
   (c) Let $u = f(s,t)$ with $s = x/y$ and $t = y/z$. Express $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, and $\frac{\partial u}{\partial z}$ in terms of the partials of f.

2. **Total Differential.**
   (a) Let $u = f(ax^2 + by^2 + cz^2)$. Find the total differential du.
   (b) Let $f(x,y,z) = (x/y)^{1/z}$. Compute the differential df evaluated at the point $(1,1,1)$.

3. **PDE Verification.** Verify the following partial differential equations using the Chain Rule.
   (a) Let $w = F(xy, yz)$, where F is a continuously differentiable function. Prove that:
   $$x\frac{\partial w}{\partial x} + z\frac{\partial w}{\partial z} = y\frac{\partial w}{\partial y}.$$
   (b) Let $z = f(xy)$, where f is a differentiable function of one variable. Prove that:
   $$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 0.$$
   (c) Let $u(x,y) = \varphi(x + at) + \psi(x - at)$, where φ, ψ are twice differentiable. Show that u satisfies the wave equation:
   $$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = 0.$$

4. **Polar Symmetry.**
   (a) Let $u = F(x,y)$ satisfy the equation $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$. Prove that in polar coordinates, F depends only on θ (i.e., $F(r\cos\theta, r\sin\theta) = h(\theta)$).
   (b) Let $F(x,y) = f(x)g(y)$. In polar coordinates, suppose $F(r\cos\theta, r\sin\theta) = h(\theta)$ (independent of r). Determine the form of $F(x,y)$.
   (c) Let $u = f(\sqrt{x^2+y^2})$ be a radial function satisfying the Laplace equation $\Delta u = 0$. Find the explicit form of $u(x,y)$.

5. $\star$ **Coordinate Independence of the Laplacian.** Let $x = x(u,v)$ and $y = y(u,v)$ be a change of variables satisfying the Cauchy-

Riemann relations:

$$\frac{\partial x}{\partial u} = \frac{\partial y}{\partial v}, \quad \frac{\partial x}{\partial v} = -\frac{\partial y}{\partial u}.$$

Let $w(x,y)$ be a twice differentiable function.

(a) Prove that if $\Delta_{x,y} w = 0$, then $\Delta_{u,v} w = 0$.
(b) Prove that $\Delta_{u,v}(xy) = 0$.

6. **Linear Dependence Criterion.** Prove that a differentiable function $z = f(x,y)$ is a function of the single variable $ax + by$ (where $ab \neq 0$) if and only if

$$b\frac{\partial z}{\partial x} = a\frac{\partial z}{\partial y}.$$

7. $\star$ **Monge-Ampère Relation.** Let $u(x,y)$ have continuous second derivatives. Let $F(s,t)$ be a function such that $F(u_x, u_y) = 0$ identically, with $\nabla F \neq \mathbf{0}$. Prove that the Hessian determinant vanishes:

$$u_{xx}u_{yy} - (u_{xy})^2 = 0.$$

8. **Composition Validity.** Let $f : \mathbb{R}^3 \to \mathbb{R}$, $f_{\text{vec}} : \mathbb{R}^2 \to \mathbb{R}^3$, $g : \mathbb{R}^2 \to \mathbb{R}$, and $g_{\text{vec}} : \mathbb{R}^3 \to \mathbb{R}^2$.

(a) Determine which compositions (e.g., $f \circ g$, $g_{\text{vec}} \circ f_{\text{vec}}$) are well-defined.
(b) For the well-defined compositions, specify the dimensions of the resulting derivative matrix.

9. **Chain Rule Calculation.** Let $f(x,y,z) = x^2 + y^2 + z^2$ and $g(x,y) = (x^2y, y^2, x+y)$.

(a) Compute the derivative of $h = f \circ g$ at a point (a,b).
(b) Compute the derivative of $k = g \circ \nabla f$ (viewing ∇f as a map $\mathbb{R}^3 \to \mathbb{R}^3$) at (x,y,z).

10. **Existence of Derivatives.** Let $f(x,y) = \sin(e^{xy})$. Is f differentiable at the origin? Compute its derivative if it exists.

11. **Product Rules.** Let $f, g : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable.

(a) If $m = 1$, prove $D(fg)(\mathbf{a})\mathbf{h} = f(\mathbf{a})Dg(\mathbf{a})\mathbf{h} + g(\mathbf{a})Df(\mathbf{a})\mathbf{h}$.
(b) If $m = 3$, prove the product rule for the cross product $f \times g$.

12. **Impossible Composition.** Prove that there is no differentiable map $g : \mathbb{R}^2 \to \mathbb{R}^2$ such that $g(1,1) = (0,0)$ and the composition $f \circ g$ has the Jacobian matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ at $(1,1)$, where $f(x,y) = (x+y, x+y)$.

13. **Asymptotic Expansion.** Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$. Suppose there exists a matrix A such that for $\mathbf{x}$ near $\mathbf{x}_0$:

$$f(\mathbf{x}) = f(\mathbf{x}_0) + A(\mathbf{x} - \mathbf{x}_0) + o(\|\mathbf{x} - \mathbf{x}_0\|).$$

Prove that f is differentiable at $\mathbf{x}_0$ and $Df(\mathbf{x}_0) = A$.

14. **Geometric Orthogonality.** Let $f : \mathbb{R} \to \mathbb{R}^3$ be a differentiable path lying on the unit sphere (i.e., $\|f(t)\| = 1$ for all t). Prove that $f'(t) \cdot f(t) = 0$ for all t. Interpret this geometrically.

15. **Jacobian Determinant of Invertible Maps.** Let $u(x,y)$ and $v(x,y)$ be continuously differentiable. Suppose they satisfy the "separation condition":

$$(u_1 - u_2)^2 + (v_1 - v_2)^2 \geq C[(x_1 - x_2)^2 + (y_1 - y_2)^2]$$

for some $C > 0$. Prove that the Jacobian determinant $\frac{\partial(u,v)}{\partial(x,y)}$ is never zero.

16. $\star$ **Derivative of Determinant (General).** Let $A(\mathbf{x})$ be an $n \times n$ matrix depending on $\mathbf{x} \in \mathbb{R}^k$. Prove the identity for the derivative of the determinant:

$$\frac{\partial}{\partial x_i} \det(A) = \operatorname{tr}\left(\operatorname{adj}(A) \frac{\partial A}{\partial x_i}\right).$$

Using this, prove that if $u = \det(V)$ where V is the Vandermonde matrix defined by $x_1, \ldots, x_n$, then $\sum_{i=1}^n \frac{\partial u}{\partial x_i} = 0$.

17. $\star$ **Hadamard's Identity.** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^2 map. Let J_f be its Jacobian matrix and C_{ij} be the cofactors of J_f. Prove that the columns of the cofactor matrix are divergence-free:

$$\sum_{i=1}^n \frac{\partial C_{ij}}{\partial x_i} = 0 \quad \text{for each } j = 1, \ldots, n.$$

18. $\star$ **Orthogonal Transformations.** Let A be an orthogonal matrix and $f : \mathbb{R}^n \to \mathbb{R}$ be twice differentiable. Let $F(\mathbf{x}) = f(A\mathbf{x})$. Prove that the Laplacian is invariant under orthogonal coordinate changes:

$$\sum_{i=1}^n \frac{\partial^2 F}{\partial x_i^2} = \sum_{j=1}^n \frac{\partial^2 f}{\partial y_j^2},$$

where $\mathbf{y} = A\mathbf{x}$.

19. **Error Propagation in Products.** Let $f(x) = \prod_{i=1}^n f_i(x)$. Using the total differential approximation $\Delta f \approx df$, prove that the relative error of the product is approximately the sum of the relative errors of the factors:

$$\frac{\Delta f}{f} \approx \sum_{i=1}^n \frac{\Delta f_i}{f_i}.$$

Extend this result to quotients.

20. **Linear Dependence of Functions.** Let $x_1(t), \ldots, x_n(t)$ satisfy a system of differential equations $x_i'(t) = \sum a_{ij} x_j(t)$ with $a_{ij} > 0$. If $x_i(t) \to 0$ as $t \to \infty$ for all i, must the functions be linearly dependent?

7
Implicit Functions and Differentiation

The functional relationships encountered in geometry and physics are frequently defined not by explicit formulas $y = f(x)$, but by equations of the form $F(x, y) = 0$ or systems of such equations. For instance, the unit circle is the locus of points satisfying $x^2 + y^2 - 1 = 0$. While one can sometimes solve for y explicitly (e.g., $y = \pm\sqrt{1 - x^2}$), this is often algebraically impossible or practically inconvenient.

The **Implicit Function Theorem** provides the conditions under which such an equation locally defines a function, even if an explicit formula cannot be found. Furthermore, it allows us to compute the derivatives of these functions directly from the defining equation. This chapter establishes the theoretical existence of such functions and develops the calculus of coordinate transformations and differential operators.

7.1 The Implicit Function Theorem

Consider a level set defined by $F(x, y) = 0$. The goal is to determine when this equation defines y as a function of x in the neighbourhood of a solution point (x_0, y_0). Geometrically, this corresponds to the curve $F(x, y) = 0$ being a graph over the x-axis locally, which requires the tangent to the curve not to be vertical.

Theorem 7.1. ***Implicit Function Theorem (Scalar Case).***
Let $D \subseteq \mathbb{R}^2$ be an open set and let $F : D \to \mathbb{R}$ be a function satisfying the following conditions:

1. F is continuously differentiable (C^1) on D;
2. There exists a point $(x_0, y_0) \in D$ such that $F(x_0, y_0) = 0$;
3. The partial derivative with respect to y is non-zero at this point:

$$\frac{\partial F}{\partial y}(x_0, y_0) \neq 0.$$

Then there exists an open interval I containing x_0 and an open interval J containing y_0 such that for every $x \in I$, there is a unique $y \in J$ satisfying $F(x, y) = 0$. This defines a function $f : I \to J$ such that $y = f(x)$.
Moreover, f is continuously differentiable on I, and its derivative is given by:

$$f'(x) = -\frac{F_x(x, f(x))}{F_y(x, f(x))}.$$

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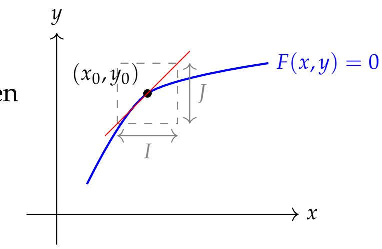

Figure 7.1: The Implicit Function Theorem guarantees that within the neighbourhood $I \times J$, the curve defines a unique function $y = f(x)$. This requires the tangent to be non-vertical ($F_y \neq 0$).

Note

Constructive Limitations: The theorem guarantees the *existence* of the function f, but it does not provide an algorithm to construct it explicitly. In many applications, an analytical expression for f is impossible to find.

Example 7.1. Kepler's Equation. In celestial mechanics, the position of a planet in its orbit is determined by **Kepler's Equation**:

$$y - x - \epsilon \sin y = 0, \quad \text{where } 0 < \epsilon < 1.$$

Here y is the eccentric anomaly and x is the mean anomaly. Let $F(x, y) = y - x - \epsilon \sin y$. At the origin, $F(0, 0) = 0$. The partial derivative with respect to y is:

$$F_y = 1 - \epsilon \cos y.$$

Since $|\epsilon| < 1$, $F_y \geq 1 - \epsilon > 0$ everywhere. Thus, the implicit function $y = y(x)$ exists globally and is differentiable. Its derivative is:

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-1}{1 - \epsilon \cos y} = \frac{1}{1 - \epsilon \cos y}.$$

Although $y(x)$ cannot be expressed in terms of elementary functions, its properties can be analysed fully (e.g., $y'(x) > 0$, so the function is monotonic).

範例

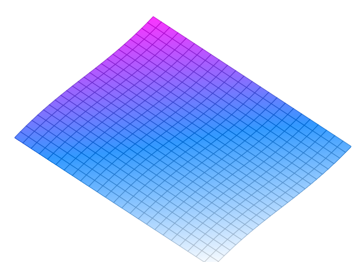

Figure 7.2: The surface $F(x, y) = y - x - \frac{1}{2} \sin y$. The zero level set (where $F = 0$) defines y implicitly as a monotonic function of x.

The differentiation formula $f'(x) = -F_x/F_y$ follows from the general chain rule applied to the identity $F(x, f(x)) \equiv 0$:

$$\frac{d}{dx} F(x, y(x)) = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = F_x + F_y y' = 0.$$

Higher-order derivatives can be computed by differentiating this relation repeatedly.

Example 7.2. Implicit Differentiation. Consider the equation defining a curve near $(1,1)$:

$$xy + 2\ln x + 3\ln y - 1 = 0.$$

Let $F(x,y) = xy + 2\ln x + 3\ln y - 1$. The conditions at $(1,1)$ are:
1. $F(1,1) = 1(1) + 2(0) + 3(0) - 1 = 0$.
2. $F_y(x,y) = x + \frac{3}{y}$. At $(1,1)$, $F_y(1,1) = 1 + 3 = 4 \neq 0$.

Thus, a differentiable function $y = f(x)$ exists near $x = 1$ with $f(1) = 1$. Its derivative is:

$$f'(x) = -\frac{F_x}{F_y} = -\frac{y + \frac{2}{x}}{x + \frac{3}{y}} = -\frac{xy+2}{x} \cdot \frac{y}{xy+3} = -\frac{y(xy+2)}{x(xy+3)}.$$

At the point $(1,1)$:

$$f'(1) = -\frac{1(1+2)}{1(1+3)} = -\frac{3}{4}.$$

範例

7.2 *Systems of Implicit Functions*

The concept extends naturally to systems of equations. Consider a mapping $\mathbf{F} : \mathbb{R}^{n+m} \to \mathbb{R}^m$, with m dependent variables (say $\mathbf{y}$) in terms of n independent variables (say $\mathbf{x}$). The solvability condition is determined by the Jacobian determinant of the dependent variables.

Theorem 7.2. ***Implicit Function Theorem (General Systems).***
Let $\mathbf{F}(\mathbf{x},\mathbf{y})$ be a C^1 function from an open set in $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^m$. Let $(\mathbf{x}_0,\mathbf{y}_0)$ be a point such that:
1. $\mathbf{F}(\mathbf{x}_0,\mathbf{y}_0) = \mathbf{0}$;
2. The Jacobian matrix with respect to $\mathbf{y}$ is invertible at this point:

$$\det\left(\frac{\partial \mathbf{F}}{\partial \mathbf{y}}\right)(\mathbf{x}_0,\mathbf{y}_0) = \det\begin{bmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_m} \end{bmatrix} \neq 0.$$

Then there exists a neighbourhood of $\mathbf{x}_0$ in $\mathbb{R}^n$ and a unique continuously differentiable function $\mathbf{g}(\mathbf{x})$ such that $\mathbf{y} = \mathbf{g}(\mathbf{x})$ and $\mathbf{F}(\mathbf{x},\mathbf{g}(\mathbf{x})) = \mathbf{0}$ for all $\mathbf{x}$ in that neighbourhood.

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The derivatives of the implicit functions can be found by solving the linear system obtained by differentiating $\mathbf{F}(\mathbf{x},\mathbf{y}(\mathbf{x})) = \mathbf{0}$. If the Jacobian matrices are denoted by $D_{\mathbf{x}}\mathbf{F}$ and $D_{\mathbf{y}}\mathbf{F}$, the chain rule in *theorem 5.6* yields:

$$D_{\mathbf{x}}\mathbf{F} + D_{\mathbf{y}}\mathbf{F} \cdot D\mathbf{y} = \mathbf{0} \implies D\mathbf{y} = -(D_{\mathbf{y}}\mathbf{F})^{-1} D_{\mathbf{x}}\mathbf{F}.$$

Example 7.3. Spherical Coordinate Derivatives. Consider the transformation from Cartesian to spherical coordinates defined by the system:

$$\begin{aligned} x &= \rho \sin\phi \cos\theta \\ y &= \rho \sin\phi \sin\theta \\ z &= \rho \cos\phi \end{aligned}$$

Suppose z is viewed as a function of x and y (locally restricted to the upper hemisphere of a fixed sphere, so ρ is constant). The goal is to calculate $\partial^2 z/\partial x^2$ using the implicit relations. From $x^2 + y^2 + z^2 = \rho^2$, differentiating with respect to x (holding y constant) gives:

$$2x + 2z\frac{\partial z}{\partial x} = 0 \implies \frac{\partial z}{\partial x} = -\frac{x}{z}.$$

Differentiating again with respect to x:

$$\frac{\partial}{\partial x}\left(-\frac{x}{z}\right) = -\frac{1 \cdot z - x \cdot z_x}{z^2} = -\frac{z - x(-x/z)}{z^2} = -\frac{z^2 + x^2}{z^3}.$$

Since $x^2 + z^2 = \rho^2 - y^2$, it follows that:

$$\frac{\partial^2 z}{\partial x^2} = -\frac{\rho^2 - y^2}{z^3}.$$

Alternatively, using the angular coordinates directly from the system requires inverting the Jacobian of the map $(x, y, z) \mapsto (\rho, \phi, \theta)$, illustrating the power of the implicit approach for constraints like $x^2 + y^2 + z^2 = \text{const}$.

範例

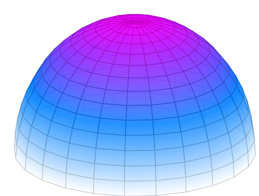

Figure 7.3: Upper hemisphere: $z = \sqrt{1 - x^2 - y^2}$. The constraint $x^2 + y^2 + z^2 = 1$ implicitly defines z as a function of (x, y) where $z > 0$.

7.3 *The Inverse Function Theorem*

A special case of the implicit function problem arises when inverting a mapping $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$. This is a specialisation of *Implicit Function Theorem (General Systems)* with $m = n$, solving $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{f}(\mathbf{x}) - \mathbf{y} = \mathbf{0}$ for $\mathbf{x}$ in terms of $\mathbf{y}$.

Theorem 7.3. ***Inverse Function Theorem.***
Let $\mathbf{f} : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 mapping. Let $\mathbf{a} \in U$ and suppose the derivative matrix $D\mathbf{f}(\mathbf{a})$ is invertible (i.e., $\det(D\mathbf{f}(\mathbf{a})) \neq 0$).
Then there exist open sets V containing $\mathbf{a}$ and W containing $\mathbf{f}(\mathbf{a})$ such

that $\mathbf{f} : V \to W$ is a bijection. Its inverse $\mathbf{g} = \mathbf{f}^{-1} : W \to V$ is continuously differentiable, and its derivative is given by:

$$D\mathbf{g}(\mathbf{y}) = [D\mathbf{f}(\mathbf{g}(\mathbf{y}))]^{-1}.$$

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Proof

A proof based on the **Contraction Mapping Principle** is included, highlighting the connection to fixed-point iterations.
Let $A = D\mathbf{f}(\mathbf{a})$. Since A is invertible, the problem can be linearised. Seek $\mathbf{x}$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{y}$. This is equivalent to finding a fixed point of the map:

$$\boldsymbol{\phi}(\mathbf{x}) = \mathbf{x} + A^{-1}(\mathbf{y} - \mathbf{f}(\mathbf{x})).$$

Observe that $\mathbf{f}(\mathbf{x}) = \mathbf{y} \iff \boldsymbol{\phi}(\mathbf{x}) = \mathbf{x}$.
Step 1: Constructing the Contraction. Differentiation yields $D\boldsymbol{\phi}(\mathbf{x}) = I - A^{-1}D\mathbf{f}(\mathbf{x}) = A^{-1}(A - D\mathbf{f}(\mathbf{x}))$. Since $\mathbf{f}$ is C^1, $D\mathbf{f}(\mathbf{x})$ is continuous. Choose a sufficiently small radius $r > 0$ such that for all $\mathbf{x}$ in the ball $B_r(\mathbf{a})$, $\|D\mathbf{f}(\mathbf{x}) - A\| < \frac{1}{2\|A^{-1}\|}$. This implies $\|D\boldsymbol{\phi}(\mathbf{x})\| \leq \|A^{-1}\|\|A - D\mathbf{f}(\mathbf{x})\| < \frac{1}{2}$. By the Mean Value Inequality, for any $\mathbf{x}_1, \mathbf{x}_2 \in B_r(\mathbf{a})$:

$$\|\boldsymbol{\phi}(\mathbf{x}_1) - \boldsymbol{\phi}(\mathbf{x}_2)\| \leq \frac{1}{2}\|\mathbf{x}_1 - \mathbf{x}_2\|.$$

Thus, $\boldsymbol{\phi}$ is a contraction mapping.
Step 2: Existence of the Inverse. For any $\mathbf{y}$ sufficiently close to $\mathbf{b} = \mathbf{f}(\mathbf{a})$, say within distance λr, the map $\boldsymbol{\phi}$ maps the closed ball $\overline{B}_r(\mathbf{a})$ into itself. By the Banach Fixed Point Theorem, there exists a unique $\mathbf{x}$ such that $\boldsymbol{\phi}(\mathbf{x}) = \mathbf{x}$, which implies $\mathbf{f}(\mathbf{x}) = \mathbf{y}$. This proves $\mathbf{f}$ is locally a bijection.
Step 3: Differentiability. Let $\mathbf{g}$ be the local inverse. Let $\mathbf{y}, \mathbf{y} + \mathbf{k} \in W$ and $\mathbf{x} = \mathbf{g}(\mathbf{y}), \mathbf{x} + \mathbf{h} = \mathbf{g}(\mathbf{y} + \mathbf{k})$. Since $\mathbf{f}$ is differentiable:

$$\mathbf{f}(\mathbf{x} + \mathbf{h}) - \mathbf{f}(\mathbf{x}) = D\mathbf{f}(\mathbf{x})\mathbf{h} + o(\|\mathbf{h}\|) = \mathbf{k}.$$

Applying the inverse matrix $T = [D\mathbf{f}(\mathbf{x})]^{-1}$:

$$\mathbf{h} = T\mathbf{k} - T(\text{error}).$$

One can show that $\|\text{error}\|/\|\mathbf{k}\| \to 0$ as $\mathbf{k} \to \mathbf{0}$, establishing that $D\mathbf{g}(\mathbf{y}) = T$.

∎

7.4 *Variable Substitution and Differential Operators*

A practical application of the chain rule and implicit differentiation is the transformation of differential operators under a change of variables. This technique is ubiquitous in simplifying partial differential equations.

When performing a substitution, it is critical to distinguish between the **independent variables** and the **dependent variables** at every step.

Example 7.4. Transforming a Second-Order Operator. Let $z = z(x, y)$ be a twice continuously differentiable function. Consider the differential equation:

$$\frac{1}{(x+y)^2}\left(\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2}\right) - \frac{1}{(x+y)^3}\left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) = 0.$$

Apply the variable substitution $u = xy$ and $v = x - y$, expressing the equation in terms of u and v.

First, express the partial differential operators ∂_x and ∂_y in terms of ∂_u and ∂_v using the chain rule.

$$\frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v} = y\frac{\partial}{\partial u} + \frac{\partial}{\partial v},$$
$$\frac{\partial}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v} = x\frac{\partial}{\partial u} - \frac{\partial}{\partial v}.$$

Observe that the sum of the operators simplifies significantly:

$$\frac{\partial}{\partial x} + \frac{\partial}{\partial y} = (x+y)\frac{\partial}{\partial u}.$$

Apply this combined operator to the function z:

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) z = (x+y)\frac{\partial z}{\partial u}.$$

Now calculate the second derivative term $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 z$. Apply the operator $(x+y)\partial_u$ to the result $(x+y)z_u$:

$$\begin{aligned}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left[(x+y)\frac{\partial z}{\partial u}\right] &= \left[\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)(x+y)\right]\frac{\partial z}{\partial u} + (x+y)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\frac{\partial z}{\partial u} \\ &= [1+1]\frac{\partial z}{\partial u} + (x+y)\left[(x+y)\frac{\partial}{\partial u}\right]\frac{\partial z}{\partial u} \\ &= 2\frac{\partial z}{\partial u} + (x+y)^2\frac{\partial^2 z}{\partial u^2}.\end{aligned}$$

Substitute these transformed derivatives back into the original equation:

$$\frac{1}{(x+y)^2}\left[2z_u + (x+y)^2 z_{uu}\right] - \frac{1}{(x+y)^3}[(x+y)z_u] = 0.$$

Simplifying the terms:

$$\left[\frac{2}{(x+y)^2}z_u + z_{uu}\right] - \frac{1}{(x+y)^2}z_u = 0 \implies z_{uu} + \frac{1}{(x+y)^2}z_u = 0.$$

Finally, express the coefficient $(x + y)^2$ in terms of u and v. Note that:

$$(x+y)^2 = (x-y)^2 + 4xy = v^2 + 4u.$$

The transformed equation is the linear ordinary differential equation (with parameter v):

$$\frac{\partial^2 z}{\partial u^2} + \frac{1}{v^2+4u}\frac{\partial z}{\partial u} = 0.$$

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Example 7.5. Canonical Form of a PDE. Consider the partial differential equation with constant coefficients:

$$a\frac{\partial^2 z}{\partial x^2} + 2b\frac{\partial^2 z}{\partial x \partial y} + c\frac{\partial^2 z}{\partial y^2} = 0,$$

where $b^2 - ac = 0$ (the parabolic case) and $c \neq 0$. A linear substitution $u = x + \alpha y$, $v = x + \beta y$ simplifies this equation.
Without loss of generality, let $c = 1$, so $a = b^2$. The operator can be factored:

$$b^2\partial_{xx} + 2b\partial_{xy} + \partial_{yy} = (b\partial_x + \partial_y)^2.$$

Choose α, β so that the operator $b\partial_x + \partial_y$ becomes proportional to a single derivative, say ∂_u. Using the chain rule:

$$\partial_x = \partial_u + \partial_v, \quad \partial_y = \alpha\partial_u + \beta\partial_v.$$

Substituting these into the operator:

$$b\partial_x + \partial_y = b(\partial_u + \partial_v) + (\alpha\partial_u + \beta\partial_v) = (b+\alpha)\partial_u + (b+\beta)\partial_v.$$

To eliminate ∂_v, choose $\beta = -b$. To keep ∂_u (and ensure the transformation is invertible), choose any $\alpha \neq -b$, for example $\alpha = 1 - b$. With these choices, the operator becomes $(b + 1 - b)\partial_u = \partial_u$. The differential equation simplifies to:

$$\frac{\partial^2 z}{\partial u^2} = 0.$$

The solution is $z = uf(v) + g(v)$, or in original coordinates, $z = (x + (1-b)y)f(x - by) + g(x - by)$.

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7.5 *Differentiation with Side Conditions*

In many applications, particularly in thermodynamics and physics, the distinction between independent and dependent variables is physically motivated but mathematically fluid. A system state may be described by n variables subject to k constraints, leaving $n - k$ degrees of freedom. The partial derivative of a quantity depends crucially on which variables are held constant.

Constrained Partial Derivatives

Consider a set of variables related by constraint equations. If z is a function of x and y, the symbol $\frac{\partial z}{\partial x}$ is unambiguous: y is held fixed. However, if x, y, z are constrained by $F(x, y, z) = 0$, any variable can be viewed as a function of the other two. The notation $\left(\frac{\partial z}{\partial x}\right)_y$ explicitly indicates that y is held constant.

Example 7.6. The Ideal Gas Law. The state of an ideal gas is described by pressure P, volume V, and temperature T, constrained by the equation of state $PV = nRT$ (where n, R are constants). The rate of change of pressure with respect to volume under isothermal conditions (constant T) is:

$$\left(\frac{\partial P}{\partial V}\right)_T = \frac{\partial}{\partial V}\left(\frac{nRT}{V}\right) = -\frac{nRT}{V^2}.$$

If P is held constant (isobaric expansion), V and T vary, but P does not change, so $\left(\frac{\partial P}{\partial V}\right)_P$ is ill-defined or zero depending on interpretation (usually zero in the context of differential forms $dP = 0$). More interestingly, consider the internal energy $U = U(P, V, T)$. Since P, V, T are not independent, the independent variables must be specified.

1. **Variables** (T, V)**:** $U = f(T, V)$. Then $\left(\frac{\partial U}{\partial T}\right)_V$ is the heat capacity at constant volume.
2. **Variables** (T, P)**:** Substitute $V = nRT/P$ to write $U = g(T, P)$. Then $\left(\frac{\partial U}{\partial T}\right)_P$ is related to the heat capacity at constant pressure.

Using the chain rule, these quantities satisfy:

$$\left(\frac{\partial U}{\partial T}\right)_P = \left(\frac{\partial U}{\partial T}\right)_V + \left(\frac{\partial U}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_P.$$

For an ideal gas, U depends only on T (Joule's Law), so $\left(\frac{\partial U}{\partial V}\right)_T = 0$, and the two derivatives with respect to T are equal. For real gases, they differ.

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The Triple Product Rule

A fundamental identity relates the partial derivatives of three variables constrained by a single equation.

Theorem 7.4. ***The Cyclic Chain Rule.***
Let x, y, z be variables related by a differentiable equation $F(x, y, z) = 0$, such that the partial derivatives F_x, F_y, F_z are non-zero. Then:

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1.$$

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Proof

From the total differential of $F(x, y, z) = 0$:

$$dF = F_x\,dx + F_y\,dy + F_z\,dz = 0.$$

To find $\left(\frac{\partial x}{\partial y}\right)_z$, set $dz = 0$:

$$F_x\,dx + F_y\,dy = 0 \implies \left(\frac{\partial x}{\partial y}\right)_z = -\frac{F_y}{F_x}.$$

By symmetry (permuting variables):

$$\left(\frac{\partial y}{\partial z}\right)_x = -\frac{F_z}{F_y}, \quad \left(\frac{\partial z}{\partial x}\right)_y = -\frac{F_x}{F_z}.$$

Multiplying these three expressions:

$$\left(-\frac{F_y}{F_x}\right)\left(-\frac{F_z}{F_y}\right)\left(-\frac{F_x}{F_z}\right) = (-1)^3 \cdot 1 = -1.$$

■

This result is counter-intuitive if one attempts to treat partial derivatives like fractions (where the product would seemingly cancel to $+1$). The sign arises because the "paths" of constant z, constant x, and constant y define a cycle on the surface $F = 0$ that reverses orientation.

7.6 *Gradients in Curvilinear Coordinates*

By *definition 5.7*, the gradient ∇f is defined independently of the coordinate system as the vector representing the differential df, while its component representation depends on the basis vectors. In Cartesian coordinates, $\nabla f = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}$. In curvilinear coordinates, account for the changing direction and scaling of the basis vectors.

Orthogonal Basis Vectors

Let (u_1, u_2, u_3) be a set of orthogonal curvilinear coordinates. The standard basis vectors $\hat{\mathbf{e}}_i$ are unit vectors tangent to the coordinate curves. Since ∇f is a vector, it can be decomposed in this basis:

$$\nabla f = (\nabla f \cdot \hat{\mathbf{e}}_1)\hat{\mathbf{e}}_1 + (\nabla f \cdot \hat{\mathbf{e}}_2)\hat{\mathbf{e}}_2 + (\nabla f \cdot \hat{\mathbf{e}}_3)\hat{\mathbf{e}}_3.$$

By *theorem 5.1*, the directional derivative is $D_{\mathbf{v}} f = \nabla f \cdot \mathbf{v}$. Thus, the coefficient of $\hat{\mathbf{e}}_i$ is the rate of change of f with respect to distance along the u_i-curve. If a small change du_i produces a displacement $ds_i = h_i du_i$ (where $h_i = \|\frac{\partial \mathbf{r}}{\partial u_i}\|$ is the scale factor), then:

$$\nabla f \cdot \hat{\mathbf{e}}_i = \frac{\partial f}{\partial s_i} = \frac{1}{h_i}\frac{\partial f}{\partial u_i}.$$

This yields the general formula:

$$\nabla f = \sum_{i=1}^{3} \frac{1}{h_i}\frac{\partial f}{\partial u_i}\hat{\mathbf{e}}_i.$$

Polar and Spherical Gradients

Polar Coordinates (r, θ): The position vector is $\mathbf{r} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}$. The scale factors are $h_r = \|\mathbf{r}_r\| = 1$ and $h_\theta = \|\mathbf{r}_\theta\| = r$. Thus:

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}}.$$

Example 7.7. Gradient of a Potential. If $f(r, \theta) = r^k \cos(k\theta)$, then:

$$\nabla f = (kr^{k-1}\cos(k\theta))\hat{\mathbf{r}} + \frac{1}{r}(-kr^k \sin(k\theta))\hat{\boldsymbol{\theta}} = kr^{k-1}(\cos(k\theta)\hat{\mathbf{r}} - \sin(k\theta)\hat{\boldsymbol{\theta}}).$$

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Spherical Coordinates (ρ, ϕ, θ): Use the physics convention: ρ is the radial distance, $\phi \in [0, \pi]$ is the polar angle from the z-axis, and $\theta \in [0, 2\pi)$ is the azimuthal angle in the xy-plane.

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi.$$

The scale factors are:
- $h_\rho = \|\partial\mathbf{r}/\partial\rho\| = 1$.
- $h_\phi = \|\partial\mathbf{r}/\partial\phi\| = \rho$.
- $h_\theta = \|\partial\mathbf{r}/\partial\theta\| = \rho\sin\phi$.

Substituting these into the general formula:

$$\nabla f = \frac{\partial f}{\partial \rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{1}{\rho\sin\phi}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}}.$$

Example 7.8. Electric Field of a Point Charge. The electric potential is $V(\rho) = \frac{q}{\rho}$. Since V is independent of ϕ and θ, the gradient is purely radial:

$$\mathbf{E} = -\nabla V = -\left(\frac{\partial}{\partial \rho}(\rho^{-1})\hat{\boldsymbol{\rho}}\right) = -(-\rho^{-2})\hat{\boldsymbol{\rho}} = \frac{1}{\rho^2}\hat{\boldsymbol{\rho}}.$$

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7.7 *Global Existence of Implicit Functions*

The Implicit Function Theorem is strictly local. By *theorem 7.1*, it guarantees a solution $y = f(x)$ within a small neighbourhood of a point. The local guarantee in *theorem 7.1* does not address global behaviour. However, questions in analysis often demand global solutions.

Proposition 7.1. ***Sufficient Condition for Global Existence.*** Let $F : (a,b) \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose the partial derivative F_y exists everywhere and satisfies $F_y(x,y) \geq m > 0$ for some constant m. Then for any $x \in (a,b)$, the equation $F(x,y) = 0$ has a unique solution $y = f(x)$, and the function f is continuous on (a,b).

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Proof

For a fixed x_0, consider the function $g(y) = F(x_0, y)$. By the Mean Value Theorem, for any $y_1 < y_2$:

$$g(y_2) - g(y_1) = F_y(x_0, \xi)(y_2 - y_1) \geq m(y_2 - y_1).$$

Letting $y_2 \to \infty$, $g(y_2) \to \infty$. Letting $y_1 \to -\infty$, $g(y_1) \to -\infty$. By the Intermediate Value Theorem, there exists a y_0 such that $g(y_0) = 0$. By the strict monotonicity ($F_y > 0$), this solution is unique. Continuity of f follows from the local Implicit Function Theorem argument extended over the domain.

■

While global existence for scalar equations relies on monotonicity, global invertibility for maps $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is subtler. The condition $\det(J_\mathbf{F}) \neq 0$ everywhere is *not* sufficient for global invertibility.

Example 7.9. Local vs. Global Invertibility. Consider the map $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by:

$$u = e^x \cos y, \quad v = e^x \sin y.$$

The Jacobian determinant is $e^{2x} \neq 0$. Thus, $\mathbf{F}$ is locally invertible

> everywhere. However, $\mathbf{F}$ is not injective (it is 2π-periodic in y), so no global inverse exists. The image of $\mathbf{F}$ is $\mathbb{R}^2 \setminus \{\mathbf{0}\}$, which is not simply connected.
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> **Theorem 7.5.** ***Hadamard's Global Inverse Function Theorem.***
> Let $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map. $\mathbf{F}$ is a diffeomorphism (a smooth bijection with a smooth inverse) if and only if:
> 1. The Jacobian determinant $\det(J_{\mathbf{F}}(\mathbf{x}))$ is never zero.
> 2. The map is **proper**, meaning $\|\mathbf{F}(\mathbf{x})\| \to \infty$ as $\|\mathbf{x}\| \to \infty$.
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7.8 *Exercises*

Implicit Differentiation and Systems

1. Implicit Differentiation. Find dy/dx for the curve defined by $\sin(x)\cos(y) - \sin(x) - \cos(y) = 0$.

2. Implicit System. Let $u(x, y)$ be determined by the system:

$$u = x + y + z + t, \quad x + y = uzt, \quad z^2 + t^2 = 1.$$

Assume appropriate solvability conditions. Determine the partial derivative $\left(\frac{\partial u}{\partial x}\right)_y$.

3. Derivatives from Equations. Let $y(x)$ be defined implicitly by the following equations. Find y' and y'':

(a) $\ln(x^2 + y^2) = \arctan(y/x)$
(b) $xy' - 2x \ln 2 + 2y = 0$
(c) $y^3 + y - x^2 = 0$ at $x = 0$.
(d) $x^3 + y^3 - 4 = 0$ at $(1, \sqrt[3]{3})$.
(e) $\sin x + 2\cos y - 1 = 0$ at $(\pi/2, 3\pi/2)$.

4. Implicit Surfaces. Let $z = z(x, y)$ be determined by the given equations. Compute the specified derivatives:

(a) $x + y + z = e^{-(x+y+z)}$. Find $\partial_x z, \partial_y z, \partial_{xx} z, \partial_{xy} z$.
(b) $x^3 + y^3 + z^3 - 3xyz - 4 = 0$. Find ∇z at $(1, 1, 2)$.
(c) $z = \sqrt{x^2 - y^2} \tan(z/\sqrt{x^2 - y^2})$. Find $\partial_x z, \partial_y z$.
(d) $x/y = \ln(z/x)$. Find the total differential dz.
(e) $xy + yz + zx = 1$. Find all first and second order partial derivatives.

5. Functional Constraints. Let f be a differentiable function. Find the derivatives of $z(x, y)$ defined by:

(a) $f(x + y + z, x^2 + y^2 + z^2) = 0$. Find dz.

(b) $f(x, x+y, x+y+z) = 0$. Find $\partial_x z, \partial_y z$.
(c) $f(x+y, y+z, z+x) = 0$. Find all second derivatives.

6. **Inverse Function Derivatives.**

(a) Given $x = e^y + u^3, y = e^u - v^3$, find $\partial_x u$ and $\partial_y v$.
(b) For the system $x = u\cos(v/u), y = u\sin(v/u)$, find the partials of the inverse map $u(x,y), v(x,y)$.
(c) Let $x = e^{v+w}, y = e^{v-w}, z = uw$. Find dz and d^2z at $(u,v) = (0,0)$.

Coordinate Transformations and PDEs

7. **Variable Substitution (Polar).** Transform the differential expression $E = x\frac{\partial u}{\partial y} - y\frac{\partial u}{\partial x}$ into polar coordinates (r,θ).

8. **Variable Substitution (PDE).** Transform the equation

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \frac{1}{\sqrt{x^2+y^2}}$$

using the substitution $x = uv$ and $y = \frac{1}{2}(u^2 - v^2)$.

9. **Dependent Variable Transformation.** Transform the equation $x^2 z_x + y^2 z_y = z^2$ by setting $x = t$, $y = \frac{t}{1+tu}$, and $z = \frac{t}{1+tv}$. Show that the transformed equation is simply $\frac{\partial v}{\partial t} = 0$.

10. **PDE Verification.** Verify that the implicitly defined functions satisfy the given PDEs:

(a) $xz_x - yz_y = 2x$, where $F(xy, z-2x) = 0$.
(b) $(x^2 - y^2 - z^2)z_x + 2xyz_y = 2xz$, where $x^2 + y^2 + z^2 = yf(z/y)$.
(c) $\star$ **Monge-Ampère Identity**. $z_{xx}z_{yy} - (z_{xy})^2 = 0$, where $x/z = \phi(y/z)$ with $\phi'' \neq 0$.

11. **Simplifying PDEs.**

(a) Transform $(x-y)z_x + yz_y = 0$ by making z a variable and y, z independent.
(b) Transform the Wave Equation $u_{tt} - c^2 u_{xx} = 0$ using $\xi = x - ct, \eta = x + ct$.
(c) Transform $xy'' - yx''$ (derivatives w.r.t t) into polar coordinates r, θ.
(d) Transform $z_x = z_y$ using $\xi = x + y, \eta = x - y$.

12. **Operator Transformations.**

(a) Show that $\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz}$ in cylindrical coordinates.
(b) Using parabolic coordinates $x = \frac{1}{2}(u^2 - v^2), y = uv$, show that

$$\Delta_{x,y} z = \frac{1}{u^2+v^2}(\Delta_{u,v} z).$$

Curvilinear Coordinates

13. Spherical Laplacian. Using the formula for ∇f in spherical coordinates, derive the expression for the Laplacian $\Delta f = \nabla \cdot \nabla f$.

14. Jacobian Determinant. Calculate the Jacobian determinant of the transformation from spherical to Cartesian coordinates and interpret it geometrically as the volume distortion factor.

Side-Condition Calculus and Thermodynamics

15. Constrained Derivatives.

(a) Let $x+y+z=0, x^2+y^2+z^2=1$. Find dx/dz at $(1/\sqrt{2}, -1/\sqrt{2}, 0)$.
(b) Let $x^3+y^3+z^3=3xyz, x+y+z=a$. Find $y'(x), z'(x), y''(x), z''(x)$.

16. Thermodynamic Relations. Re-derive the cyclic identity using the total differential:

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1.$$

Verify this identity explicitly for the ideal gas law $PV = nRT$.

Theoretical Foundations and Global Issues

17. Existence and Uniqueness.

(a) For $F(x,y) = (x-y)^2 = 0$ at $(0,0)$, $F_y = 0$, yet $y = x$ is a unique smooth solution. Why does this not contradict the Implicit Function Theorem?
(b) Prove that $y - x - \frac{1}{2}\sin y = 0$ defines a unique smooth function $y(x)$ on all of $\mathbb{R}$.
(c) Let $x = y + \phi(y)$ with $\phi(0) = 0$ and $|\phi'| \leq k < 1$. Prove local invertibility near $y = 0$.

18. Global Invertibility. Discuss why the map $f(x) = x^3$ is a global diffeomorphism on $\mathbb{R}$, whereas $f(x) = x - x^3$ is not, despite both being polynomials. Relate this to Hadamard's Theorem.

19. ⋆ **Iterative Solution**. Consider the map $u = \frac{1}{2}(x^2 - y^2), v = xy$. Let $\mathbf{a} = (1,1)$. Calculate the Jacobian J at $\mathbf{a}$. Use the Newton iteration formula

$$\mathbf{x}_{n+1} = \mathbf{x}_n + J^{-1}(\mathbf{y} - \mathbf{f}(\mathbf{x}_n))$$

to approximate the inverse map near $(u,v) = (0,1)$.

20. ⋆ **Global Invertibility**.

(a) Let $F(x,y) = (x + e^y, y - e^x)$. Show J_F is never zero. Is F globally invertible? (Check properness).
(b) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be C^1. If $\det(J_f) \neq 0$ everywhere and $|f(\mathbf{x})| \to \infty$ as $|\mathbf{x}| \to \infty$, prove f is surjective.

21. $\star$ **Legendre Transformation**. Let $y = f(x)$ be strictly convex. Let $p = f'(x)$ and define $g(p) = px - f(x)$. Prove that $g'(p) = x$ and $g''(p) = 1/f''(x)$.

8
Geometric Applications of Differentiation

The differential calculus developed in the preceding chapters provides a robust framework for analysing local linear behaviour. Having established the machinery of gradients, Jacobians, and the Implicit Function Theorem, we now turn to the geometry of curves and surfaces in $\mathbb{R}^3$. The study of tangent spaces extends to more complex configurations, such as curves defined by the intersection of surfaces, and to the geometric relationships — angles and orthogonality — between these objects.

8.1 Tangent Analysis of Space Curves

In *theorem* 6.1, we identified the derivative $\mathbf{r}'(t)$ of a path $\mathbf{r}(t)$ as the velocity vector, which is geometrically tangent to the trajectory. We now formalise the geometric structures associated with this vector.

Curves Defined Parametrically

Let Γ be a smooth curve in $\mathbb{R}^3$ parameterised by $\mathbf{r}(t) = (x(t), y(t), z(t))$ for $t \in [a, b]$. At a point $\mathbf{p}_0 = \mathbf{r}(t_0)$ where $\mathbf{r}'(t_0) \neq \mathbf{0}$, the **tangent vector** is $\boldsymbol{\tau} = \mathbf{r}'(t_0)$.

The **tangent line** to Γ at $\mathbf{p}_0$ is the line passing through $\mathbf{p}_0$ parallel to $\boldsymbol{\tau}$. Its parametric equation is $\mathbf{l}(s) = \mathbf{p}_0 + s\boldsymbol{\tau}$. In symmetric form, provided the components τ_x, τ_y, τ_z are non-zero:

$$\frac{x - x_0}{\tau_x} = \frac{y - y_0}{\tau_y} = \frac{z - z_0}{\tau_z}.$$

The plane passing through $\mathbf{p}_0$ and orthogonal to the tangent vector is called the **normal plane**. Its equation is derived from the condition $(\mathbf{x} - \mathbf{p}_0) \cdot \boldsymbol{\tau} = 0$:

$$\tau_x(x - x_0) + \tau_y(y - y_0) + \tau_z(z - z_0) = 0.$$

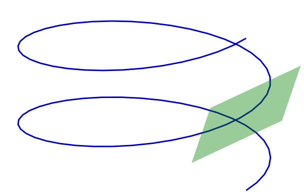

Figure 8.1: A helix Γ in $\mathbb{R}^3$. At each point, $\boldsymbol{\tau} = \mathbf{r}'(t)$ is tangent to the curve, and the normal plane is orthogonal to $\boldsymbol{\tau}$.

Curves Defined by Intersections

Frequently, a curve arises not from an explicit parametrisation but as the intersection of two surfaces. Let Γ be the locus of points satisfying the system:

$$F(x, y, z) = 0, \quad G(x, y, z) = 0.$$

Assume F and G are continuously differentiable. A point $\mathbf{p}_0 \in \Gamma$ is regular if the gradients $\nabla F(\mathbf{p}_0)$ and $\nabla G(\mathbf{p}_0)$ are linearly independent.

Theorem 8.1. ***Tangent to an Intersection Curve.***
Let Γ be the intersection of the level surfaces $F = 0$ and $G = 0$. At a regular point $\mathbf{p}_0$, the tangent vector $\boldsymbol{\tau}$ to Γ is parallel to the cross product of the gradients:

$$\boldsymbol{\tau} = \nabla F(\mathbf{p}_0) \times \nabla G(\mathbf{p}_0).$$

Explicitly, via the Jacobian determinants:

$$\boldsymbol{\tau} = \left\langle \frac{\partial(F,G)}{\partial(y,z)}, \frac{\partial(F,G)}{\partial(z,x)}, \frac{\partial(F,G)}{\partial(x,y)} \right\rangle \Bigg|_{\mathbf{p}_0}.$$

定理

Proof

Let $\mathbf{r}(t)$ be a parametrisation of the curve Γ with $\mathbf{r}(0) = \mathbf{p}_0$. Since the curve lies on both surfaces, we have $F(\mathbf{r}(t)) \equiv 0$ and $G(\mathbf{r}(t)) \equiv 0$. Differentiating with respect to t using *theorem* 6.1 yields:

$$\nabla F(\mathbf{p}_0) \cdot \mathbf{r}'(0) = 0 \quad \text{and} \quad \nabla G(\mathbf{p}_0) \cdot \mathbf{r}'(0) = 0.$$

The tangent vector $\mathbf{r}'(0)$ is orthogonal to both normal vectors ∇F and ∇G. Therefore, it must be parallel to their cross product $\nabla F \times \nabla G$. The components follow from the definition of the cross product:

$$\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ F_x & F_y & F_z \\ G_x & G_y & G_z \end{vmatrix} = \begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix} \mathbf{i} - \begin{vmatrix} F_x & F_z \\ G_x & G_z \end{vmatrix} \mathbf{j} + \begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix} \mathbf{k}.$$

■

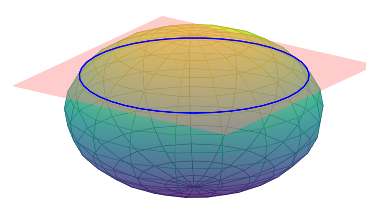

Figure 8.2: A sphere $x^2 + y^2 + z^2 = r^2$ intersected by a plane $z = c$. The intersection curve is a circle with tangent vector $\boldsymbol{\tau} = \nabla F \times \nabla G$.

Example 8.1. Slope of a Lifted Line. Consider a curve Γ defined by the intersection of the surface $z = f(x, y)$ and the vertical plane passing through (x_0, y_0) with angle α to the x-axis. We wish to find the tangent of the angle ϕ that the tangent line to Γ makes with the xy-plane.

Let the defining functions be:

$$F(x, y, z) = z - f(x, y) = 0,$$

$$G(x, y, z) = (x - x_0)\sin\alpha - (y - y_0)\cos\alpha = 0.$$

The gradients are:

$$\nabla F = \langle -f_x, -f_y, 1\rangle, \quad \nabla G = \langle \sin\alpha, -\cos\alpha, 0\rangle.$$

The tangent vector is:

$$\boldsymbol{\tau} = \nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -f_x & -f_y & 1 \\ \sin\alpha & -\cos\alpha & 0 \end{vmatrix} = \langle \cos\alpha, \sin\alpha, f_x\cos\alpha + f_y\sin\alpha\rangle.$$

The vector $\boldsymbol{\tau}$ has horizontal component $\mathbf{u} = \langle \cos\alpha, \sin\alpha\rangle$ (a unit vector) and vertical component $v = f_x\cos\alpha + f_y\sin\alpha$. The angle ϕ with the horizontal plane satisfies:

$$\tan\phi = \frac{\text{vertical rise}}{\text{horizontal run}} = \frac{f_x\cos\alpha + f_y\sin\alpha}{\sqrt{\cos^2\alpha + \sin^2\alpha}} = f_x\cos\alpha + f_y\sin\alpha.$$

範例

Remark.

This result matches precisely the definition of the **directional derivative** $D_{\mathbf{u}}f$ in *definition* 5.1. The geometric intersection method confirms the analytical definition.

8.2 *Surface Geometry and Limits*

We previously established that the gradient ∇F serves as the normal vector to the level surface $F(x, y, z) = 0$. Here we explore more advanced behaviours, such as the limiting behaviour of tangent planes near singularities or boundaries.

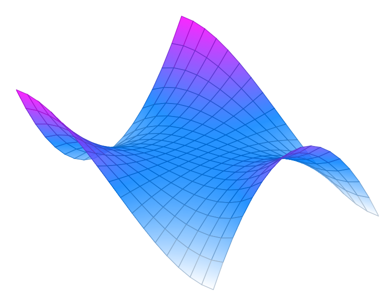

Figure 8.3: The monkey saddle $z = x^3 - 3xy^2$. Near the origin, the tangent plane degenerates as $\mathbf{r}_u \times \mathbf{r}_v \to \mathbf{0}$.

Example 8.2. Limit Position of a Tangent Plane. Consider the surface defined parametrically by:

$$x = u + v, \quad y = u^2 + v^2, \quad z = u^3 + v^3.$$

We investigate the behaviour of the tangent plane as the parameter point (u, v) approaches the boundary line $u = v$. For $u \neq v$, we compute the normal vector $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$. The partial derivatives are:

$$\mathbf{r}_u = \langle 1, 2u, 3u^2\rangle, \quad \mathbf{r}_v = \langle 1, 2v, 3v^2\rangle.$$

The cross product components are determined by the determinants:

$$N_x = \frac{\partial(y,z)}{\partial(u,v)} = 2u(3v^2) - 2v(3u^2) = 6uv(v-u),$$
$$N_y = \frac{\partial(z,x)}{\partial(u,v)} = 3u^2(1) - 3v^2(1) = 3(u^2 - v^2) = -3(v-u)(u+v),$$
$$N_z = \frac{\partial(x,y)}{\partial(u,v)} = 1(2v) - 1(2u) = 2(v-u).$$

Since $u \neq v$, we can divide the vector $\mathbf{N}$ by the common factor $v-u$ to obtain a parallel normal vector $\mathbf{n}$:

$$\mathbf{n} = \langle 6uv, -3(u+v), 2 \rangle.$$

The equation of the tangent plane at $\mathbf{p} = (x,y,z)$ is $\mathbf{n} \cdot (\mathbf{X} - \mathbf{p}) = 0$:

$$6uv(X-x) - 3(u+v)(Y-y) + 2(Z-z) = 0.$$

As $(u,v) \to (u_0,u_0)$, the point $\mathbf{p}$ approaches $(2u_0, 2u_0^2, 2u_0^3)$ and the normal vector approaches:

$$\mathbf{n}_0 = \langle 6u_0^2, -6u_0, 2 \rangle.$$

The limiting tangent plane is:

$$6u_0^2(X - 2u_0) - 6u_0(Y - 2u_0^2) + 2(Z - 2u_0^3) = 0.$$

Simplifying:

$$6u_0^2 X - 6u_0 Y + 2Z = 12u_0^3 - 12u_0^3 + 4u_0^3 = 4u_0^3.$$

Dividing by 2 yields the final equation: $3u_0^2 X - 3u_0 Y + Z = 2u_0^3$.

範例

8.3 *Angles and Orthogonality*

The angle between two surfaces at an intersection point is defined as the angle between their respective tangent planes, or equivalently, the angle between their normal vectors.

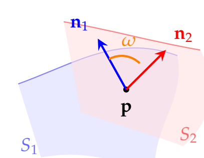

Figure 8.4: The angle ω between surfaces S_1 and S_2 equals the angle between their normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$.

Definition 8.1. ***Angle Between Surfaces.***
Let S_1 and S_2 be surfaces with normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ at a point $\mathbf{p}$. The angle ω between the surfaces is given by:

$$\cos\omega = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|}.$$

The surfaces are **orthogonal** if $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$.

定義

This concept allows us to construct orthogonal coordinate systems in $\mathbb{R}^3$, where coordinate surfaces intersect at right angles everywhere.

Example 8.3. Orthogonal Surface Families. Consider three families of surfaces defined by parameters u, v, w:

1. $S_1(u)$: $\frac{xy}{z} = u \implies F = xy - uz = 0$.
2. $S_2(v)$: $\sqrt{x^2+z^2} + \sqrt{y^2+z^2} = v$.
3. $S_3(w)$: $\sqrt{x^2+z^2} - \sqrt{y^2+z^2} = w$.

We verify that these surfaces are mutually orthogonal at any intersection point $\mathbf{p} = (x, y, z)$. First, compute the gradients (normals). For S_1, use the implicit form $F = xy/z - u = 0$.

$$\mathbf{n}_1 = \nabla\left(\frac{xy}{z}\right) = \left\langle \frac{y}{z}, \frac{x}{z}, -\frac{xy}{z^2} \right\rangle = \frac{1}{z^2}\langle yz, xz, -xy \rangle.$$

For S_2 and S_3, let $A = \sqrt{x^2+z^2}$ and $B = \sqrt{y^2+z^2}$. $S_2 : A + B = v$. Gradient $\mathbf{n}_2 = \nabla A + \nabla B$. $S_3 : A - B = w$. Gradient $\mathbf{n}_3 = \nabla A - \nabla B$. We check orthogonality of $\mathbf{n}_2$ and $\mathbf{n}_3$:

$$\mathbf{n}_2 \cdot \mathbf{n}_3 = (\nabla A + \nabla B) \cdot (\nabla A - \nabla B) = \|\nabla A\|^2 - \|\nabla B\|^2.$$

Compute $\|\nabla A\|^2$:

$$\nabla A = \left\langle \frac{x}{A}, 0, \frac{z}{A} \right\rangle \implies \|\nabla A\|^2 = \frac{x^2+z^2}{A^2} = 1.$$

Similarly $\|\nabla B\|^2 = 1$. Thus $\mathbf{n}_2 \cdot \mathbf{n}_3 = 1 - 1 = 0$. Families S_2 and S_3 are orthogonal.
Now check $\mathbf{n}_1$ against $\mathbf{n}_2$. It suffices to check $\mathbf{n}_1 \cdot \nabla A$ and $\mathbf{n}_1 \cdot \nabla B$.

$$\mathbf{n}_1 \cdot \nabla A \propto \langle yz, xz, -xy \rangle \cdot \langle x, 0, z \rangle = xyz - xyz = 0.$$

$$\mathbf{n}_1 \cdot \nabla B \propto \langle yz, xz, -xy \rangle \cdot \langle 0, y, z \rangle = xyz - xyz = 0.$$

Thus $\mathbf{n}_1$ is orthogonal to both ∇A and ∇B, and consequently to $\mathbf{n}_2$ and $\mathbf{n}_3$. The three families form a triply orthogonal system.

範例

8.4 *Further Applications of the Gradient*

These applications connect the gradient to global properties of functions, such as extrema on compact sets and homogeneity.

Example 8.4. Extremal Values on a Sphere. Find the points on the unit sphere $x^2 + y^2 + z^2 = 1$ where the directional derivative of $f(x, y, z) = x + y + z$ along the outer normal is maximised.

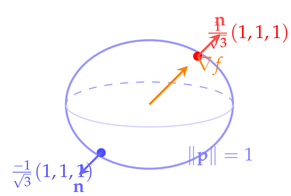

Figure 8.5: On the unit sphere, $D_{\mathbf{n}} f = \nabla f \cdot \mathbf{n}$ is maximised where $\mathbf{n} \parallel \nabla f$.

The outer normal to the sphere at $\mathbf{p} = (x, y, z)$ is the unit vector $\mathbf{n} = \mathbf{p}$ (since the radius is 1). The directional derivative is:

$$D_{\mathbf{n}} f = \nabla f \cdot \mathbf{n}.$$

Here $\nabla f = \langle 1, 1, 1 \rangle$.

$$D_{\mathbf{n}} f = \langle 1, 1, 1 \rangle \cdot \langle x, y, z \rangle = x + y + z.$$

We must maximise $u = x + y + z$ subject to $x^2 + y^2 + z^2 = 1$. By the Cauchy-Schwarz inequality:

$$|x + y + z| \leq \sqrt{1^2 + 1^2 + 1^2}\sqrt{x^2 + y^2 + z^2} = \sqrt{3} \cdot 1 = \sqrt{3}.$$

Equality holds when $\mathbf{p}$ is parallel to $\langle 1, 1, 1 \rangle$. Thus, the maximum directional derivative is $\sqrt{3}$ at $\frac{1}{\sqrt{3}}(1, 1, 1)$, and the minimum is $-\sqrt{3}$ at $\frac{-1}{\sqrt{3}}(1, 1, 1)$.

範例

Example 8.5. A Characterisation of Constant Functions. Suppose a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies the partial differential equation:

$$x f_x(x, y) + y f_y(x, y) = 0.$$

We prove that f must be constant. From Euler's Homogeneous Function Theorem, this equation implies f is homogeneous of degree $k = 0$.

$$f(tx, ty) = t^0 f(x, y) = f(x, y).$$

Taking the limit as $t \to 0$:

$$f(x, y) = \lim_{t \to 0} f(tx, ty) = f(0, 0).$$

Thus $f(x, y)$ is constant everywhere. Alternatively, in polar coordinates, the operator $x\partial_x + y\partial_y$ is equivalent to $r\partial_r$. The equation becomes $r\frac{\partial f}{\partial r} = 0$, implying f depends only on θ. Continuity at the origin then forces f to be constant.

範例

8.5 *Optimisation with Constraints: Lagrange Multipliers*

In single-variable calculus, finding extrema of a function often involves identifying critical points where the derivative vanishes.
For functions of several variables, this idea generalises to critical points where the gradient is zero. However, many practical problems involve finding extrema subject to certain side conditions or con-

straints. These constraints restrict the domain to a curve or surface, analogous to finding extrema on a closed interval in one dimension. The method of Lagrange multipliers provides a powerful technique for locating potential extrema of a function f subject to a constraint $g = 0$. While it does not guarantee the existence of such extrema, it identifies candidate points where they might occur.

Definition 8.2. ***Local and Global Extrema.***
Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function.

1. f has a **local maximum** at $\mathbf{a} \in D$ if there exists an open ball $B(\mathbf{a}, \delta)$ such that $f(\mathbf{x}) \leq f(\mathbf{a})$ for all $\mathbf{x} \in B(\mathbf{a}, \delta) \cap D$.
2. f has a **local minimum** at $\mathbf{a} \in D$ if there exists an open ball $B(\mathbf{a}, \delta)$ such that $f(\mathbf{x}) \geq f(\mathbf{a})$ for all $\mathbf{x} \in B(\mathbf{a}, \delta) \cap D$.
3. If $S \subseteq D$ and $f(\mathbf{a}) \geq f(\mathbf{x})$ for all $\mathbf{x} \in S$, then $f(\mathbf{a})$ is a **maximum** of f on S.
4. If $S \subseteq D$ and $f(\mathbf{a}) \leq f(\mathbf{x})$ for all $\mathbf{x} \in S$, then $f(\mathbf{a})$ is a **minimum** of f on S.

Maximum and minimum values are collectively called **extreme values**. If the domain S is the entire domain of f, these are called **global** maximum or minimum.

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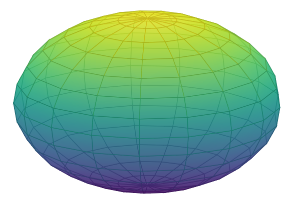

Figure 8.6: Constrained optimisation on $g(\mathbf{x}) = c$. At an extremum, $\nabla f \parallel \nabla g$: the objective's steepest direction aligns with the constraint's normal.

Theorem 8.2. ***The Method of Lagrange Multipliers.***
Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ and $g : D \subseteq \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable functions with $\nabla g(\mathbf{p}) \neq \mathbf{0}$ for all $\mathbf{p}$ on the level set $S = \{\mathbf{x} \in D \mid g(\mathbf{x}) = c\}$. If f attains a local extremum on S at a point $\mathbf{p}_0 \in S$, then there exists a scalar λ (the Lagrange multiplier) such that:

$$\nabla f(\mathbf{p}_0) = \lambda \nabla g(\mathbf{p}_0).$$

定理

We present the proof for the $n = 2$ and $n = 3$ cases, which are illustrative of the general principle.

Case $n = 2$ (Constraint Curve)

Suppose f has a local maximum at $\mathbf{p}_0 = (x_0, y_0)$ on the level curve $g(x, y) = c$. Let $\mathbf{r}(t)$ be a smooth path parameterising $g(x, y) = c$ such that $\mathbf{r}(0) = \mathbf{p}_0$. Since $\mathbf{r}(t)$ lies on the level curve, $g(\mathbf{r}(t)) = c$ for all t in some interval. The function $h(t) = f(\mathbf{r}(t))$ has a local maximum at $t = 0$. By Fermat's Theorem from single-variable calculus, $h'(0) = 0$. Applying the Chain Rule, we get:

$$h'(t) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t).$$

So, $h'(0) = \nabla f(\mathbf{p}_0) \cdot \mathbf{r}'(0) = 0$. This means $\nabla f(\mathbf{p}_0)$ is orthogonal to the tangent vector $\mathbf{r}'(0)$ of the curve. Similarly, since $g(\mathbf{r}(t)) = c$, differentiating with respect to t gives:

$$\nabla g(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 0.$$

At $t = 0$, $\nabla g(\mathbf{p}_0) \cdot \mathbf{r}'(0) = 0$. Thus, both $\nabla f(\mathbf{p}_0)$ and $\nabla g(\mathbf{p}_0)$ are orthogonal to the tangent vector $\mathbf{r}'(0)$. In $\mathbb{R}^2$, if two vectors are both orthogonal to the same non-zero vector, they must be collinear. Therefore, there exists a scalar λ such that $\nabla f(\mathbf{p}_0) = \lambda \nabla g(\mathbf{p}_0)$.

証明終

Case $n = 3$ (Constraint Surface)

Suppose f has a local maximum at $\mathbf{p}_0 = (x_0, y_0, z_0)$ on the level surface $g(x, y, z) = c$. Let $\mathbf{r}(t)$ be any smooth path on the surface $g(x, y, z) = c$ such that $\mathbf{r}(0) = \mathbf{p}_0$. As in the $n = 2$ case, $h(t) = f(\mathbf{r}(t))$ has a local maximum at $t = 0$, so $h'(0) = \nabla f(\mathbf{p}_0) \cdot \mathbf{r}'(0) = 0$. Also, $\nabla g(\mathbf{p}_0) \cdot \mathbf{r}'(0) = 0$. These conditions imply that both $\nabla f(\mathbf{p}_0)$ and $\nabla g(\mathbf{p}_0)$ are orthogonal to the tangent vector $\mathbf{r}'(0)$ of any curve passing through $\mathbf{p}_0$ on the surface. Since this holds for all such curves, it means $\nabla f(\mathbf{p}_0)$ and $\nabla g(\mathbf{p}_0)$ are both normal to the tangent plane of the surface $g = c$ at $\mathbf{p}_0$. As established in the Theorem on Normal Vector to Level Surfaces, such normal vectors must be collinear. Therefore, there exists a scalar λ such that $\nabla f(\mathbf{p}_0) = \lambda \nabla g(\mathbf{p}_0)$.

証明終

The method does not guarantee that the solutions correspond to maxima or minima, nor does it guarantee their existence. If the constraint set is closed and bounded (i.e., compact), and f is continuous, then extrema are guaranteed to exist by the Extreme Value Theorem.

Examples of Lagrange Multipliers

Example 8.6. Distance to a Hyperbola. Find the points on the hyperbola $x^2 - y^2 = 1$ closest to the origin.

We minimise the squared distance $f(x, y) = x^2 + y^2$ subject to the constraint $g(x, y) = x^2 - y^2 - 1 = 0$. The gradients are $\nabla f = \langle 2x, 2y \rangle$ and $\nabla g = \langle 2x, -2y \rangle$. The Lagrange multiplier equation $\nabla f = \lambda \nabla g$ yields:

$$2x = 2\lambda x \implies x(1 - \lambda) = 0$$
$$2y = -2\lambda y \implies y(1 + \lambda) = 0$$

From $x(1 - \lambda) = 0$, we have either $x = 0$ or $\lambda = 1$. If $x = 0$, the constraint becomes $-y^2 = 1$, which has no real solutions. So $x \neq 0$. Thus, we must have $\lambda = 1$. Substituting $\lambda = 1$ into $y(1 + \lambda) = 0$

gives $y(1+1) = 0 \implies 2y = 0 \implies y = 0$. Substitute $y = 0$ into the constraint: $x^2 - 0^2 = 1 \implies x^2 = 1 \implies x = \pm 1$. The candidate points are $(1,0)$ and $(-1,0)$. The distance squared at these points is $f(1,0) = 1^2 + 0^2 = 1$ and $f(-1,0) = (-1)^2 + 0^2 = 1$. The minimum distance to the origin is 1. The hyperbola is unbounded, so there is no maximum distance.

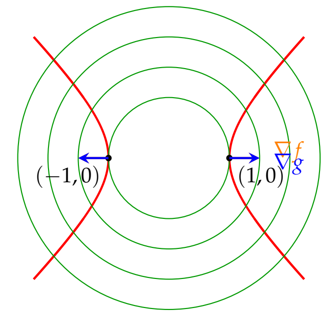

Figure 8.7: Level curves of $f(x,y) = x^2 + y^2$ (green circles) and constraint $g(x,y) = x^2 - y^2 = 1$ (red hyperbola). Gradients align at the extremal points.

範例

Example 8.7. Extrema on the Unit Circle. Find the extrema of $f(x,y) = x^2 - y^2$ on the unit circle $g(x,y) = x^2 + y^2 - 1 = 0$.
The gradients are $\nabla f = \langle 2x, -2y \rangle$ and $\nabla g = \langle 2x, 2y \rangle$. The Lagrange multiplier equations are:

$$2x = 2\lambda x \implies x(1-\lambda) = 0$$
$$-2y = 2\lambda y \implies y(1+\lambda) = 0$$

From these, we have:

- If $\lambda = 1$: Then $y(1+1) = 0 \implies y = 0$. The constraint $x^2 + y^2 = 1$ implies $x^2 = 1 \implies x = \pm 1$. Candidate points: $(\pm 1, 0)$.
- If $\lambda = -1$: Then $x(1-(-1)) = 0 \implies x = 0$. The constraint $x^2 + y^2 = 1$ implies $y^2 = 1 \implies y = \pm 1$. Candidate points: $(0, \pm 1)$.

Evaluating f at these points: $f(\pm 1, 0) = (\pm 1)^2 - 0^2 = 1$. $f(0, \pm 1) = 0^2 - (\pm 1)^2 = -1$. The maximum value is 1 (at $(\pm 1, 0)$), and the minimum value is -1 (at $(0, \pm 1)$). The unit circle is compact, so these extrema are guaranteed to exist.

範例

Example 8.8. Closest Point on a Plane. Find the point on the plane $2x - 2y + 6z = 12$ closest to the point $(2,3,4)$.
We minimise the squared distance $f(x,y,z) = (x-2)^2 + (y-3)^2 + (z-4)^2$ subject to $g(x,y,z) = 2x - 2y + 6z - 12 = 0$. The gradients are $\nabla f = \langle 2(x-2), 2(y-3), 2(z-4) \rangle$ and $\nabla g = \langle 2, -2, 6 \rangle$. The Lagrange multiplier equation gives:

$$2(x-2) = 2\lambda \implies x - 2 = \lambda$$
$$2(y-3) = -2\lambda \implies y - 3 = -\lambda$$
$$2(z-4) = 6\lambda \implies z - 4 = 3\lambda$$

From these, we express x, y, z in terms of λ: $x = 2 + \lambda$, $y = 3 - \lambda$, $z = 4 + 3\lambda$. Substitute these into the plane equation: $2(2+\lambda) - 2(3-\lambda) + 6(4+3\lambda) = 12$ $4 + 2\lambda - 6 + 2\lambda + 24 + 18\lambda = 12$ $22 + 22\lambda = 12$ $22\lambda = -10 \implies \lambda = -10/22 = -5/11$. Substitute λ back to find the point: $x = 2 - 5/11 = 17/11$ $y = 3 + 5/11 = 38/11$ $z = 4 + 3(-5/11) = 4 - 15/11 = 29/11$. The closest point is $(17/11, 38/11, 29/11)$. This problem is one where common sense tells us a minimum exists (a point to a plane). The Lagrange method identifies this unique point.

範例

Example 8.9. Maximising Box Volume. A rectangular box without a lid is made from 12 m^2 of cardboard. Find the maximum volume of such a box.
Let the dimensions be x, y, z. The volume to maximise is $V(x, y, z) = xyz$. The surface area (without a lid) is $A(x, y, z) = xy + 2xz + 2yz = 12$. This is our constraint $g(x, y, z) = xy + 2xz + 2yz - 12 = 0$. The gradients are $\nabla V = \langle yz, xz, xy\rangle$ and $\nabla g = \langle y + 2z, x + 2z, 2x + 2y\rangle$. The Lagrange equations are:

$$yz = \lambda(y + 2z) \quad (1)$$
$$xz = \lambda(x + 2z) \quad (2)$$
$$xy = \lambda(2x + 2y) \quad (3)$$

Assume $x, y, z \neq 0$ (otherwise $V = 0$, which is not a maximum). Also, if $\lambda = 0$, then $yz = 0$, which implies $x, y, z \neq 0$ is violated. So $\lambda \neq 0$. From (1), $xyz = \lambda(xy + 2xz)$. From (2), $xyz = \lambda(xy + 2yz)$. From (3), $xyz = \lambda(2xz + 2yz)$. Equating the first two expressions: $\lambda(xy + 2xz) = \lambda(xy + 2yz) \implies 2xz = 2yz$. Since $z \neq 0$, $x = y$. Now equate the second and third expressions: $\lambda(xy + 2yz) = \lambda(2xz + 2yz) \implies xy = 2xz$. Since $x \neq 0$, $y = 2z$. So we have $x = y$ and $y = 2z$, which implies $x = 2z$. Substitute $x = 2z$ and $y = 2z$ into the constraint equation: $(2z)(2z) + 2(2z)z + 2(2z)z = 12$ $4z^2 + 4z^2 + 4z^2 = 12$ $12z^2 = 12 \implies z^2 = 1$. Since z is a length, $z = 1$. Then $x = 2z = 2$ and $y = 2z = 2$. The dimensions are $2 \times 2 \times 1$. The maximum volume is $V = (2)(2)(1) = 4$ m^3.

範例

Extrema of Quadratic Forms on a Circle

A particularly illuminating application of Lagrange multipliers is finding the extrema of a quadratic form on a circle. This result forms the basis for the second derivative test in higher dimensions.
Consider the quadratic form $Q(x, y) = ax^2 + 2bxy + cy^2$ and the constraint $g(x, y) = x^2 + y^2 = R^2$. The gradients are $\nabla Q = \langle 2ax + 2by, 2bx + 2cy\rangle$ and $\nabla g = \langle 2x, 2y\rangle$. The Lagrange equations $\nabla Q = \lambda \nabla g$ are:

$$2ax + 2by = 2\lambda x \implies (a - \lambda)x + by = 0 \quad (1)$$
$$2bx + 2cy = 2\lambda y \implies bx + (c - \lambda)y = 0 \quad (2)$$

We also have the constraint $x^2 + y^2 = R^2$. This system of linear equations in x and y has non-trivial solutions (i.e., not $x = y = 0$) if and only if the determinant of the coefficient matrix is zero:

$$\begin{vmatrix} a-\lambda & b \\ b & c-\lambda \end{vmatrix} = (a-\lambda)(c-\lambda) - b^2 = 0.$$

This is the **characteristic equation**. It is a quadratic equation in λ:

$$\lambda^2 - (a+c)\lambda + (ac - b^2) = 0.$$

The solutions λ are always real numbers. Let λ_1, λ_2 be the two (possibly repeated) real solutions. For each λ, we find the corresponding x, y values satisfying equations (1), (2), and the constraint $x^2 + y^2 = R^2$.

Theorem 8.3. ***Extrema of Quadratic Forms on a Circle.*** Suppose $Q(x, y) = ax^2 + 2bxy + cy^2$. For a circle S_R with equation $x^2 + y^2 = R^2$ ($R > 0$), the characteristic equation $(a - \lambda)(c - \lambda) - b^2 = 0$ has two real solutions, λ_1 and λ_2. The extreme values of Q on S_R are $\lambda_1 R^2$ and $\lambda_2 R^2$. If $\lambda_1 = \lambda_2$, $Q(x, y)$ is constant on S_R with value $\lambda_1 R^2$. If $\lambda_1 \neq \lambda_2$, let $\lambda_1 < \lambda_2$. Then the minimum value of Q on S_R is $\lambda_1 R^2$ and the maximum value is $\lambda_2 R^2$.

定理

Proof

The derivation of the characteristic equation and the reality of its roots has been shown above. To demonstrate that $Q(x, y) = \lambda R^2$ at the points identified by λ, multiply equation (1) by x and equation (2) by y:

$$(a - \lambda)x^2 + bxy = 0$$
$$bxy + (c - \lambda)y^2 = 0$$

Summing these two equations: $(a - \lambda)x^2 + 2bxy + (c - \lambda)y^2 = 0$ $ax^2 + 2bxy + cy^2 - \lambda(x^2 + y^2) = 0$ $Q(x, y) - \lambda R^2 = 0 \implies Q(x, y) = \lambda R^2$. This proves that at any point (x, y) satisfying the Lagrange multiplier conditions for a given λ, the value of the quadratic form $Q(x, y)$ is λR^2. Since Q is a continuous function on a compact domain (the circle), its extrema exist and must be among these values. The smallest λ gives the minimum, and the largest λ gives the maximum.

■

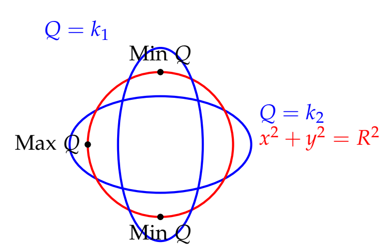

Figure 8.8: Extrema of $Q(x, y)$ on $x^2 + y^2 = R^2$. The level curves of Q (ellipses) are tangent to the constraint circle at the extrema.

Example 8.10. Extrema of $f(x, y) = xy$ on an Ellipse. Let $f(x, y) = xy$. Find the extrema of f on the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$.
The constraint is $g(x, y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$. $\nabla f = \langle y, x \rangle$, $\nabla g =$

$\langle x/4, y\rangle$. The Lagrange equations:

$$y = \lambda\frac{x}{4} \quad (1)$$
$$x = \lambda y \quad (2)$$

Substitute (2) into (1): $y = \lambda\frac{\lambda y}{4} = \frac{\lambda^2}{4}y$. So $y(1 - \frac{\lambda^2}{4}) = 0 \implies y = 0$ or $\lambda^2 = 4 \implies \lambda = \pm 2$.

- If $y = 0$: From (1), $0 = \lambda x/4$. If $\lambda = 0$, then $x = 0$. $(0,0)$ is not on the ellipse. If $\lambda \neq 0$, then $x = 0$. $(0,0)$ is not on the ellipse.
- If $\lambda = 2$: From (2), $x = 2y$. Substitute into the ellipse equation: $\frac{(2y)^2}{8} + \frac{y^2}{2} = 1 \implies \frac{4y^2}{8} + \frac{y^2}{2} = 1 \implies \frac{y^2}{2} + \frac{y^2}{2} = 1 \implies y^2 = 1 \implies y = \pm 1$. If $y = 1$, $x = 2$. Point $(2,1)$. $f(2,1) = 2(1) = 2$. If $y = -1$, $x = -2$. Point $(-2,-1)$. $f(-2,-1) = (-2)(-1) = 2$.
- If $\lambda = -2$: From (2), $x = -2y$. Substitute into the ellipse equation: $\frac{(-2y)^2}{8} + \frac{y^2}{2} = 1 \implies \frac{4y^2}{8} + \frac{y^2}{2} = 1 \implies y^2 = 1 \implies y = \pm 1$. If $y = 1$, $x = -2$. Point $(-2,1)$. $f(-2,1) = (-2)(1) = -2$. If $y = -1$, $x = 2$. Point $(2,-1)$. $f(2,-1) = (2)(-1) = -2$.

The maximum value of f is 2, occurring at $(2,1)$ and $(-2,-1)$. The minimum value of f is -2, occurring at $(-2,1)$ and $(2,-1)$.

範例

The nature of the critical points of a quadratic form $Q(x,y)$ at the origin $(0,0)$ (which is a critical point since $\nabla Q(0,0) = \mathbf{0}$) is determined by the values of λ_1, λ_2 from the characteristic equation:

1. If $\lambda_1, \lambda_2 > 0$: $Q(0,0)$ is a local minimum (paraboloid opens upward).
2. If $\lambda_1, \lambda_2 < 0$: $Q(0,0)$ is a local maximum (paraboloid opens downward).
3. If $\lambda_1 < 0 < \lambda_2$: $Q(0,0)$ is a saddle point (hyperbolic paraboloid).
4. If $\lambda_1 = 0$ and $\lambda_2 > 0$: $Q(0,0)$ is a non-isolated local minimum, a parabolic trough opening upward.
5. If $\lambda_1 = 0$ and $\lambda_2 < 0$: $Q(0,0)$ is a non-isolated local maximum, a parabolic trough opening downward.

This classification is crucial for the multivariate second derivative test, which we will derive using Taylor series expansions.

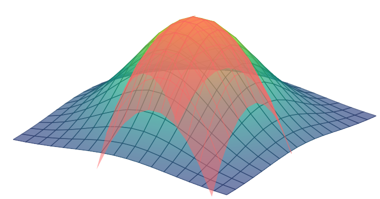
Figure 8.9: The Gaussian $z = e^{-(x^2+y^2)}$ (blue) and its quadratic Taylor approximation $z = 1 - x^2 - y^2$ (red) near the origin.

8.6 *Multivariate Taylor's Theorem*

While the linear approximation $L(\mathbf{x})$ provided by the differential is sufficient for analysing local behaviour such as tangency and sensitivity, it fails to capture curvature or the nature of critical points.

To distinguish between maxima, minima, and saddle points, or to achieve higher precision in numerical estimation, we require higher-order derivatives.
The extension of Taylor's Theorem to multiple variables can be derived by restricting a function $f(\mathbf{x})$ to a line segment passing through a point $\mathbf{a}$. Parameterising the segment as $\mathbf{r}(t) = \mathbf{a} + t\mathbf{h}$, the composite function $g(t) = f(\mathbf{a} + t\mathbf{h})$ is a single-variable function to which the standard Taylor's Theorem applies.

The Operator Notation

Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function of class C^k (continuous partial derivatives up to order k). We introduce the differential operator representing the directional derivative along a vector $\mathbf{h} = (h_1, \ldots, h_n)$:

$$(\mathbf{h} \cdot \nabla) = \sum_{i=1}^{n} h_i \frac{\partial}{\partial x_i}.$$

Applying this operator k times yields the symbolic power:

$$(\mathbf{h} \cdot \nabla)^k = \left(\sum_{i=1}^{n} h_i \frac{\partial}{\partial x_i} \right)^k.$$

For the bivariate case $\mathbf{h} = (h, k)$, the binomial expansion gives:

$$(h\partial_x + k\partial_y)^2 f = h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}.$$

Theorem 8.4. ***Taylor's Theorem (Multivariate).***
Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ be of class C^{m+1} on an open convex set D. Let $\mathbf{a} \in D$ and let $\mathbf{h}$ be a vector such that $\mathbf{a} + \mathbf{h} \in D$. Then:

$$f(\mathbf{a} + \mathbf{h}) = \sum_{k=0}^{m} \frac{1}{k!} [(\mathbf{h} \cdot \nabla)^k f](\mathbf{a}) + R_m(\mathbf{h}),$$

where the remainder term $R_m(\mathbf{h})$ can be expressed in the Lagrange form:

$$R_m(\mathbf{h}) = \frac{1}{(m+1)!} [(\mathbf{h} \cdot \nabla)^{m+1} f](\mathbf{a} + \theta\mathbf{h})$$

for some $\theta \in (0, 1)$. Alternatively, the Peano remainder form states that $R_m(\mathbf{h}) = o(\|\mathbf{h}\|^m)$ as $\mathbf{h} \to \mathbf{0}$.

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Remark.

In the quadratic case ($m = 2$), this expansion reveals the structure governing local extrema. For a point $\mathbf{a}$, letting $\mathbf{x} = \mathbf{a} + \mathbf{h}$:

$$f(\mathbf{x}) \approx f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + \frac{1}{2} (\mathbf{x} - \mathbf{a})^T H_f(\mathbf{a}) (\mathbf{x} - \mathbf{a}),$$

where $H_f(\mathbf{a})$ is the **Hessian matrix** of second partial derivatives.

Example 8.11. Higher-Order Expansion by Substitution. Compute the Taylor expansion of $f(x,y) = e^x \sin(y)$ about the origin up to degree 3.
Rather than computing all partial derivatives manually, we may utilise the known single-variable series for e^x and $\sin y$:

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$$

$$\sin y = y - \frac{1}{6}y^3 + \cdots$$

Multiplying these series and retaining terms with total degree ≤ 3:

$$\begin{aligned} f(x,y) &= \left(1 + x + \frac{1}{2}x^2 + \dots\right)\left(y - \frac{1}{6}y^3 + \dots\right) \\ &= y - \frac{1}{6}y^3 + xy + \frac{1}{2}x^2y + \dots \\ &= y + xy + \frac{1}{2}x^2y - \frac{1}{6}y^3 + o(\rho^3). \end{aligned}$$

This result implies, for instance, that $\frac{\partial^3 f}{\partial x^2 \partial y}(0,0) = 1$ (from the term $\frac{1}{2}x^2y$, noting the coefficient in Taylor series is $\frac{1}{2!1!}f_{xxy}$).

範例

Proposition 8.1. ***Uniqueness of Taylor Series.***
If a function f of class C^m admits an expansion $f(\mathbf{a}+\mathbf{h}) = P_m(\mathbf{h}) + o(\|\mathbf{h}\|^m)$ where P_m is a polynomial of degree at most m, then P_m is the Taylor polynomial of f at $\mathbf{a}$.

命題

This proposition justifies the substitution method used above.

Example 8.12. Limit Calculation via Expansion. Consider the function

$$f(x,y) = \begin{cases} \frac{1-e^{x^2+y^2}}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}.$$

We wish to find the Taylor polynomial of order 2. Let $u = x^2 + y^2$. Since $e^u = 1 + u + \frac{1}{2}u^2 + o(u^2)$, we have:

$$\frac{1 - (1 + u + \frac{1}{2}u^2)}{u} = -1 - \frac{1}{2}u + o(u) = -1 - \frac{1}{2}(x^2+y^2) + o(x^2+y^2).$$

Thus, the polynomial is $P_2(x,y) = -1 - \frac{1}{2}x^2 - \frac{1}{2}y^2$. It follows immediately that $f_{xx}(0,0) = -1$ and $f_{xy}(0,0) = 0$.

範例

8.7 *Classification of Critical Points*

For a differentiable function f, a point $\mathbf{a}$ is a **stationary point** (or critical point) if $\nabla f(\mathbf{a}) = \mathbf{0}$. At such a point, the linear term in the Taylor expansion vanishes, and the local behaviour is dominated by the quadratic term:

$$f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) \approx \frac{1}{2}\mathbf{h}^T H_f(\mathbf{a})\mathbf{h}.$$

The nature of the stationary point is determined by the definiteness of the Hessian matrix $H_f(\mathbf{a})$.

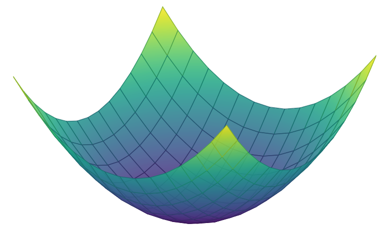

Figure 8.10: A local minimum at the origin for $z = x^2 + y^2$. The Hessian is positive definite with eigenvalues $\lambda_1 = \lambda_2 = 2$.

Theorem 8.5. ***The Second Derivative Test.***
Let $\mathbf{a}$ be a critical point of a C^2 function $f : \mathbb{R}^n \to \mathbb{R}$. Let $H = H_f(\mathbf{a})$ be the Hessian matrix.

1. If H is **positive definite** (all eigenvalues > 0), then f has a local **minimum** at $\mathbf{a}$.
2. If H is **negative definite** (all eigenvalues < 0), then f has a local **maximum** at $\mathbf{a}$.
3. If H is **indefinite** (eigenvalues have mixed signs), then f has a **saddle point** at $\mathbf{a}$.
4. If H is singular (at least one zero eigenvalue) and semi-definite, the test is inconclusive (higher order terms are required).

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The Bivariate Case

For $n = 2$, Sylvester's Criterion allows us to classify points using the determinant. Let $A = f_{xx}$, $B = f_{xy}$, $C = f_{yy}$ evaluated at $\mathbf{a}$. The Hessian is $H = \begin{bmatrix} A & B \\ B & C \end{bmatrix}$. Let $\Delta = \det(H) = AC - B^2$.

- If $\Delta > 0$ and $A > 0$: Local Minimum (Positive Definite).
- If $\Delta > 0$ and $A < 0$: Local Maximum (Negative Definite).
- If $\Delta < 0$: Saddle Point (Indefinite).
- If $\Delta = 0$: Inconclusive.

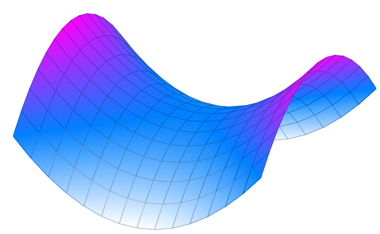

Figure 8.11: A saddle point at the origin for $z = x^2 - y^2$. The Hessian has eigenvalues $\lambda_1 = 2, \lambda_2 = -2$.

Example 8.13. Classifying Extrema. Find and classify the critical points of $f(x,y) = x^3 - 12xy + 8y^3$.
Step 1: Find critical points.

$$\nabla f = \langle 3x^2 - 12y, -12x + 24y^2 \rangle = \mathbf{0}.$$

From the first equation, $y = x^2/4$. Substituting into the second:

$$-12x + 24(x^2/4)^2 = -12x + \frac{3}{2}x^4 = 0 \implies x(x^3 - 8) = 0.$$

The real solutions are $x = 0$ and $x = 2$. If $x = 0, y = 0$. Point $P_1(0,0)$. If $x = 2, y = 1$. Point $P_2(2,1)$.
Step 2: Analyse the Hessian.

$$H(x,y) = \begin{bmatrix} 6x & -12 \\ -12 & 48y \end{bmatrix}.$$

At $P_1(0,0)$:

$$H = \begin{bmatrix} 0 & -12 \\ -12 & 0 \end{bmatrix}, \quad \Delta = 0 - 144 = -144 < 0.$$

P_1 is a **saddle point**.
At $P_2(2,1)$:

$$H = \begin{bmatrix} 12 & -12 \\ -12 & 48 \end{bmatrix}, \quad \Delta = (12)(48) - 144 = 576 - 144 = 432 > 0.$$

Since $\Delta > 0$ and $f_{xx} = 12 > 0$, P_2 is a **local minimum**. The value is $f(2,1) = 8 - 24 + 8 = -8$.

範例

Example 8.14. A Degenerate Case. Consider $f(x,y) = (x+y)^2$. The gradient is $\nabla f = \langle 2(x+y), 2(x+y) \rangle$.
The critical points are all points on the line $y = -x$. The Hessian is $H = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$. Here $\Delta = 4 - 4 = 0$. The test is inconclusive. However, inspection of the function reveals that $f(x,y) \geq 0$ everywhere, and $f(x,y) = 0$ on the line $y = -x$. Thus, every point on this line is a global minimum (non-strict). This describes a parabolic trough.

範例

8.8 *Global Optimisation Strategies*

While the Hessian classifies local extrema, finding the global maximum and minimum on a domain D requires a global comparison.

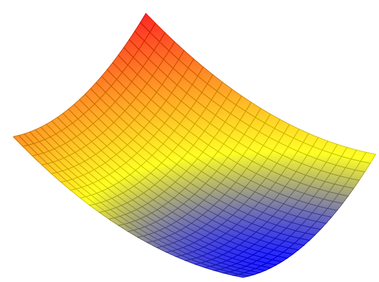

Figure 8.12: A coercive function $f(x,y) = \frac{1}{2}(x^2 + y^2) - 2x + y + 3$. As $\|\mathbf{x}\| \to \infty$, the quadratic terms dominate, guaranteeing a global minimum.

Optimisation on Compact Sets

If D is closed and bounded (compact) and f is continuous, the Extreme Value Theorem guarantees the existence of global extrema. The algorithm is:

1. Find all critical points in the interior of D and evaluate f.
2. Find all points where ∇f does not exist and evaluate f.

3. Find the extrema of f restricted to the boundary ∂D.
4. Compare values.

Example 8.15. Global Extrema on a Region. Find the global extrema of $f(x,y) = x^2 - xy + y^2 - 2x + y$ on the region bounded by $x = 0, y = 0, x + y = 3$. First consider the unbounded case to illustrate coercivity.
Consider f on $\mathbb{R}^2$. Stationary point calculation:

$$2x - y - 2 = 0, \quad -x + 2y + 1 = 0 \implies (x,y) = (1,0).$$

Hessian at $(1,0)$: $f_{xx} = 2, f_{xy} = -1, f_{yy} = 2 \implies \Delta = 3 > 0$. Local minimum. Value $f(1,0) = -1$. Is this a global minimum? We examine the behaviour at infinity. Converting to polar coordinates or completing the square:

$$f(x,y) = \frac{1}{2}(x-y)^2 + \frac{1}{2}x^2 + \frac{1}{2}y^2 - 2x + y.$$

The quadratic terms dominate the linear terms. As $\|\mathbf{x}\| \to \infty$, $f(\mathbf{x}) \to \infty$. Thus, the function is **coercive**. A coercive continuous function on $\mathbb{R}^n$ must attain a global minimum. Since $(1,0)$ is the only critical point, it is the global minimum.
Now restrict to the triangle $D = \{x \geq 0, y \geq 0, x + y \leq 3\}$. The interior critical point $(1,0)$ remains feasible with value $f(1,0) = -1$. On the boundary segment $x = 0$, $f(0,y) = y^2 + y$ for $y \in [0,3]$, so the minimum is 0 at $y = 0$ and the maximum is 12 at $y = 3$. On the boundary segment $y = 0$, $f(x,0) = x^2 - 2x = (x-1)^2 - 1$ for $x \in [0,3]$, so the minimum is -1 at $x = 1$ and the maximum is 3 at $x = 3$. On the boundary segment $x + y = 3$, write $y = 3 - x$ with $x \in [0,3]$ and obtain $f(x, 3-x) = 3(x-2)^2$, so the minimum is 0 at $x = 2$ and the maximum is 12 at $x = 0$. Comparing values, the global minimum on D is -1 at $(1,0)$ and the global maximum is 12 at $(0,3)$.

範例

Example 8.16. Least Squares Regression. We apply minimisation to a fundamental problem in data analysis. Given data points $\{(x_i, y_i)\}_{i=1}^n$, we wish to fit a line $y = ax + b$ minimising the sum of squared errors:

$$E(a,b) = \sum_{i=1}^{n}(ax_i + b - y_i)^2.$$

This is a convex function of (a,b). We find the critical point by differentiating E with respect to a and b:

$$\frac{\partial E}{\partial a} = 2\sum(ax_i + b - y_i)x_i = 0 \implies a\sum x_i^2 + b\sum x_i = \sum x_i y_i,$$
$$\frac{\partial E}{\partial b} = 2\sum(ax_i + b - y_i) = 0 \implies a\sum x_i + b \cdot n = \sum y_i.$$

This linear system (the Normal Equations) has a unique solution provided the x_i are not all identical. The Hessian matrix is:

$$H = \begin{bmatrix} E_{aa} & E_{ab} \\ E_{ba} & E_{bb} \end{bmatrix} = \begin{bmatrix} 2\sum x_i^2 & 2\sum x_i \\ 2\sum x_i & 2n \end{bmatrix}.$$

The determinant is $4n\sum x_i^2 - 4(\sum x_i)^2$, which is strictly positive by the Cauchy-Schwarz inequality (unless all x_i are equal). Since $E_{bb} = 2n > 0$, the Hessian is positive definite everywhere. Thus, the solution is a global minimum.

範例

Example 8.17. The Fermat-Torricelli Problem. Given three non-collinear points A, B, C in the plane, find the point P that minimises the sum of distances $S(P) = PA + PB + PC$.
Assuming P is not one of the vertices, the gradient ∇S must be zero.

$$\nabla S = \nabla(\|\mathbf{r} - \mathbf{a}\| + \|\mathbf{r} - \mathbf{b}\| + \|\mathbf{r} - \mathbf{c}\|) = \mathbf{u}_A + \mathbf{u}_B + \mathbf{u}_C = \mathbf{0},$$

where $\mathbf{u}_A$ is the unit vector from P to A. The condition $\mathbf{u}_A + \mathbf{u}_B + \mathbf{u}_C = \mathbf{0}$ implies that the three unit vectors form an equilateral triangle in vector space, meaning the angles between the segments PA, PB, PC must all be $120°$. This point exists inside the triangle only if no angle of $\triangle ABC$ exceeds $120°$. If an angle is $\geq 120°$, the minimum occurs at that vertex.

範例

8.9 Global Optimisation

While the method of Lagrange multipliers identifies stationary points on the boundary or constraint manifold, it does not, by itself, determine the global extrema of a function. To find the absolute maximum and minimum of a differentiable function f on a compact domain D, we must employ a strategy analogous to the "closed interval method" from single-variable calculus.

The Closed Set Method

The Extreme Value Theorem guarantees that a continuous function on a closed and bounded (compact) set $D \subset \mathbb{R}^n$ attains its global maximum and minimum values. Since any local extremum in the interior of D must be a stationary point, and any extremum on the boundary ∂D must be a constrained extremum, the search can be systematised as follows.

Theorem 8.6. ***Global Optimisation Algorithm.***
Let $f : D \to \mathbb{R}$ be a continuous function on a compact set D with a piecewise smooth boundary ∂D. To find the global extrema:

1. **Interior Analysis**: Find all critical points of f in the interior of D (where $\nabla f = \mathbf{0}$ or is undefined). Evaluate f at these points.
2. **Boundary Analysis**: Find the extrema of f restricted to the boundary ∂D. This may be achieved by parameterising the boundary or using Lagrange multipliers. Evaluate f at these points.
3. **Comparison**: Compare all function values found in steps 1 and 2. The largest is the global maximum; the smallest is the global minimum.

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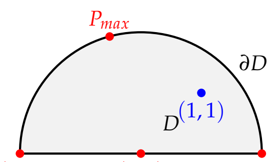

Figure 8.13: Candidates for global extrema on the half-disk include interior critical points (blue) and boundary extrema (red).

Example 8.18. Extrema on a Half-Disk. Find the global maximum and minimum of $f(x,y) = x^4 + y^4 - 4xy + 1$ on the half-disk $D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 4, y \ge 0\}$.
Step 1: Interior Critical Points. We solve $\nabla f = \mathbf{0}$:

$$f_x = 4x^3 - 4y = 0 \implies y = x^3,$$

$$f_y = 4y^3 - 4x = 0 \implies x = y^3.$$

Substituting $y = x^3$ into the second equation yields $x = x^9$, so $x(x^8 - 1) = 0$. The real solutions are $x = 0, 1, -1$.

- $x = 0 \implies y = 0$. Point $(0,0)$.
- $x = 1 \implies y = 1$. Point $(1,1)$.
- $x = -1 \implies y = -1$. Point $(-1,-1)$.

We examine these points with respect to D: $(1,1)$ lies in the interior $(1^2 + 1^2 = 2 < 4, 1 > 0)$. $f(1,1) = 1 + 1 - 4 + 1 = -1$. $(-1,-1)$ is not in D (since $y < 0$). $(0,0)$ lies on the boundary, so we will handle it in Step 2.
Step 2: Boundary Analysis. The boundary ∂D consists of the diameter segment and the semicircular arc.
(i) The Diameter Segment: $y = 0, x \in [-2,2]$. Let $g(x) = f(x,0) = x^4 + 1$. On $[-2,2]$, $g'(x) = 4x^3$. The critical point is $x = 0$. Values: $g(0) = 1$, $g(\pm 2) = 17$.

(ii) The Semicircular Arc: $x = 2\cos t, y = 2\sin t$ for $t \in [0, \pi]$. Let $h(t) = f(2\cos t, 2\sin t) = 16(\cos^4 t + \sin^4 t) - 16\cos t \sin t + 1$. Using the identities $\cos^4 t + \sin^4 t = 1 - 2\sin^2 t\cos^2 t = 1 - \frac{1}{2}\sin^2(2t)$ and $\sin(2t) = 2\sin t\cos t$:

$$h(t) = 16\left(1 - \frac{1}{2}\sin^2(2t)\right) - 8\sin(2t) + 1 = 17 - 8\sin^2(2t) - 8\sin(2t).$$

Let $u = \sin(2t)$. As t ranges from 0 to π, $2t$ ranges from 0 to 2π, so u takes all values in $[-1, 1]$. We maximise $q(u) = 17 - 8u^2 - 8u$ on $[-1, 1]$. $q'(u) = -16u - 8 = 0 \implies u = -1/2$.

- Critical value: $q(-1/2) = 17 - 8(1/4) - 8(-1/2) = 17 - 2 + 4 = 19$.
- Endpoints $u = 1$: $q(1) = 17 - 8 - 8 = 1$.
- Endpoints $u = -1$: $q(-1) = 17 - 8 + 8 = 17$.

Step 3: Comparison. Collecting all candidate values:

- Interior: $f(1, 1) = -1$.
- Diameter: $f(0, 0) = 1$, $f(\pm 2, 0) = 17$.
- Arc: Max value 19 (where $\sin 2t = -1/2$), Min value (on arc) 1.

Conclusion: The global maximum is 19, and the global minimum is -1.

範例

Inequalities via Optimisation

Constrained optimisation is a potent tool for proving inequalities. By finding the maximum of a function subject to a constraint, we establish an upper bound for all points satisfying that constraint.

Example 8.19. A Weighted AM-GM Type Inequality. Prove that for any positive numbers a, b, c:

$$ab^2c^3 \leq 108\left(\frac{a+b+c}{6}\right)^6.$$

範例

Proof

Let $x^2 = a, y^2 = b, z^2 = c$. The inequality is equivalent to finding the maximum of $u = x^2y^4z^6$ subject to $x^2 + y^2 + z^2 = K$. For simplicity, consider the equivalent problem of maximising

$$f(x, y, z) = \ln(x^2y^4z^6) = 2\ln x + 4\ln y + 6\ln z$$

subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 = 6r^2$ (where $x, y, z > 0$). Using Lagrange multipliers:

$$\nabla f = \lambda \nabla g \implies \left\langle \frac{2}{x}, \frac{4}{y}, \frac{6}{z} \right\rangle = \lambda \langle 2x, 2y, 2z \rangle.$$

This yields the system:

$$\frac{2}{x} = 2\lambda x \implies x^2 = \frac{1}{\lambda}, \quad \frac{4}{y} = 2\lambda y \implies y^2 = \frac{2}{\lambda}, \quad \frac{6}{z} = 2\lambda z \implies z^2 = \frac{3}{\lambda}.$$

Substituting into the constraint:

$$\frac{1}{\lambda} + \frac{2}{\lambda} + \frac{3}{\lambda} = 6r^2 \implies \frac{6}{\lambda} = 6r^2 \implies \lambda = \frac{1}{r^2}.$$

Thus the optimal point satisfies $x^2 = r^2$, $y^2 = 2r^2$, $z^2 = 3r^2$. Evaluating the original objective function $P(x, y, z) = x^2 y^4 z^6$ at this point:

$$P_{max} = (r^2)(2r^2)^2(3r^2)^3 = r^2 \cdot 4r^4 \cdot 27r^6 = 108r^{12}.$$

Since $6r^2 = a + b + c$, we have $r^2 = \frac{a+b+c}{6}$. Thus, $ab^2c^3 \leq 108 \left(\frac{a+b+c}{6}\right)^{12/2} = 108 \left(\frac{a+b+c}{6}\right)^6$.

■

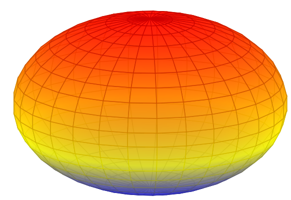

Figure 8.14: An ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. The extrema of z occur where ∇F is vertical, i.e., $F_x = F_y = 0$.

8.10 Extrema of Implicit Functions

Frequently, the objective variable z is not given explicitly as $z = f(x, y)$ but is defined implicitly by an equation $F(x, y, z) = 0$. While one could use the Implicit Function Theorem (*theorem 7.1*) to compute derivatives and set them to zero, it is often more efficient to treat the problem as a constrained optimisation where we optimise the coordinate function z subject to the constraint $F(x, y, z) = 0$.

The Method of Lagrange Multipliers for Implicit Surfaces

To find the extrema of z subject to $F(x, y, z) = 0$, we set up the Lagrangian with objective function $h(x, y, z) = z$:

$$\mathcal{L}(x, y, z, \lambda) = z + \lambda F(x, y, z).$$

The stationarity conditions are:

$$\begin{cases} \mathcal{L}_x = \lambda F_x = 0 \\ \mathcal{L}_y = \lambda F_y = 0 \\ \mathcal{L}_z = 1 + \lambda F_z = 0 \\ F(x, y, z) = 0 \end{cases}$$

From the third equation, $\lambda = -1/F_z$, which implies $\lambda \neq 0$ (assuming F_z is finite). Thus, the conditions $F_x = 0$ and $F_y = 0$ must hold. This recovers the direct method result: the extrema of the implicit surface $z(x, y)$ occur where the normal vector to the surface $F = 0$ is vertical (parallel to the z-axis), i.e., $\nabla F = \langle 0, 0, F_z \rangle$.

Example 8.20. Extrema of an Implicit Surface. Find the extrema of the function $z = z(x, y)$ determined by the equation:

$$2x^2 + y^2 + z^2 + 2xy - 2x - 2y - 4z + 4 = 0.$$

Let $F(x, y, z)$ be the left-hand side. We identify the stationary points where $F_x = 0$ and $F_y = 0$:

$$F_x = 4x + 2y - 2 = 0 \implies 2x + y = 1.$$

$$F_y = 2y + 2x - 2 = 0 \implies x + y = 1.$$

Solving this linear system yields $x = 0, y = 1$. Substitute these coordinates back into the defining equation to find the corresponding z values:

$$2(0)^2 + (1)^2 + z^2 + 2(0)(1) - 2(0) - 2(1) - 4z + 4 = 0$$

$$1 + z^2 - 2 - 4z + 4 = 0 \implies z^2 - 4z + 3 = 0.$$

Factoring gives $(z - 1)(z - 3) = 0$. The stationary points on the surface are $P_1(0, 1, 1)$ and $P_2(0, 1, 3)$. To classify them, we could compute the second derivatives of the implicit function, or simply observe the geometry. Since the surface is a quadric (an ellipsoid), $z = 1$ corresponds to the bottom (minimum) and $z = 3$ to the top (maximum). Specifically, using the Lagrange multiplier $\lambda = -1/F_z$: At $P_1(0, 1, 1)$, $F_z = 2z - 4 = -2$, so $\lambda = 1/2$. At $P_2(0, 1, 3)$, $F_z = 2z - 4 = 2$, so $\lambda = -1/2$. The positive multiplier at the minimum and negative at the maximum is consistent with the orientation of the gradient of the constraint.

範例

8.11 A Higher-Dimensional Rolle's Theorem

We conclude this chapter with a theoretical result that generalises Rolle's Theorem to vector-valued functions. In one dimension, Rolle's Theorem states that if a differentiable function f vanishes at the endpoints of an interval, its derivative must vanish somewhere inside. In higher dimensions, the "vanishing derivative" condition is subtler.

Proposition 8.2. ***Rolle's Theorem for Balls.***
Let $B = B(\mathbf{0}, r) \subset \mathbb{R}^n$ be a ball and let $F : \bar{B} \to \mathbb{R}^m$ be continuous on the closure and differentiable in the interior. Suppose there exists a non-zero vector $\mathbf{v} \in \mathbb{R}^m$ such that:

$$\mathbf{v} \cdot F(\mathbf{x}) = 0 \quad \text{for all } \mathbf{x} \in \partial B.$$

Then there exists a point $\boldsymbol{\xi} \in B$ such that for all $\mathbf{u} \in \mathbb{R}^n$:

$$\mathbf{v} \cdot [DF(\boldsymbol{\xi})\mathbf{u}] = 0.$$

命題

Proof

Define the scalar function $g : \bar{B} \to \mathbb{R}$ by $g(\mathbf{x}) = \mathbf{v} \cdot F(\mathbf{x})$. By hypothesis, g is identically zero on the boundary ∂B. Since g is continuous on a compact set, it attains a global maximum and minimum. If g is constant (zero) everywhere, then $\nabla g(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x}$. If g is not constant, it must attain a non-zero extremum at some interior point $\boldsymbol{\xi} \in B$. At this interior extremum, the gradient vanishes: $\nabla g(\boldsymbol{\xi}) = \mathbf{0}$. By the chain rule (or properties of the dot product derivative), the directional derivative of g along any vector $\mathbf{u}$ is:

$$D_{\mathbf{u}} g(\boldsymbol{\xi}) = \nabla g(\boldsymbol{\xi}) \cdot \mathbf{u} = \mathbf{v} \cdot [DF(\boldsymbol{\xi})\mathbf{u}].$$

Since $\nabla g(\boldsymbol{\xi}) = \mathbf{0}$, this quantity is zero for all $\mathbf{u}$. ∎

Remark (Geometric Interpretation).

Let $n = 2$ and $m = 3$. The image $F(B)$ is a surface in $\mathbb{R}^3$. The condition $\mathbf{v} \cdot F(\mathbf{x}) = 0$ on the boundary means the edge of the surface lies in the plane passing through the origin with normal $\mathbf{v}$. The conclusion states that there is an interior point $\boldsymbol{\xi}$ where the tangent plane to the surface is orthogonal to $\mathbf{v}$ (i.e., parallel to the boundary plane). This perfectly recovers the intuition of the Mean Value Theorem in geometry.

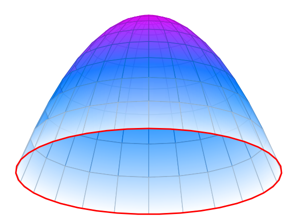

Figure 8.15: The surface $F(B)$ with boundary in the plane $\mathbf{v} \cdot \mathbf{y} = 0$. At the apex $\boldsymbol{\xi}$, the tangent plane is parallel to this boundary plane.

8.12 Exercises

1. **Tangent Lines.** Find the parametric equations for the tangent line to the curve $\mathbf{r}(t) = [t, t^2, t^3]$ at the point where $t = 1$. Show that this line intersects the plane $z = 0$ at the point $(-2/3, 1/3, 0)$.
2. **Normal Planes.** Find the equation of the normal plane to the twisted cubic $\mathbf{r}(t) = [t, t^2, t^3]$ at the origin. Show that the normal planes at t and $-t$ are perpendicular if and only if $t^2 = \dots$

(determine the value).

3. **Intersecting Surfaces.** The surfaces $x^2 + y^2 = 2$ (a cylinder) and $z = x^2 - y^2$ (a hyperbolic paraboloid) intersect to form a curve Γ.
   (a) Find the tangent vector to Γ at the point $(1, 1, 0)$ using the cross product of the gradients.
   (b) Determine the angle at which the curve Γ intersects the plane $x + y + z = 2$.
4. **A Constant Sum Property.** Consider the surface defined by $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{a}$ for $x, y, z > 0$. Prove that the sum of the intercepts of any tangent plane to this surface with the coordinate axes is constant and equal to a.
5. **Orthogonal Families.** Show that the family of spheres $x^2 + y^2 + z^2 = ax$ and the family of spheres $x^2 + y^2 + z^2 = by$ are orthogonal at every point of intersection (excluding the origin).
6. **The Gradient and Level Sets.** Let $f : \mathbb{R}^3 \to \mathbb{R}$ be differentiable.
   (a) Prove that if $\nabla f(\mathbf{x}_0) \neq \mathbf{0}$, then $\nabla f(\mathbf{x}_0)$ is orthogonal to the tangent plane of the level surface $f(x, y, z) = c$ passing through $\mathbf{x}_0$.
   (b) Find the points on the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$ where the tangent plane is parallel to the plane $3x - y + 3z = 1$.
7. **Steepest Ascent.** The temperature at a point (x, y, z) is given by $T(x, y, z) = 200e^{-x^2-3y^2-9z^2}$.
   (a) Determine the direction of fastest temperature increase at the point $P(2, -1, 2)$.
   (b) A mosquito flies from P at a speed of v. Show that to cool down as quickly as possible, it should initially fly towards the origin, but with a trajectory bent towards the x-axis.
8. **The Discrete Laplacian.** Let $u(x, y)$ have continuous second-order partial derivatives in a neighbourhood of (x_0, y_0). Prove that as $h \to 0$:

$$\frac{u(x_0 + h, y_0) + u(x_0 - h, y_0) + u(x_0, y_0 + h) + u(x_0, y_0 - h) - 4u(x_0, y_0)}{h^2} = \Delta u(x_0, y_0) + O(h^2)$$

   where $\Delta u = u_{xx} + u_{yy}$.
9. **Stationary Point Analysis.** For the following functions, determine if $(0, 0)$ is a stationary point. If so, classify it as a local maximum, local minimum, or saddle point.
   (a) $f(x, y) = x^2 - 4xy + 5y^2 - 1$
   (b) $f(x, y) = \sqrt{x^2 + y^2}$

(c) $f(x,y) = (x+y)^2 - y^2$

10. Classification of Critical Points. Find all stationary points for the following functions and classify them using the Hessian matrix.

(a) $u(x,y) = x^2(y-1)^2$

(b) $u(x,y) = 3x^2y - x^4 - 2y^2$

(c) $u(x,y) = (1+e^y)\cos x - ye^y$

11. Local vs Global Extrema. Consider the function $f(x,y) = x^3 - 4x^2 + 2xy - y^2$.

(a) Prove that f has exactly one critical point in $\mathbb{R}^2$ and that this point is a local maximum.

(b) Prove that f does *not* have a global maximum on $\mathbb{R}^2$.

> *Remark.*
>
> Consider the behaviour of the function along a specific line, such as $y = 4x$.

12. The Monkey Saddle. Analyse the critical point of $f(x,y) = x^3 - 3xy^2$ at the origin.

(a) Show that the determinant of the Hessian is zero.

(b) Sketch the regions where $f(x,y) > 0$ and $f(x,y) < 0$ in the xy-plane to show it is a saddle point.

13. Quadratic Approximation. Compute the second-order Taylor polynomial $P_2(x,y)$ for the function $f(x,y) = \frac{\cos x}{\cos y}$ near the origin. Use this to estimate $f(0.1, 0.1)$.

14. Constrained Quartics. Find the global extrema of $f(x,y,z) = x^4 + y^4 + z^4$ subject to the constraint $xyz = 1$.

> *Remark.*
>
> Are the extrema maxima or minima? Is the set compact?

15. Linear on a Sphere. Find the extrema of $u(x,y,z) = x - 2y + 2z$ subject to the constraint $x^2 + y^2 + z^2 = 1$. Interpret this geometrically as the distance of a plane from the origin.

16. Trigonometric Constraints. Find the extrema of $z(x,y) = \cos^2 x + \cos^2 y$ subject to the condition $x - y = \frac{\pi}{4}$ within the square $0 \leq x, y \leq \pi$.

17. Euclidean Distance to a Hyperplane. Use Lagrange multipliers to find the minimum of $f(x,y,z) = x^2 + y^2 + z^2$ subject to $ax + by + cz = k$, where $a, b, c, k > 0$. Confirm your result matches the standard formula for the distance from the origin to a plane.

18. **Extrema on Compact Sets.** Find the maximum and minimum of

$$f(x,y) = \sin x \sin y \sin(x+y)$$

on the domain $D = \{(x,y) \mid x \geq 0, y \geq 0, x+y \leq \pi\}$.

19. **Temperature Extremes.** Suppose that the temperature T in the xy-plane satisfies the differential relations:

$$\frac{\partial T}{\partial x} = 8x - 4y, \quad \frac{\partial T}{\partial y} = 8y - 4x.$$

Use the method of Lagrange multipliers to find the maximum and minimum temperatures on the unit circle $x^2 + y^2 = 1$.

> *Remark.*
>
> One must first integrate the gradient to find $T(x,y)$ up to a constant, or formulate the Lagrange condition directly in terms of the gradient components.

20. **Proximity to a Plane.** Use the method of Lagrange multipliers to find the point on the plane $x + 2y - 3z = 10$ which is closest to the point $(8, 8, 8)$.

> *Remark.*
>
> Minimise the squared distance function to simplify computations.

21. **Triply Orthogonal System.** Let a, b, c be non-zero constants. Prove that the following three surfaces are mutually perpendicular at every point of intersection:

$$S_1 : xy = az^2, \quad S_2 : x^2 + y^2 + z^2 = b, \quad S_3 : z^2 + 2x^2 = c(z^2 + 2y^2).$$

> *Remark.*
>
> Compute the gradients and verify their pairwise dot products vanish.

22. **Optimal Design.** A cylindrical soda can is to be designed to contain a fixed volume V. Assume the cost of production is directly proportional to the surface area. Determine the ratio of the height h to the radius r that minimises the cost.

23. **High-Degree Spherical Extrema.** Find the global extreme values of the function $f(x,y,z) = xy^2z$ on the sphere $x^2 + y^2 + z^2 = R^2$.

> *Remark.*
>
> Consider the symmetry of the function and the constraint.

24. **Optimisation Under Threat.** An agent is investigating an estate on the xy-plane ($z = 0$) and must assume a position that minimises the rate of damage inflicted by three antagonists located at specific coordinates:

- Antagonist A at $(1, 0, 0)$ inflicts damage at a rate of $5/d_A^2$, where d_A is the distance to the agent.
- Antagonist B at $(-1, 1, 0)$ inflicts a constant damage rate of 3 within a radius of 2, and 0 otherwise.
- Antagonist C at $(1, 1, 3)$ inflicts damage at a rate of $5d_C^2$, where d_C is the distance to the agent.

Find the optimal location $(x, y, 0)$ for the agent.

25. **Intersecting Constraints.** Find the extrema of the quadratic form $u = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$ (where $a > b > c > 0$) subject to the two constraints:

$$x^2 + y^2 + z^2 = 1$$

$$x \cos \alpha + y \cos \beta + z \cos \gamma = 0$$

where $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$.

> *Remark.*
>
> This determines the axes of the ellipse formed by the intersection of an ellipsoid and a plane through the centre.

26. **Cyclic Inequality.** Let $a > 0$ be fixed. Find the minimum of $\sum_{i=1}^{n} x_i$ for $x_i > 0$ subject to:

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_n}{x_1} = \frac{1}{a}.$$

Under what condition on a does a solution exist?

27. **Heron's Formula via Lagrange.** Prove that for a triangle with fixed perimeter $2s$, the area is maximised when the triangle is equilateral. Use the area formula $A = \sqrt{s(s-x)(s-y)(s-z)}$ where x, y, z are the side lengths.

28. **Maximum Volume in Ellipsoid.** Find the maximum volume of a rectangular box with sides parallel to the axes that can be inscribed in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

29. **Hölder's Inequality.** Let $p, q > 1$ such that $1/p + 1/q = 1$.

    (a) Maximise $\sum_{i=1}^{n} a_i x_i$ subject to $\sum_{i=1}^{n} x_i^p = 1$ and $x_i \geq 0$.

    (b) Deduce that for any vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, $|\mathbf{a} \cdot \mathbf{b}| \leq \|\mathbf{a}\|_p \|\mathbf{b}\|_q$.

30. **Euler's Homogeneous Function Theorem.** A function $f : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}$ is *homogeneous of degree k* if $f(t\mathbf{x}) = t^k f(\mathbf{x})$ for all $t > 0$.

    (a) Prove that if f is differentiable and homogeneous of degree k, then $\mathbf{x} \cdot \nabla f(\mathbf{x}) = kf(\mathbf{x})$.

    (b) Prove the converse: if $\mathbf{x} \cdot \nabla f(\mathbf{x}) = kf(\mathbf{x})$, then f is homogeneous of degree k.

    > *Remark.*
    > Differentiate $g(t) = t^{-k} f(t\mathbf{x})$ with respect to t.

31. **The Envelope Theorem.** Let $f(x, \alpha)$ be a C^2 function where $x \in \mathbb{R}^n$ is the variable and $\alpha \in \mathbb{R}$ is a parameter. Let $x^*(\alpha)$ be the point where $f(\cdot, \alpha)$ attains its global maximum. Let $V(\alpha) = f(x^*(\alpha), \alpha)$ be the value function. Prove that, assuming $x^*(\alpha)$ is differentiable:

$$V'(\alpha) = \frac{\partial f}{\partial \alpha}(x^*(\alpha), \alpha).$$

> *Remark.*
> This states that the derivative of the maximum value is the partial derivative of the objective function evaluated at the optimum; the variation of x^* does not contribute to the first order change.

32. **Implicit Function Theorem Practice.** The equation $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1 = 0$ defines z implicitly as a function of x and y.

    (a) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

    (b) Find $\frac{\partial^2 z}{\partial x \partial y}$ at the point $(0, 0, 1)$.

33. $\star$ **Convexity and the Hessian.** Let $f : D \to \mathbb{R}$ be a C^2 function on a convex set $D \subseteq \mathbb{R}^n$. Prove that f is convex (i.e., $f(t\mathbf{x} + (1-t)\mathbf{y}) \le tf(\mathbf{x}) + (1-t)f(\mathbf{y})$) if and only if the Hessian matrix $H_f(\mathbf{x})$ is positive semi-definite for all $\mathbf{x} \in D$.

34. $\star$ **Differentiation under the Integral.** Let $F(y) = \int_a^b f(x, y)\, dx$. Using the definition of the partial derivative and the Mean Value Theorem, prove Leibniz's Rule: if f and $\frac{\partial f}{\partial y}$ are continuous, then

$$F'(y) = \int_a^b \frac{\partial f}{\partial y}(x, y)\, dx.$$

    Apply this to compute $g'(y)$ for $g(y) = \int_0^\infty e^{-x^2} \cos(2xy)\, dx$, and hence evaluate the integral.

35. **Conservative Fields and Clairaut's Theorem.** Consider the vector field $\mathbf{F}(x, y) = \langle y^3 + x, x^2 + y \rangle$. Use Clairaut's Theorem (equality of

mixed partials) to prove that there exists no scalar function f such that $\mathbf{F} = \nabla f$.

36. **Zero Gradient Property.** Let $U \subseteq \mathbb{R}^n$ be a path-connected open set. Prove that if $\nabla f(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in U$, then f is constant on U.

> *Remark.*
>
> Use the single-variable Mean Value Theorem along a path connecting any two points in U.

37. **The Multivariate Mean Value Theorem.** Prove that if $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable on an open set containing the line segment connecting points $\mathbf{p}$ and $\mathbf{q}$, then there exists a point $\mathbf{c}$ on the segment such that:

$$f(\mathbf{q}) - f(\mathbf{p}) = \nabla f(\mathbf{c}) \cdot (\mathbf{q} - \mathbf{p}).$$

> *Remark.*
>
> Parametrise the segment as $\mathbf{r}(t) = \mathbf{p} + t(\mathbf{q} - \mathbf{p})$ for $t \in [0, 1]$ and apply the Chain Rule.

38. **The Method of Characteristics.** Consider the system of differential equations:

$$\frac{dx}{dt} = -y, \quad \frac{dy}{dt} = x.$$

Eliminate dt to form a differential equation relating x and y. Solve this equation to find the geometric shape of the trajectories (level curves) along which the motion occurs.

9
Multiple Integration

Following our exploration of local geometric behaviour via differentiation, we now turn to the global properties of functions on $\mathbb{R}^n$: volume, mass, and accumulation. Just as the single-variable integral aggregates values over an interval, the multiple integral aggregates values over a region in higher dimensions. We extend the Riemann sum construction to planar and spatial domains, establishing rigorous conditions for integrability and providing computational tools via iterated integration.

9.1 *The Riemann Integral in* $\mathbb{R}^2$

Let $D \subset \mathbb{R}^2$ be a bounded closed region. We seek to define the integral of a function $f : D \to \mathbb{R}$. The geometric intuition—the volume of the solid bounded by the graph $z = f(x, y)$ and the xy-plane—relies on approximating the region with elementary shapes.

Partitions and Measurability

Definition 9.1. ***Partition and Riemann Sum.***
Let D be a closed, bounded region in $\mathbb{R}^2$. A **partition** T of D is a collection of subregions $\{\sigma_1, \ldots, \sigma_n\}$ with measurable areas $\Delta\sigma_i$, whose interiors are disjoint and whose union is D. The **norm** of the partition, denoted $\|T\|$, is the maximum diameter of any subregion σ_i.
For a function $f : D \to \mathbb{R}$, a **Riemann sum** with respect to T is given by:

$$S(f, T) = \sum_{i=1}^{n} f(\xi_i, \eta_i)\Delta\sigma_i,$$

where (ξ_i, η_i) is an arbitrary sample point chosen within the subregion σ_i.

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Definition 9.2. ***Double Integral.***
The function f is **Riemann integrable** on D if the limit of the Riemann sums exists as the norm of the partition approaches zero, independent of the choice of sample points. We write:

$$\iint_D f(x,y)\,dA = \lim_{\|T\|\to 0} \sum_{i=1}^{n} f(\xi_i, \eta_i)\Delta\sigma_i.$$

In Cartesian coordinates, the area element is $dA = dx\,dy$.

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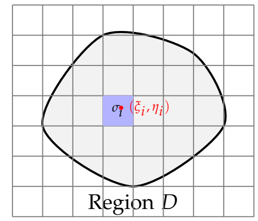

Figure 9.1: A partition of a general region D into subregions. Subregions intersecting the boundary require careful handling regarding measurability.

To ensure this definition is well-posed, the boundary of D must not contribute to the sum in the limit. This requires the concept of **measure zero**. A set $S \subset \mathbb{R}^2$ has measure zero if, for every $\epsilon > 0$, S can be covered by a countable collection of rectangles whose total area is less than ϵ.

Proposition 9.1. ***Lebesgue's Criterion for Integrability.***
Let D be a bounded closed region whose boundary ∂D has measure zero (Jordan measurable). A bounded function $f : D \to \mathbb{R}$ is Riemann integrable on D if and only if the set of discontinuities of f in D has measure zero.

命題

This proposition implies that continuous functions, or bounded functions with discontinuities limited to a finite number of smooth curves, are integrable.

9.2 *Iterated Integrals and Fubini's Theorem*

While the definition relies on the limit of sums, the practical computation of multiple integrals is achieved by reducing them to successive single-variable integrals.

Theorem 9.1. ***Fubini's Theorem.***
Let $R = [a,b] \times [c,d]$ be a rectangular region. If f is continuous on R, then:

$$\iint_R f(x,y)\,dA = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy.$$

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The integrals on the right-hand side are called **iterated integrals**. In the inner integral $\int_c^d f(x,y)\,dy$, the variable x is treated as a constant. The value of this inner integral depends on x, forming the integrand for the outer integral.

Example 9.1. Order of Integration. Let $R = [0,2] \times [0,\pi/2]$. We evaluate the integral of $f(x,y) = y\cos(xy)$ over R.
Using the order $dx\,dy$:

$$\int_0^2 \int_0^{\pi/2} y\cos(xy)\,dx\,dy = \int_0^2 \left[\frac{y}{y}\sin(xy)\right]_{x=0}^{x=\pi/2} dy$$
$$= \int_0^2 \sin\left(\frac{\pi y}{2}\right) dy$$
$$= \left[-\frac{2}{\pi}\cos\left(\frac{\pi y}{2}\right)\right]_0^2$$
$$= -\frac{2}{\pi}(\cos(\pi) - \cos(0)) = \frac{4}{\pi}.$$

If we were to integrate with respect to y first ($\int y\cos(xy)\,dy$), we would require integration by parts, which is significantly more laborious. This highlights the strategic importance of choosing the optimal order of integration.

範例

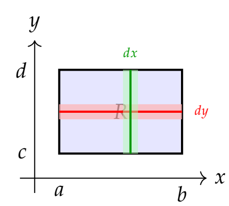

Figure 9.2: The region R for *theorem 9.1*. Integrate by horizontal (dx first) or vertical (dy first) strips.

General Regions

For integrable f (or nonnegative f by Tonelli), Fubini's Theorem extends to non-rectangular regions D by introducing the characteristic function $\chi_D(x,y)$ which is 1 if $(x,y) \in D$ and 0 otherwise, effectively integrating over a bounding rectangle.

- **Type I Region:** $D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$.

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

- **Type II Region:** $D = \{(x,y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$.

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

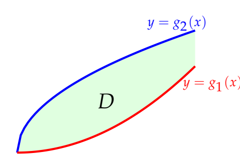

Figure 9.3: A Type I region bounded by $y = g_1(x)$ (lower) and $y = g_2(x)$ (upper). Integration proceeds as $\int_a^b \int_{g_1(x)}^{g_2(x)} dy\,dx$.

Example 9.2. Volume of a Solid. Calculate the volume of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 2 - x$.
The domain D is the unit disk $x^2 + y^2 \le 1$. The height of the solid is given by $f(x,y) = 2 - x$.

$$V = \iint_D (2 - x)\,dA.$$

Using linearity of the integral and symmetry:

$$V = 2\iint_D dA - \iint_D x\,dA.$$

The first term is $2 \times (\text{Area of } D) = 2\pi$. The second term integrates the odd function x over a region symmetric about the y-axis, so it vanishes. Thus, $V = 2\pi$.

範例

Applications and Interpretations

The multiple integral is a versatile tool for aggregating density functions.

Definition 9.3. ***Physical Quantities.***
Let $\rho(x, y, z)$ be a density function defined on a region $B \subset \mathbb{R}^3$.

1. **Volume:** If $\rho = 1$, $V = \iiint_B dV$.
2. **Mass:** If ρ is mass density (mass/volume), $M = \iiint_B \rho(x, y, z)\, dV$.
3. **Charge:** If ρ is charge density, $Q = \iiint_B \rho(x, y, z)\, dV$.

定義

Example 9.3. Mass of a Variable Density Box. Let $B = [0, 1] \times [0, 2] \times [0, 3]$ be a solid with density $\rho(x, y, z) = xyz$. The total mass is:

$$M = \int_0^3 \int_0^2 \int_0^1 xyz\, dx\, dy\, dz.$$

Since the integrand factors as $f(x)g(y)h(z)$ and the limits are constants, the integral separates:

$$M = \left(\int_0^1 x\, dx\right) \left(\int_0^2 y\, dy\right) \left(\int_0^3 z\, dz\right) = \left[\frac{1}{2}\right] [2] \left[\frac{9}{2}\right] = \frac{9}{2}.$$

範例

9.3 *Advanced Methods of Integration*

These techniques streamline computation by reshaping domains and simplifying integrands.

Strategic Order of Integration

Although Fubini's Theorem guarantees that the order of integration does not affect the result for continuous functions, the choice of order can significantly impact the computational complexity.

Example 9.4. Computational Complexity. Consider the integral over the unit square $A = [0, 1] \times [0, 1]$:

$$I = \iint_A \frac{y}{(1 + x^2 + y^2)^{3/2}}\, dA.$$

Integrating with respect to y first:

$$\begin{aligned} I &= \int_0^1 \left(\int_0^1 \frac{y}{(1+x^2+y^2)^{3/2}}\, dy \right) dx \\ &= \int_0^1 \left[-(1+x^2+y^2)^{-1/2} \right]_{y=0}^{y=1} dx \\ &= \int_0^1 \left(\frac{1}{\sqrt{1+x^2}} - \frac{1}{\sqrt{2+x^2}} \right) dx. \end{aligned}$$

Using standard integrals $\int (u^2+a^2)^{-1/2} du = \ln(u+\sqrt{u^2+a^2})$, we obtain:

$$I = \left[\ln(x+\sqrt{1+x^2}) - \ln(x+\sqrt{2+x^2}) \right]_0^1 = \ln\left(\frac{1+\sqrt{2}}{1}\right) - \ln\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right) = \ln\frac{2+\sqrt{2}}{1+\sqrt{3}}.$$

Reverse order (integrating x first) requires evaluating $\int (1 + x^2 + y^2)^{-3/2} dx$. Letting $x = \sqrt{1+y^2}\tan\theta$ yields a more involved trigonometric substitution, confirming that the first order was preferable.

範例

It is crucial to recognise that Fubini's Theorem relies on integrability. If the conditions are not met, the iterated integrals may behave pathologically.

Example 9.5. Failure of Fubini's Theorem. Define f on $A = [0,1] \times [0,1]$ by a variant of the Dirichlet function:

$$f(x,y) = \begin{cases} 1 & \text{if } x \text{ is irrational,} \\ 2y & \text{if } x \text{ is rational.} \end{cases}$$

For any fixed $y \neq 1/2$, the function $x \mapsto f(x,y)$ is discontinuous everywhere, so $\int_0^1 f(x,y)\,dx$ does not exist. Consequently, the iterated integral $\int_0^1 dy \int_0^1 f(x,y)\,dx$ is undefined. However, for a fixed rational x, $\int_0^1 f(x,y)\,dy = \int_0^1 2y\,dy = 1$. For a fixed irrational x, $\int_0^1 f(x,y)\,dy = \int_0^1 1\,dy = 1$. Thus, the inner integral exists for all x, and $\int_0^1 dx \int_0^1 f(x,y)\,dy = \int_0^1 1\,dx = 1$. The double integral $\iint_A f\,dA$ does not exist because the set of discontinuities is not of measure zero (it is the entire square), violating Lebesgue's criterion (*proposition* 9.1).

範例

Change of Variables

The geometric flexibility of integration is greatly enhanced by coordinate transformations. A change of variables substitutes (x, y) with functions of new parameters (u, v), deforming the region of integration to a simpler shape (e.g., transforming a disk to a rectangle).

> **Theorem 9.2.** ***Change of Variables Formula.***
> Let $T : D' \to D$ be a one-to-one transformation given by $x = x(u, v)$ and $y = y(u, v)$, where x, y have continuous partial derivatives on D'. If the **Jacobian determinant**
>
> $$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$
>
> is non-zero on the interior of D', then for an integrable function f:
>
> $$\iint_D f(x, y)\, dx\, dy = \iint_{D'} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv.$$
>
> 定理

The term $|J|\, du\, dv$ represents the local scaling of area elements under the transformation.

> **Example 9.6.** Area of a Lemniscate. We wish to find the area enclosed by the curve
>
> $$(x^2/a^2 + y^2/b^2)^2 = x^2/a^2 - y^2/b^2.$$
>
> This equation suggests a **generalised polar coordinate** transformation:
>
> $$x = a\rho \cos\theta, \quad y = b\rho \sin\theta.$$
>
> The Jacobian is:
>
> $$J = \det \begin{bmatrix} a\cos\theta & -a\rho\sin\theta \\ b\sin\theta & b\rho\cos\theta \end{bmatrix} = ab\rho(\cos^2\theta + \sin^2\theta) = ab\rho.$$
>
> Substituting into the curve equation:
>
> $$(\rho^2)^2 = \rho^2\cos^2\theta - \rho^2\sin^2\theta \implies \rho^4 = \rho^2\cos(2\theta) \implies \rho^2 = \cos(2\theta).$$
>
> For the right loop of the lemniscate, $-\pi/4 \le \theta \le \pi/4$. Due to symmetry, we integrate over the first quadrant ($0 \le \theta \le \pi/4$) and multiply by 4.
>
> $$\text{Area} = \iint_D dA = 4\int_0^{\pi/4}\int_0^{\sqrt{\cos 2\theta}} (ab\rho)\, d\rho\, d\theta.$$
>
> $$= 4ab\int_0^{\pi/4}\left[\frac{1}{2}\rho^2\right]_0^{\sqrt{\cos 2\theta}} d\theta = 2ab\int_0^{\pi/4}\cos(2\theta)\, d\theta = 2ab\left[\frac{1}{2}\sin(2\theta)\right]_0^{\pi/4} = ab.$$
>
> 範例

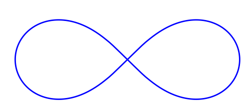

Figure 9.4: The lemniscate of Bernoulli: $\rho^2 = \cos(2\theta)$. When scaled by a and b, the enclosed area is ab.

Example 9.7. A Linear Transformation. Calculate

$$I = \iint_{\Omega} (x+y)\, dA,$$

where Ω is the region bounded by $y^2 = 2x$, $x+y=4$, and $x+y=12$.

The boundaries suggest the substitution $u = x+y$ and $v = y$. The lines become $u = 4$ and $u = 12$. The parabola $y^2 = 2x$ becomes $v^2 = 2(u-v)$, or $u = v^2/2 + v$. To apply the formula, we need the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$. Inverting the map: $x = u - v$, $y = v$.

$$J = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1.$$

The integration bounds for v are determined by the parabola equation $2u = v^2 + 2v \implies v^2 + 2v - 2u = 0$. Solving for v: $v = -1 \pm \sqrt{1+2u}$. Since we consider the region typically in the upper half plane or bounded naturally, we assume the segment between the two branches:

$$I = \int_4^{12} \int_{-1-\sqrt{1+2u}}^{-1+\sqrt{1+2u}} u \cdot 1\, dv\, du.$$

$$I = \int_4^{12} u \left(2\sqrt{1+2u}\right)\, du.$$

Let $t = \sqrt{1+2u}$, so $u = (t^2-1)/2$ and $du = t\,dt$. Limits: $u = 4 \implies t = 3$; $u = 12 \implies t = 5$.

$$I = \int_3^5 \frac{t^2-1}{2}(2t)t\, dt = \int_3^5 (t^4 - t^2)\, dt = \left[\frac{t^5}{5} - \frac{t^3}{3}\right]_3^5 = \frac{8156}{15}.$$

範例

Polar Coordinates and Region Decomposition

Polar coordinates ($x = r\cos\theta, y = r\sin\theta$) are the most common substitution, with Jacobian $J = r$. They are particularly effective when domains are circular sectors or when the integrand involves $x^2 + y^2$.

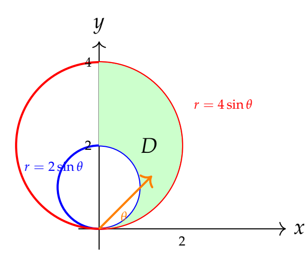

Figure 9.5: The region D bounded by $r = 2\sin\theta$ and $r = 4\sin\theta$. It is simple as a θ-type region but requires decomposition as an r-type region.

Example 9.8. Decomposing a Region. Consider the region

$$D = \{(x,y) \mid 2y \le x^2 + y^2 \le 4y, x > 0\}.$$

In Cartesian coordinates, this region is the difference between two circles tangent to the x-axis at the origin: $x^2 + (y-1)^2 \le 1$ and $x^2 + (y-2)^2 \ge 4$. In polar coordinates:

$$2r\sin\theta \le r^2 \le 4r\sin\theta \implies 2\sin\theta \le r \le 4\sin\theta.$$

Since $x > 0$, we have $\theta \in (0, \pi/2)$. This description defines D as a θ-**type region** (where r bounds depend on θ):

$$\iint_D f\, dA = \int_0^{\pi/2} \int_{2\sin\theta}^{4\sin\theta} f(r\cos\theta, r\sin\theta) r\, dr\, d\theta.$$

Conversely, if we view it as an r-**type region** (where θ bounds depend on r), we must split the integral. The radius r ranges from 0 to 4. However, the condition $2\sin\theta \le r \le 4\sin\theta$ implies $\sin\theta \le r/2$ and $\sin\theta \ge r/4$. From geometric inspection (*figure* 9.5):

- For $0 \le r \le 2$: θ runs from $\arcsin(r/4)$ to $\arcsin(r/2)$.
- For $2 \le r \le 4$: θ runs from $\arcsin(r/4)$ to $\pi/2$.

This decomposition illustrates why choosing the correct "type" of region is vital.

範例

Example 9.9. Integration over a Triangle using Polar Coordinates. Let $D = \{(x, y) \mid 0 \le y \le x \le 1\}$. We compute

$$\iint_D f(\sqrt{x^2 + y^2})\, dA.$$

In polar coordinates, the line $x = 1$ becomes $r\cos\theta = 1 \implies r = \sec\theta$. The line $y = x$ is $\theta = \pi/4$. The region is $0 \le \theta \le \pi/4$ and $0 \le r \le \sec\theta$.

$$I = \int_0^{\pi/4} \int_0^{\sec\theta} f(r) r\, dr\, d\theta.$$

Alternatively, we can integrate with respect to θ first. The maximum radius in the region is $\sqrt{1^2 + 1^2} = \sqrt{2}$. The region splits at $r = 1$:

- For $0 \le r \le 1$: θ ranges from 0 to $\pi/4$.
- For $1 \le r \le \sqrt{2}$: The boundary is $x = 1$, so $\sec\theta \ge r \implies \cos\theta \le 1/r \implies \theta \ge \arccos(1/r)$. Thus $\arccos(1/r) \le \theta \le \pi/4$.

$$I = \int_0^1 \left(\int_0^{\pi/4} d\theta \right) f(r) r\, dr + \int_1^{\sqrt{2}} \left(\int_{\arccos(1/r)}^{\pi/4} d\theta \right) f(r) r\, dr.$$

This form converts the double integral into a single-variable integral involving inverse trigonometric functions.

範例

Symmetry in Integration

Just as $\int_{-a}^{a} f(x)\, dx = 0$ for odd functions, symmetry in $\mathbb{R}^2$ simplifies double integrals.

Proposition 9.2. ***Symmetry Principles.***
Let D be a region symmetric about an axis or the origin.

1. **x-axis symmetry**: If $(x,y) \in D \iff (x,-y) \in D$.
   - If $f(x,-y) = -f(x,y)$, then $\iint_D f\,dA = 0$.
   - If $f(x,-y) = f(x,y)$, then $\iint_D f\,dA = 2\iint_{D\cap\{y\geq 0\}} f\,dA$.
2. **Origin symmetry**: If $(x,y) \in D \iff (-x,-y) \in D$.
   - If $f(-x,-y) = -f(x,y)$, then $\iint_D f\,dA = 0$.

命題

These properties are immediate consequences of the change of variables $(u,v) = (x,-y)$ or $(u,v) = (-x,-y)$ with Jacobian $|J| = 1$. Use of symmetry is highly recommended to eliminate terms before embarking on laborious integration.

9.4 *Triple Integrals*

The extension of the Riemann integral to functions of three variables, $f : B \subset \mathbb{R}^3 \to \mathbb{R}$, follows the same constructive logic as the double integral. For a bounded solid region B, we define the triple integral

$$\iiint_B f(x,y,z)\,dV$$

as the limit of Riemann sums $\sum f(\xi_i,\eta_i,\zeta_i)\Delta V_i$ over partitions of B, where ΔV_i represents the volume of a sub-block.

Iterated Integration in $\mathbb{R}^3$

Just as Fubini's Theorem (*theorem* 9.1) reduces double integrals to iterated single integrals, the evaluation of triple integrals relies on projecting the three-dimensional region onto a coordinate plane.

Definition 9.4. ***Elementary Regions in*** $\mathbb{R}^3$**.**
A region $E \subset \mathbb{R}^3$ is **z-simple** (or Type I) if it lies between two continuous functions of x and y:

$$E = \{(x,y,z) \mid (x,y) \in D,\, u_1(x,y) \leq z \leq u_2(x,y)\},$$

where D is the projection of E onto the xy-plane. Analogous definitions apply for **x-simple** and **y-simple** regions.

定義

Definition 9.5. ***Method of Projection (Filament Summation).***
Let $\Omega \subset \mathbb{R}^3$ be a region bounded above and below by surfaces $z = u_2(x,y)$ and $z = u_1(x,y)$. The triple integral represents the summa-

tion of vertical filaments:

$$\iiint_{\Omega} f(x,y,z)\,dV = \iint_{D} \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right] dA,$$

where D is the orthogonal projection of Ω onto the xy-plane. This corresponds to the standard reduction over a Type I region.

定義

Definition 9.6. ***Method of Cross-Sections (Slice Summation).*** Alternatively, if Ω lies between planes $x = a$ and $x = b$, we may view the integral as the summation of planar slices. Let $D(x)$ be the cross-section of Ω at a fixed x. Then:

$$\iiint_{\Omega} f(x,y,z)\,dV = \int_a^b \left[\iint_{D(x)} f(x,y,z)\,dy\,dz \right] dx.$$

This method is particularly effective when the cross-sections $D(x)$ are simple regions (e.g., disks or rectangles) whose parameters vary continuously with x, or when the integrand allows for immediate simplification over the slice.

定義

For a z-simple region E, the triple integral reduces to:

$$\iiint_{E} f(x,y,z)\,dV = \iint_{D} \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right] dA.$$

Example 9.10. Volume of a Region. Let E be the region bounded by the parabolic cylinder $z = y^2$ and the planes $z = 0$, $x = 0$, $x = 1$, $y = -1$, and $y = 1$. We calculate the volume $V = \iiint_E dV$.
The region is bounded above by $z = y^2$ and below by $z = 0$. The projection onto the xy-plane is the rectangle $D = [0,1] \times [-1,1]$.

$$V = \int_{-1}^{1} \int_0^1 \int_0^{y^2} dz\,dx\,dy = \int_{-1}^{1} \int_0^1 y^2\,dx\,dy = \int_{-1}^{1} y^2\,dy = \left[\frac{y^3}{3} \right]_{-1}^{1} = \frac{2}{3}.$$

範例

Symmetry and Order of Integration

The strategic use of symmetry, as discussed in *proposition* 9.2 for $\mathbb{R}^2$, is even more critical in $\mathbb{R}^3$ to reduce computational burden.

Example 9.11. Cylindrical Wedge and Symmetry. Let B be the solid bounded by the cylinder $x^2 + y^2 = 1$, the plane $z = 0$, and the plane $z = x + y + 1$. We evaluate $\iiint_B x\,dV$.
The geometric description implies $0 \leq z \leq x + y + 1$. The domain D

in the xy-plane is the unit disk $x^2 + y^2 \leq 1$.

$$I = \iint_D \left(\int_0^{x+y+1} x\,dz \right) dA = \iint_D x(x+y+1)\,dA = \iint_D (x^2+xy+x)\,dA.$$

Using the symmetry of the unit disk D:

- $f(x,y) = x$ is odd with respect to x; its integral over D is 0.
- $f(x,y) = xy$ is odd in x (with y fixed) and odd in y (with x fixed); its integral over D is 0.

Thus, only the x^2 term survives.

$$I = \iint_D x^2\,dA.$$

Using polar coordinates for the remaining term ($x = r\cos\theta, dA = r\,dr\,d\theta$):

$$I = \int_0^{2\pi} \int_0^1 (r^2\cos^2\theta) r\,dr\,d\theta = \left(\int_0^1 r^3\,dr \right) \left(\int_0^{2\pi} \cos^2\theta\,d\theta \right) = \frac{1}{4} \cdot \pi = \frac{\pi}{4}.$$

範例

Example 9.12. Integration via Cross-Sections. Calculate $I = \iiint_\Omega (x+y)\,dV$, where Ω is the region bounded by the planes $x = 0$, $x = 1$, and the surface $1 + x^2 = \frac{y^2}{a^2} + \frac{z^2}{b^2}$.

The bounding surface describes a hyperboloid-like tube expanding along the x-axis. The cross-section $D(x)$ at a fixed $x \in [0,1]$ is the region:

$$\frac{y^2}{a^2} + \frac{z^2}{b^2} \leq 1 + x^2.$$

This is an ellipse with semi-axes $A = a\sqrt{1+x^2}$ and $B = b\sqrt{1+x^2}$.

Applying the method of cross-sections:

$$I = \int_0^1 \left[\iint_{D(x)} (x+y)\,dy\,dz \right] dx.$$

The inner integral splits into two terms:

(i) $\iint_{D(x)} x\,dy\,dz = x \cdot \text{Area}(D(x))$. The area of the ellipse is $\pi AB = \pi(a\sqrt{1+x^2})(b\sqrt{1+x^2}) = \pi ab(1+x^2)$. Thus, this term contributes $\pi abx(1+x^2)$.

(ii) $\iint_{D(x)} y\,dy\,dz$. The domain $D(x)$ is symmetric with respect to the z-axis (i.e., $(y,z) \in D(x) \iff (-y,z) \in D(x)$). Since the integrand $g(y,z) = y$ is odd, this integral vanishes.

Substituting back:

$$I = \int_0^1 \pi abx(1+x^2)\,dx = \pi ab \int_0^1 (x+x^3)\,dx = \pi ab \left[\frac{1}{2}x^2 + \frac{1}{4}x^4 \right]_0^1 = \pi ab \left(\frac{1}{2} + \frac{1}{4} \right) = \frac{3}{4}\pi ab.$$

Attempting this via the projection method onto the yz-plane would require handling the projection of the bounding surfaces $x = 0$ and

$x = 1$ within the hyperbolic constraints, which is significantly more laborious.

範例

Example 9.13. The Tetrahedron. Let T be the tetrahedron bounded by the coordinate planes and the plane $x + y + z = 1$. We calculate the volume.

The plane equation yields $z = 1 - x - y$. The projection onto the xy-plane is the triangle bounded by $x = 0, y = 0$, and $x + y = 1$ (obtained by setting $z = 0$).

$$V = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\, dy\, dx = \int_0^1 \int_0^{1-x} (1 - x - y)\, dy\, dx.$$

$$= \int_0^1 \left[(1-x)y - \frac{1}{2}y^2\right]_0^{1-x} dx = \int_0^1 \frac{1}{2}(1-x)^2\, dx = \left[-\frac{1}{6}(1-x)^3\right]_0^1 = \frac{1}{6}.$$

範例

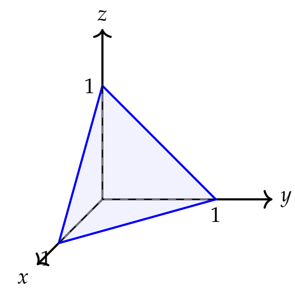

Figure 9.6: The tetrahedron bounded by $x + y + z = 1$. The order of integration corresponds to slicing the solid parallel to the axes.

Example 9.14. Changing Order of Integration. Consider the iterated integral $I = \int_0^1 \int_0^z \int_0^{x+z} 6xz\, dy\, dx\, dz$.

Here, the region E is defined by $0 \leq z \leq 1$, $0 \leq x \leq z$, and $0 \leq y \leq x + z$. Integrating first with respect to y:

$$I = \int_0^1 \int_0^z 6xz(x+z)\, dx\, dz = \int_0^1 \int_0^z (6x^2z + 6xz^2)\, dx\, dz.$$

$$= \int_0^1 \left[2x^3z + 3x^2z^2\right]_{x=0}^{x=z} dz = \int_0^1 (2z^4 + 3z^4)\, dz = \int_0^1 5z^4\, dz = 1.$$

Arbitrarily changing the bounds without geometric analysis is impossible. For instance, to integrate dz first, one must invert the dependencies $x \leq z \leq 1$ and determine how y bounds z (since $y \leq x + z \implies z \geq y - x$).

範例

Geometric Analysis and Visualisation

The successful evaluation of multiple integrals often depends on the initial geometric analysis before any calculus is performed.

1. **Schematic Construction:** Always sketch the region. For 3D regions, identify the projection onto the coordinate planes and the "entry" and "exit" surfaces for the innermost variable.

2. **Intersection Analysis:** Explicitly calculate the curves of intersection between boundary surfaces. For example, the intersection of $x^2 + y^2 + z^2 = R^2$ and $x^2 + y^2 = Rx$ is a curve on the sphere that projects to a circle in the xy-plane.

3. **Symmetry Exploitation:** Before integrating, check for invariance under reflections ($x \to -x$). If the region is symmetric and the function is odd, the integral is zero. If even, the domain can be reduced.

> *Remark.*
>
> In the context of mechanics, the method of cross-sections corresponds to summing the moments of thin laminae, whereas the method of projection corresponds to summing the moments of linear filaments. The choice of method should align with the "natural" decomposition of the physical object.

9.5 *Coordinate Systems in* $\mathbb{R}^3$

While Cartesian coordinates suffice for rectangular boxes, regions with rotational symmetry require coordinate systems that respect their geometry. *theorem* 9.2 extends naturally to $\mathbb{R}^3$.

Definition 9.7. ***Jacobian in*** $\mathbb{R}^3$**.**
Let $T : S \to R$ be a transformation given by $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$. The Jacobian determinant is:

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}.$$

定義

Theorem 9.3. ***Change of Variables in*** $\mathbb{R}^3$**.**
If T is a C^1 diffeomorphism (except possibly on a set of measure zero), then:

$$\iiint_R f(x, y, z)\, dV = \iiint_S f(T(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw.$$

定理

Cylindrical Coordinates

Cylindrical coordinates extend polar coordinates by adding the z-axis.

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

The Jacobian is:

$$J = \det \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = r.$$

Thus, the volume element is $dV = r\,dz\,dr\,d\theta$.

Example 9.15. Volume of a Solid Between Surfaces. Find the volume of the region E bounded above by the paraboloid $z = 18 - 2x^2 - 2y^2$ and below by the plane $z = 0$.

In cylindrical coordinates, the paraboloid is $z = 18 - 2r^2$. The intersection with $z = 0$ occurs at $18 - 2r^2 = 0 \implies r = 3$.

$$V = \int_0^{2\pi} \int_0^3 \int_0^{18-2r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^3 r(18 - 2r^2)\,dr.$$

$$= 2\pi \left[9r^2 - \frac{1}{2}r^4\right]_0^3 = 2\pi \left(81 - \frac{81}{2}\right) = 81\pi.$$

範例

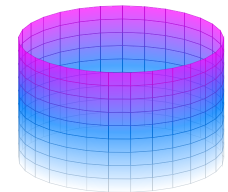

Figure 9.7: A cylinder in $\mathbb{R}^3$. In cylindrical coordinates (r, θ, z), the volume element is $dV = r\,dr\,d\theta\,dz$.

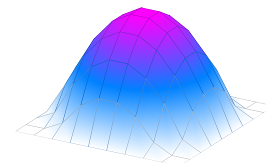

Figure 9.8: The paraboloid $z = 18 - 2r^2$ bounded below by $z = 0$. The volume beneath is computed via cylindrical coordinates.

Spherical Coordinates

Spherical coordinates (ρ, θ, ϕ) are defined by:

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi.$$

Here $\rho \geq 0$ is the distance from the origin, $0 \leq \phi \leq \pi$ is the polar angle (from the positive z-axis), and $0 \leq \theta \leq 2\pi$ is the azimuthal angle.

Claim 9.1. Jacobian of Spherical Coordinates.

$$\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = -\rho^2 \sin\phi.$$

Note

The sign depends on the chosen parameter ordering; the volume element uses $|J|$.

Taking the absolute value, the volume element is $dV = \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$.

主張

Proof

Expanding the determinant:

$$J = \det \begin{bmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{bmatrix}.$$

Straightforward computation (using $\cos^2 + \sin^2 = 1$) yields $-\rho^2 \sin\phi$.

■

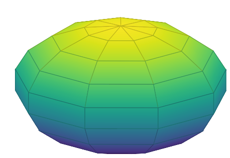

Figure 9.9: A sphere in spherical coordinates (ρ, ϕ, θ). The volume element is $dV = \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$.

Example 9.16. Moment of Inertia of a Sphere. Let B be a ball of radius R with constant density δ. We calculate the moment of inertia about the z-axis,

$$I_z = \iiint_B \delta(x^2 + y^2)\, dV.$$

In spherical coordinates, $x^2 + y^2 = (\rho \sin\phi)^2$. The region is $0 \le \rho \le R$, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$.

$$I_z = \delta \int_0^{2\pi} \int_0^{\pi} \int_0^{R} (\rho^2 \sin^2\phi)(\rho^2 \sin\phi)\, d\rho\, d\phi\, d\theta.$$

Separating variables:

$$I_z = \delta \left(\int_0^{2\pi} d\theta \right) \left(\int_0^{R} \rho^4\, d\rho \right) \left(\int_0^{\pi} \sin^3\phi\, d\phi \right).$$

The integrals are 2π, $R^5/5$, and $\int_0^{\pi} (1 - \cos^2\phi) \sin\phi\, d\phi = 4/3$.

$$I_z = \delta(2\pi) \frac{R^5}{5} \frac{4}{3} = \frac{8\pi\delta R^5}{15}.$$

Since the mass $M = \frac{4}{3}\pi R^3 \delta$, we can rewrite this as $I_z = \frac{2}{5} M R^2$.

範例

Example 9.17. Volume of an Ellipsoid. Calculate the volume of the region E defined by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$.
We use the generalised coordinate transformation $x = au, y = bv, z = cw$. The Jacobian is abc. The region E transforms into the unit ball B in uvw-space.

$$V = \iiint_E dV = \iiint_B (abc)\, du\, dv\, dw = abc \times \text{Vol}(B) = \frac{4}{3}\pi abc.$$

範例

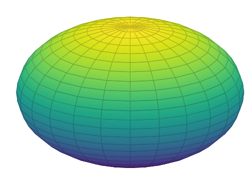

Figure 9.10: An ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Its volume is $V = \frac{4}{3}\pi abc$.

Applications: Center of Mass

For a solid E with density function $\delta(x, y, z)$, the mass is $M = \iiint_E \delta\, dV$ and the centre of mass $(\bar{x}, \bar{y}, \bar{z})$ is given by the moment integrals:

$$\bar{z} = \frac{1}{M} \iiint_E z\delta(x, y, z)\, dV.$$

Example 9.18. Center of Mass of a Conical Solid. Find the centre of mass of the solid E bounded above by the sphere $x^2 + y^2 + z^2 = 1$ and below by the cone $z = \sqrt{x^2 + y^2}$, assuming constant density $\delta = 1$.
By symmetry, $\bar{x} = \bar{y} = 0$. In spherical coordinates, the sphere is $\rho =$

1. The cone $z = r$ corresponds to $\phi = \pi/4$. Thus $0 \le \phi \le \pi/4$.

$$M = \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = 2\pi \left[\frac{\rho^3}{3}\right]_0^1 [-\cos\phi]_0^{\pi/4} = \frac{2\pi}{3}\left(1 - \frac{\sqrt{2}}{2}\right).$$

The moment M_{xy} is:

$$\iiint_E z \, dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 (\rho\cos\phi)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = 2\pi \left[\frac{\rho^4}{4}\right]_0^1 \left[\frac{\sin^2\phi}{2}\right]_0^{\pi/4} = \frac{\pi}{8}.$$

Thus, $\bar{z} = \frac{\pi/8}{M} = \frac{3}{16(1-1/\sqrt{2})} \approx 0.64$.

範例

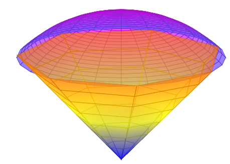

Figure 9.11: The ice-cream cone region: bounded above by the sphere $\rho = 1$ and below by the cone $\phi = \pi/4$.

9.6 Techniques in Triple Integration

While *theorems* 9.1 and 9.3 provide the theoretical infrastructure for multiple integration, complex problems often require a synthesis of geometric decomposition and algebraic manipulation. We now examine regions formed by the intersection of curved surfaces and integrals involving quadratic forms, which bridge calculus and linear algebra.

Decomposition of Complex Regions

When a region Ω is defined by the intersection of multiple surfaces, the boundary description often changes within the domain. In such cases, one must decompose Ω into subregions where the bounds are defined by single analytic functions, effectively applying the additivity of the integral.

Example 9.19. Intersection of Two Spheres. Evaluate

$$I = \iiint_\Omega z^2 \, dV,$$

where Ω is the region common to the spheres $x^2 + y^2 + z^2 \le R^2$ and $x^2 + y^2 + z^2 \le 2Rz$.

The second inequality may be rewritten as $x^2 + y^2 + (z - R)^2 \le R^2$, representing a sphere of radius R centred at $(0, 0, R)$. The intersection of the boundary surfaces $x^2 + y^2 + z^2 = R^2$ and $x^2 + y^2 + z^2 = 2Rz$ occurs where $R^2 = 2Rz$, yielding the plane $z = R/2$. Geometrically, Ω is a lens-shaped region symmetric about the z-axis.

The region naturally splits at $z = R/2$:

1. Ω_1 (lower part): $0 \le z \le R/2$. Bounded by the sphere $x^2 + y^2 + z^2 \le 2Rz$.

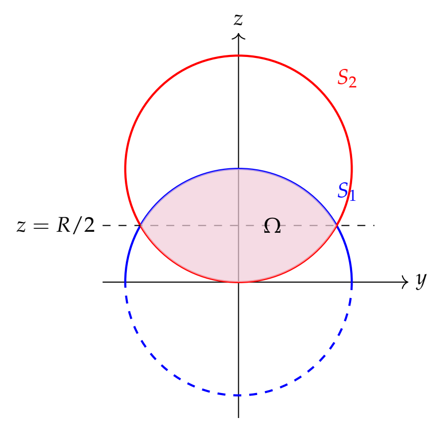

Figure 9.12: Cross-section of the region Ω in the yz-plane. The integration splits at the intersection plane $z = R/2$.

2. Ω_2 (upper part): $R/2 \le z \le R$. Bounded by the sphere $x^2 + y^2 + z^2 \le R^2$.

Using the "slicing" method, we fix z and integrate over the circular cross-section $D(z)$. For Ω_1, the condition $x^2 + y^2 \le 2Rz - z^2$ implies the area of $D(z)$ is $\pi(2Rz - z^2)$. For Ω_2, the condition $x^2 + y^2 \le R^2 - z^2$ implies the area of $D(z)$ is $\pi(R^2 - z^2)$. Thus:

$$I = \int_0^{R/2} z^2 \left[\pi(2Rz - z^2)\right] dz + \int_{R/2}^{R} z^2 \left[\pi(R^2 - z^2)\right] dz.$$

Evaluating these single-variable integrals:

$$I_1 = \pi \int_0^{R/2} (2Rz^3 - z^4)\, dz = \pi \left[\frac{1}{2}Rz^4 - \frac{1}{5}z^5\right]_0^{R/2} = \pi R^5 \left(\frac{1}{32} - \frac{1}{160}\right) = \frac{\pi R^5}{40}.$$

$$I_2 = \pi \int_{R/2}^{R} (R^2z^2 - z^4)\, dz = \pi \left[\frac{1}{3}R^2z^3 - \frac{1}{5}z^5\right]_{R/2}^{R} = \pi R^5 \left(\left(\frac{1}{3} - \frac{1}{5}\right) - \left(\frac{1}{24} - \frac{1}{160}\right)\right).$$

Summing these yields $I = \frac{59}{480}\pi R^5$.

範例

Quadratic Forms and Linear Algebra

Coordinate transformations are not limited to geometric classes (cylindrical, spherical). Spectral theory allows us to simplify integrals involving general quadratic forms.

Example 9.20. Gaussian Integral with a Positive Definite Matrix. Let A be a 3×3 symmetric positive definite matrix, and let $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^3$. We compute:

$$I = \iiint_{Q(\mathbf{x})\le 1} e^{\sqrt{Q(\mathbf{x})}}\, dV.$$

By the Spectral Theorem, there exists an orthogonal matrix P ($P^TP = I, \det P = 1$) such that $P^TAP = \Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$, where $\lambda_i > 0$ are the eigenvalues of A. Consider the linear transformation $\mathbf{x} = P\mathbf{y}$. Since P is orthogonal, the Jacobian is 1. The quadratic form becomes:

$$\mathbf{x}^T A\mathbf{x} = (P\mathbf{y})^T A(P\mathbf{y}) = \mathbf{y}^T(P^TAP)\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2.$$

To standardise the region, apply the scaling transformation $y_i = u_i/\sqrt{\lambda_i}$. The Jacobian of this second transformation is $(\lambda_1\lambda_2\lambda_3)^{-1/2} = (\det A)^{-1/2}$. The region $Q(\mathbf{x}) \le 1$ transforms into the unit ball $B = \{\mathbf{u} \mid u_1^2 + u_2^2 + u_3^2 \le 1\}$. The integrand becomes $e^{\sqrt{|\mathbf{u}|^2}} = e^{\rho}$.

$$I = \iiint_B e^{\rho}(\det A)^{-1/2}\, du_1\, du_2\, du_3.$$

Switching to spherical coordinates for the unit ball:

$$I = \frac{1}{\sqrt{\det A}} \int_0^{2\pi} \int_0^{\pi} \int_0^1 e^{\rho} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

The angular integrals contribute 4π. The radial integral is $\int_0^1 \rho^2 e^{\rho} \, d\rho = [e^{\rho}(\rho^2 - 2\rho + 2)]_0^1 = e - 2$. Thus, $I = \frac{4\pi(e-2)}{\sqrt{\det A}}$.

範例

9.7 Integration in Higher Dimensions

Generalisation to $\mathbb{R}^n$

The Riemann integral generalises naturally to $\mathbb{R}^n$. Let $\Omega \subset \mathbb{R}^n$ be a bounded region. We define

$$\int_{\Omega} f(\mathbf{x}) \, d^n\mathbf{x}$$

via the limit of sums over n-dimensional hyperrectangles. The computational tools—*theorems* 9.1 and 9.3—apply verbatim, provided one can compute the Jacobian determinant of the transformation $T : \mathbb{R}^n \to \mathbb{R}^n$.

Hyperspherical Coordinates

To integrate over n-dimensional spheres, we employ generalised spherical coordinates. For $\mathbb{R}^4$, with variables (x, y, z, w), we define:

$$\begin{aligned} x &= r \sin\psi \sin\phi \cos\theta \\ y &= r \sin\psi \sin\phi \sin\theta \\ z &= r \sin\psi \cos\phi \\ w &= r \cos\psi \end{aligned}$$

where $r \geq 0$, $\psi, \phi \in [0, \pi]$, and $\theta \in [0, 2\pi)$. The Jacobian determinant is given by:

$$J = \frac{\partial(x, y, z, w)}{\partial(r, \psi, \phi, \theta)} = r^3 \sin^2\psi \sin\phi.$$

Example 9.21. Volume of a 4-Ball. Let $B_4(a)$ be the region $x^2 + y^2 + z^2 + w^2 \leq a^2$. The volume is:

$$V_4 = \int_0^{2\pi} d\theta \int_0^{\pi} d\phi \int_0^{\pi} d\psi \int_0^{a} r^3 \sin^2\psi \sin\phi \, dr.$$

Separating the integrals:

$$V_4 = \left(\int_0^a r^3 \, dr \right) \left(\int_0^{2\pi} d\theta \right) \left(\int_0^{\pi} \sin\phi \, d\phi \right) \left(\int_0^{\pi} \sin^2\psi \, d\psi \right).$$

Evaluating each component:

- Radial: $[\frac{1}{4}r^4]_0^a = \frac{a^4}{4}$.
- Theta: 2π.
- Phi: $[-\cos\phi]_0^\pi = 2$.
- Psi: $\int_0^\pi \frac{1-\cos(2\psi)}{2}\, d\psi = \frac{\pi}{2}$.

$$V_4 = \frac{a^4}{4} \cdot 2\pi \cdot 2 \cdot \frac{\pi}{2} = \frac{\pi^2 a^4}{2}.$$

This result is consistent with the general formula for the volume of an n-ball, $V_n(a) = \frac{\pi^{n/2}}{\Gamma(n/2+1)} a^n$.

範例

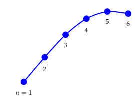

Figure 9.13: Volume of the unit n-ball. The volume peaks near $n \approx 5$ then decreases as dimension increases.

Example 9.22. The Hypercylinder. Consider the region $E \subset \mathbb{R}^4$ defined by $x^2 + y^2 \leq R^2$ and $0 \leq z, w \leq h$. This represents a cylinder in the xy-plane extended linearly in z and w.
We use polar coordinates for (x, y) and keep z, w Cartesian.

$$dV = r\, dr\, d\theta\, dz\, dw.$$

$$V = \int_0^h \int_0^h \int_0^{2\pi} \int_0^R r\, dr\, d\theta\, dz\, dw = h \cdot h \cdot 2\pi \cdot \frac{R^2}{2} = \pi R^2 h^2.$$

範例

9.8 *The Algebra of Volume Elements*

The change of variables formula relies on the Jacobian determinant, often justified by linear approximation of volume elements. A more robust algebraic framework is provided by the **wedge product** ($\wedge$) of differential forms. This formalism explains the geometric orientation and the "stretching" factors inherent in coordinate transformations.

Definition 9.8. ***The Wedge Product.***
The wedge product is an associative, anticommutative operation on differentials. The fundamental properties are:

1. $dx \wedge dy = -dy \wedge dx$ (Anticommutativity).
2. $dx \wedge dx = 0$ (Nilpotency).
3. $df = \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv$ (Total Differential).

The volume element in $\mathbb{R}^n$ is the wedge product of the coordinate differentials: $dV = dx_1 \wedge \cdots \wedge dx_n$.

定義

This algebra automatically generates the Jacobian determinant. For a linear transformation represented by a matrix A, if $\mathbf{x} = A\mathbf{u}$, then $dx_1 \wedge \cdots \wedge dx_n = (\det A)\, du_1 \wedge \cdots \wedge du_n$.

Example 9.23. Deriving the Polar Area Element. Let $x = r\cos\theta$ and $y = r\sin\theta$. Taking differentials:

$$dx = \cos\theta\, dr - r\sin\theta\, d\theta,$$
$$dy = \sin\theta\, dr + r\cos\theta\, d\theta.$$

Computing the wedge product:

$$\begin{aligned} dx \wedge dy &= (\cos\theta\, dr - r\sin\theta\, d\theta) \wedge (\sin\theta\, dr + r\cos\theta\, d\theta) \\ &= (\cos\theta\sin\theta) dr \wedge dr + (r\cos^2\theta) dr \wedge d\theta - (r\sin^2\theta) d\theta \wedge dr - (r^2\sin\theta\cos\theta) d\theta \wedge d\theta. \end{aligned}$$

Using $dr \wedge dr = 0$, $d\theta \wedge d\theta = 0$, and $d\theta \wedge dr = -dr \wedge d\theta$:

$$dx \wedge dy = (r\cos^2\theta + r\sin^2\theta) dr \wedge d\theta = r\, dr \wedge d\theta.$$

This recovers the familiar area element $dA = r\, dr\, d\theta$.

範例

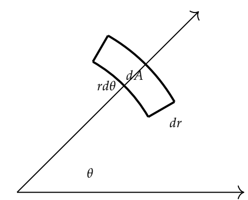

Figure 9.14: The area element dA in polar coordinates corresponds to the differential form $r\, dr \wedge d\theta$.

Example 9.24. Spherical Volume Element. Using spherical coordinates $x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta, z = \rho\cos\phi$.
The differential form is $dV = dx \wedge dy \wedge dz$. The expansion involves significant algebra, but systematic application of the wedge rules simplifies terms like $d\phi \wedge d\phi$ to zero immediately.

$$dx \wedge dy \wedge dz = \det\left(\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}\right) d\rho \wedge d\phi \wedge d\theta = \rho^2 \sin\phi\, d\rho \wedge d\phi \wedge d\theta.$$

This algebraic approach validates the geometric intuition that the volume of the infinitesimal block is the product of its side lengths: $d\rho$, $\rho d\phi$, and $\rho\sin\phi d\theta$.

範例

9.9 *Integrals over Unbounded Regions*

Let $D \subset \mathbb{R}^n$ be an unbounded set. We seek to define $\int_D f\, dV$ by approximating D with a sequence of bounded, measurable subregions.

Definition 9.9. ***Exhaustion of a Region.***
A sequence of bounded, measurable, closed regions $\{K_m\}_{m=1}^{\infty}$ is an **exhaustion** of D if:

1. $K_m \subset K_{m+1} \subset D$ for all m.
2. $D = \bigcup_{m=1}^{\infty} K_m$.
3. Any bounded measurable subset of D is contained in some K_m.

定義

Definition 9.10. ***Improper Integrability.***
Let $f : D \to \mathbb{R}$ be locally integrable (integrable on every bounded measurable subset of D). We say f is **improperly integrable** on D with integral I if, for every exhaustion $\{K_m\}$ of D:

$$I = \lim_{m\to\infty} \int_{K_m} f(\mathbf{x})\, d^n\mathbf{x}.$$

If the limit depends on the choice of exhaustion or fails to exist, the integral diverges.

定義

A critical distinction between $\mathbb{R}^1$ and $\mathbb{R}^n$ is the role of absolute convergence. In $\mathbb{R}^1$, the conditional integral $\lim_{b\to\infty} \int_{-b}^{b} x\, dx = 0$ exists, even though $\int_{-\infty}^{\infty} |x|\, dx$ diverges. In $\mathbb{R}^n$, requiring the limit to be independent of the exhaustion enforces absolute convergence.

Theorem 9.4. ***Absolute Convergence Requirement.***
Let f be locally integrable on an unbounded region D. The improper integral $\int_D f\, dV$ converges if and only if $\int_D |f|\, dV$ converges.

定理

This is stronger than the symmetric-limit notion in one dimension; it matches absolute (Lebesgue) integrability.
This theorem simplifies the analysis significantly: for non-negative functions, we need only test a single convenient exhaustion (typically balls B_R or hypercubes $[-R, R]^n$).

Convergence Tests

Since absolute convergence is required, comparison tests form the primary machinery for determining integrability.

Theorem 9.5. ***Comparison Test.***
Let $f, g : D \to \mathbb{R}$ be locally integrable functions on an unbounded domain D.

1. If $|f(\mathbf{x})| \leq g(\mathbf{x})$ almost everywhere and $\int_D g\, dV$ converges, then $\int_D f\, dV$ converges.
2. If $0 \leq g(\mathbf{x}) \leq f(\mathbf{x})$ almost everywhere and $\int_D g\, dV$ diverges, then $\int_D f\, dV$ diverges.

定理

For regions extending to infinity, the decay rate of the function relative to the volume growth determines convergence. In $\mathbb{R}^2$, the area element $r\, dr\, d\theta$ implies that f must decay faster than $1/r^2$.

Corollary 9.1. Cauchy Test for Radial Decay

推論

Let D be the exterior of a disk, $D = \{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\| \geq R_0\}$.
1. If $|f(\mathbf{x})| \leq \frac{C}{\|\mathbf{x}\|^p}$ for some $p > 2$, then $\iint_D f \, dA$ converges.
2. If $|f(\mathbf{x})| \geq \frac{C}{\|\mathbf{x}\|^p}$ for some $p \leq 2$, then $\iint_D f \, dA$ diverges.

Proof

In polar coordinates, the comparison integral is $\int_{R_0}^{\infty} r^{-p} r \, dr = \int_{R_0}^{\infty} r^{1-p} \, dr$. This converges if and only if $1 - p < -1$, i.e., $p > 2$. ■

Example 9.25. A Power-Decay Integral on an Unbounded Region. Evaluate

$$I = \iint_D \frac{1}{(x+y)^p} \, dA$$

for $D = \{(x, y) \mid 0 \leq x \leq 1, x + y \geq 1\}$.
The bounds are $x \in [0, 1]$ and $y \geq 1 - x$, so the region is **unbounded** in the y direction. Let us define the exhaustion $D_n = \{(x, y) \in D \mid x + y \leq n\}$. We employ the change of variables $u = x, v = x + y$. The Jacobian is $\partial(x, y)/\partial(u, v) = 1$. The region D_n maps to $0 \leq u \leq 1$ and $1 \leq v \leq n$.

$$I_n = \int_0^1 \int_1^n \frac{1}{v^p} \, dv \, du = \left[\frac{v^{1-p}}{1-p} \right]_1^n = \frac{1}{1-p}(n^{1-p} - 1).$$

For the limit $n \to \infty$ to exist, we require $1 - p < 0 \implies p > 1$. In this case, $I = \frac{1}{p-1}$.

範例

9.10 *The Gaussian Probability Integral*

The evaluation of $\int_{-\infty}^{\infty} e^{-x^2} \, dx$ is a prototypical application of improper multiple integration, famously solved by Poisson. It relies on the transition between Cartesian exhaustions (squares) and polar exhaustions (disks).

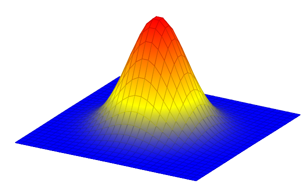

Figure 9.15: The Gaussian surface $z = e^{-(x^2+y^2)}$. The volume beneath equals π, yielding $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.

Example 9.26. The Gaussian Integral. We aim to compute

$$I = \int_{-\infty}^{\infty} e^{-x^2} \, dx.$$

Consider the integral of $f(x, y) = e^{-(x^2+y^2)}$ over $\mathbb{R}^2$. Let $S_R = [-R, R] \times [-R, R]$ be a square exhaustion. By Fubini's Theorem:

$$I_R = \iint_{S_R} e^{-(x^2+y^2)} \, dA = \left(\int_{-R}^{R} e^{-x^2} \, dx \right) \left(\int_{-R}^{R} e^{-y^2} \, dy \right) = \left(\int_{-R}^{R} e^{-x^2} \, dx \right)^2.$$

Thus, $\lim_{R\to\infty} I_R = I^2$.
Now consider the disk exhaustion $B_R = \{(x,y) \mid x^2+y^2 \le R^2\}$. Using polar coordinates:

$$J_R = \iint_{B_R} e^{-(x^2+y^2)}\, dA = \int_0^{2\pi}\int_0^R e^{-r^2} r\, dr\, d\theta.$$

Using the substitution $u = r^2$:

$$J_R = 2\pi\left[-\frac{1}{2}e^{-r^2}\right]_0^R = \pi(1-e^{-R^2}).$$

Taking the limit, $\lim_{R\to\infty} J_R = \pi$. Since $f > 0$, the improper integral exists and is unique. Thus $I^2 = \pi$, implying $I = \sqrt{\pi}$.

範例

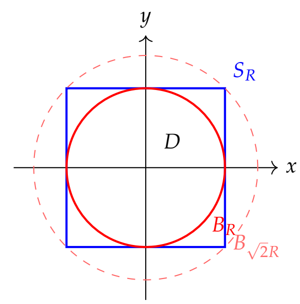

Figure 9.16: The exhaustion comparison. We have $B_R \subset S_R \subset B_{\sqrt{2}R}$. Since $f \ge 0$, the limits over squares and disks must coincide.

9.11 *Singular Functions on Bounded Regions*

The second type of improper integral arises when the domain D is bounded but the function f is unbounded (singular) at some point or along a curve.

Definition 9.11. ***Singular Points.***
Let D be a bounded region and f be continuous on $D \setminus \{p_0\}$. We define

$$\iint_D f(\mathbf{x})\, dA = \lim_{\epsilon\to 0^+} \iint_{D\setminus B_\epsilon(p_0)} f(\mathbf{x})\, dA,$$

where $B_\epsilon(p_0)$ is a ball of radius ϵ centred at the singularity p_0.

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Analogous to the decay test at infinity, we have a growth test near the singularity. For a singularity at the origin in $\mathbb{R}^2$, the integral converges if $f(r) \sim 1/r^p$ with $p < 2$.

9.12 *Counter-examples and Conditional Convergence*

We defined improper integrability via independence of the exhaustion. This effectively rules out conditionally convergent integrals. The following example demonstrates why this restriction is necessary.

Example 9.27. Dependence on Exhaustion. Consider

$$f(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}$$

over $\mathbb{R}^2 \setminus B_1(0)$.
Let us test convergence on the region E defined by $x \ge 1, y \ge 1$

and $2y \le x \le 3y$. This is an infinite sector. In polar coordinates, $f(r,\theta) = \frac{r^2\cos(2\theta)}{r^4} = \frac{\cos(2\theta)}{r^2}$. While the $1/r^2$ decay suggests possible divergence (the boundary case of the Cauchy test), we look at the absolute value on the specific sector. On the sector $2y \le x \le 3y$, we have $x > y$, so $x^2 - y^2 > 0$. Additionally, $y/x \in [1/3, 1/2]$, so $\frac{x^2-y^2}{x^2+y^2} = \frac{1-(y/x)^2}{1+(y/x)^2} \ge \frac{3}{5}$. Thus $|f| \ge \frac{3}{5}\frac{1}{r^2}$. Integrating this over the sector:

$$\int_{\theta_1}^{\theta_2} \int_1^\infty \frac{C}{r^2} r\, dr\, d\theta = C' \int_1^\infty \frac{1}{r}\, dr,$$

which diverges logarithmically. However, if one were to integrate over a symmetric square $[-R, R]^2$, symmetry arguments might lead to a cancellation of positive and negative infinite contributions, yielding a false "limit". Since the absolute integral diverges, the improper integral is undefined.

範例

9.13 *Geometric Applications of Integration*

Having established the rigorous framework for multiple integration, we now turn to its applications. While the calculation of volume and mass follows directly from the definition, more sophisticated geometric and physical quantities require the summation of differential elements that are not merely rectangular blocks. This method, often termed the **method of differential elements**, constructs the integral by summing local contributions—weighted by geometry or physics—over the domain.

Surface Area

To define the area of a curved surface S, we approximate it locally by its tangent plane, analogous to approximating a curve by its tangent line.

Definition 9.12. ***Surface Area Integral.***
Let S be a surface defined by the graph $z = f(x, y)$ over a region $D \subset \mathbb{R}^2$, where f is continuously differentiable. The area of S is given by:

$$\text{Area}(S) = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dA.$$

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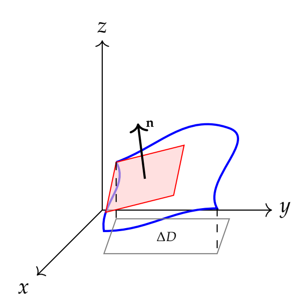

Figure 9.17: Approximating a surface patch ΔS by the area of the tangent parallelogram $\Delta\sigma$. The ratio of areas is determined by the angle γ between the normal $\mathbf{n}$ and the z-axis.

Derivation

Consider a small rectangle ΔD in the xy-plane with area ΔA.

The corresponding patch on the surface ΔS is approximated by the portion of the tangent plane above ΔD. If $\mathbf{n}$ is the unit normal to the surface, and γ is the angle between $\mathbf{n}$ and the z-axis (the normal to D), elementary geometry yields the projection relation $\Delta A = \Delta S|\cos\gamma|$. For $z = f(x,y)$, a normal vector is $\mathbf{N} = (f_x, f_y, -1)$. Thus:

$$|\cos\gamma| = \frac{|\mathbf{N}\cdot\mathbf{k}|}{\|\mathbf{N}\|} = \frac{1}{\sqrt{1+f_x^2+f_y^2}}.$$

The differential element of surface area is therefore $dS = \frac{dA}{|\cos\gamma|} = \sqrt{1+f_x^2+f_y^2}\,dA$.

■

More generally, for a surface defined parametrically by $\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v))$, the area element is determined by the cross product of the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$.

Theorem 9.6. ***Parametric Surface Area.***
If S is parameterised by $\mathbf{r} : D \to \mathbb{R}^3$, the surface area is:

$$\text{Area}(S) = \iint_D \|\mathbf{r}_u \times \mathbf{r}_v\|\,du\,dv = \iint_D \sqrt{EG - F^2}\,du\,dv,$$

where $E = \mathbf{r}_u\cdot\mathbf{r}_u$, $F = \mathbf{r}_u\cdot\mathbf{r}_v$, and $G = \mathbf{r}_v\cdot\mathbf{r}_v$ are the coefficients of the first fundamental form.

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Example 9.28. Surface Area of Revolution. Let Σ be the surface generated by rotating the curve $z = \varphi(x)$ ($a \le x \le b$) about the z-axis. We parameterise using cylindrical coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = \varphi(r),$$

where $r \in [a,b]$ and $\theta \in [0, 2\pi]$. The derivatives are $\mathbf{r}_r = (\cos\theta, \sin\theta, \varphi'(r))$ and $\mathbf{r}_\theta = (-r\sin\theta, r\cos\theta, 0)$. The metric coefficients are:

$$E = 1 + (\varphi'(r))^2, \quad F = 0, \quad G = r^2.$$

Thus, the surface area is:

$$S = \int_0^{2\pi} d\theta \int_a^b \sqrt{r^2(1+\varphi'(r)^2)}\,dr = 2\pi\int_a^b r\sqrt{1+(\varphi'(r))^2}\,dr.$$

This recovers the standard formula from single-variable calculus, derived via the summation of frustums.

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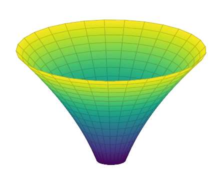

Figure 9.18: A surface of revolution generated by rotating $z = \sqrt{r}$ about the z-axis. Its area is computed via $2\pi\int r\sqrt{1+\varphi'(r)^2}\,dr$.

Volumes of Moving Surfaces

A powerful application of the differential element method is calculating the volume swept by a continuously deforming surface. This generalises the concept of a solid of revolution.

Theorem 9.7. ***Volume Swept by a Moving Surface.***
Let V be the solid swept by a family of surfaces Σ_t defined by $\varphi(x, y, z) = t$ as t varies from a to b. The volume of V is given by:

$$\text{Vol}(V) = \int_a^b \left(\iint_{\Sigma_t} \frac{1}{\|\nabla\varphi\|} \, dS \right) dt.$$

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Proof

Consider the displacement of the surface Σ_t to $\Sigma_{t+\Delta t}$. Let $\mathbf{v} = (x'(t), y'(t), z'(t))$ be the velocity of a point on the surface. Since $\varphi(x(t), y(t), z(t)) = t$, differentiating with respect to t yields:

$$\nabla\varphi \cdot \mathbf{v} = 1.$$

The normal velocity v_n of the surface is the projection of $\mathbf{v}$ onto the unit normal $\mathbf{n} = \frac{\nabla\varphi}{\|\nabla\varphi\|}$:

$$v_n = \mathbf{v} \cdot \mathbf{n} = \mathbf{v} \cdot \frac{\nabla\varphi}{\|\nabla\varphi\|} = \frac{1}{\|\nabla\varphi\|}.$$

The volume of the thin shell swept out in time Δt is approximately the surface area times the normal displacement $v_n \Delta t$. Summing these contributions yields the integral.

∎

Remark.

This result contains the **Generalised Guldin Formula**. If Σ_t describes a planar region moving through space, the volume is the product of the area of the region and the path length of its geometric centroid, provided the motion is orthogonal to the plane of the region.

9.14 *Physical Applications*

Moments and Inertia

While the mass of a body V is the zeroth moment of its density $\mu(x, y, z)$, higher-order moments characterise its rotational resistance and distribution. The **moments of inertia** about the coordinate axes are defined as:

$$I_x = \iiint_V (y^2 + z^2)\mu \, dV, \quad I_y = \iiint_V (x^2 + z^2)\mu \, dV, \quad I_z = \iiint_V (x^2 + y^2)\mu \, dV.$$

These integrals represent the summation of mass elements $\mu \, dV$ weighted by the square of their distance from the axis of rotation.

Example 9.29. The Inverse Centroid Problem. Suppose the x-coordinate of the centroid of the region bounded by $x = 0$, $x = a$, $y = 0$, and a positive curve $y = f(x)$ is given by a known function $g(a)$. We wish to reconstruct the curve $f(x)$.

The definition of the centroid $\bar{x}$ for a planar region with uniform density is:

$$g(a) = \frac{\int_0^a x f(x) \, dx}{\int_0^a f(x) \, dx}.$$

Rearranging, we have the integral equation:

$$g(a) \int_0^a f(x) \, dx = \int_0^a x f(x) \, dx.$$

Differentiating with respect to a (using the Fundamental Theorem of Calculus):

$$g'(a) \int_0^a f(x) \, dx + g(a) f(a) = a f(a).$$

Let $F(a) = \int_0^a f(x) \, dx$. Then $F'(a) = f(a)$. The equation becomes a linear differential equation for F:

$$g'(a)F(a) = (a - g(a))F'(a) \implies \frac{F'(a)}{F(a)} = \frac{g'(a)}{a - g(a)}.$$

Integrating implies $\ln F(a) = \int \frac{g'(a)}{a - g(a)} \, da$. Once $F(a)$ is found, $f(a)$ is recovered by differentiation:

$$f(a) = \frac{d}{da} \left[A \exp \left(\int \frac{g'(a)}{a - g(a)} \, da \right) \right],$$

where A is a constant determined by boundary conditions.

範例

9.15 *Analytic Applications and Inequalities*

Multiple integrals are a potent tool for proving inequalities in functional analysis, often by introducing auxiliary variables to symmetrise the problem.

Integral Inequalities

We can derive bounds for single-variable integrals by embedding them into higher dimensions.

Example 9.30. A Reverse Inequality. Let f be a positive continuous function on $[0,1]$ with bounds $0 < m \leq f(x) \leq M$. We prove:

$$1 \leq \left(\int_0^1 \frac{dx}{f(x)}\right)\left(\int_0^1 f(x)\,dx\right) \leq \frac{(m+M)^2}{4mM}.$$

範例

Proof

Let I be the product of integrals. We write this as a double integral over the unit square $D = [0,1]^2$:

$$I = \int_0^1 \frac{dx}{f(x)} \cdot \int_0^1 f(y)\,dy = \iint_D \frac{f(y)}{f(x)}\,dA.$$

Symmetrising by swapping x and y and averaging:

$$I = \frac{1}{2}\iint_D \left(\frac{f(y)}{f(x)} + \frac{f(x)}{f(y)}\right)\,dA.$$

Since $m \leq f(x) \leq M$, we have $(f(x)-m)(M-f(x)) \geq 0$, so

$$f(x)^2 - (M+m)f(x) + mM \leq 0.$$

Dividing by $f(x) > 0$ gives

$$f(x) + \frac{mM}{f(x)} \leq M + m.$$

Integrating over $[0,1]$ yields

$$\int_0^1 f(x)\,dx + mM\int_0^1 \frac{dx}{f(x)} \leq M + m.$$

Let $A = \int_0^1 f(x)\,dx$ and $B = \int_0^1 \frac{dx}{f(x)}$. By AM-GM,

$$A + mMB \geq 2\sqrt{mM\,AB}.$$

Thus $2\sqrt{mM\,I} \leq M + m$, and

$$I \leq \frac{(M+m)^2}{4mM}.$$

■

The Hölder Inequality

The generalisation of the Cauchy-Schwarz inequality to L^p spaces is naturally handled via double integration arguments.

Theorem 9.8. ***Hölder's Inequality.***
Let $\Omega \subset \mathbb{R}^n$ be a measurable region. If $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ with $1/p + 1/q = 1$ and $p, q > 1$, then:

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

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Proof

Normalise the functions by letting $u = |f|/\|f\|_p$ and $v = |g|/\|g\|_q$. We apply Young's inequality for real numbers, $uv \leq u^p/p + v^q/q$. Integrating over Ω:

$$\int_\Omega uv\, dV \leq \frac{1}{p}\int_\Omega u^p\, dV + \frac{1}{q}\int_\Omega v^q\, dV = \frac{1}{p}\cdot 1 + \frac{1}{q}\cdot 1 = 1.$$

Substituting the definitions of u and v yields the result.

■

Example 9.31. Continuity Estimates via Iterated Integration. Multiple integrals allow us to convert bounds on partial derivatives into bounds on the function's oscillation (a precursor to Sobolev embedding theorems). Suppose $\partial_t f = \partial_{xx} f$ on $[0,1]^2$ and $|\partial_x f| \leq 1$. We can prove f is Hölder continuous in t. Writing the difference as an integral:

$$f(x,t_2) - f(x,t_1) = \int_{t_1}^{t_2} \partial_t f(x,\tau)\, d\tau = \int_{t_1}^{t_2} \partial_{xx} f(x,\tau)\, d\tau.$$

Integrating this identity with respect to x from a reference point $\bar{x}$ to x:

$$\int_{\bar{x}}^{x} (f(z,t_2) - f(z,t_1))\, dz = \int_{t_1}^{t_2} (\partial_x f(x,\tau) - \partial_x f(\bar{x},\tau))\, d\tau.$$

Using $|\partial_x f| \leq 1$, the right-hand side is bounded by $2|t_2 - t_1|$. By carefully choosing $\bar{x}$ (using the Mean Value Theorem for integrals), one can derive that $|f(x,t_2) - f(x,t_1)| \leq C|t_2 - t_1|^{1/2}$.

範例

9.16 *Exercises*

1. **Algebra of Integrable Functions.** Let D be a region in $\mathbb{R}^2$.
   (a) Let f, g be integrable on D. Prove that the product $f \cdot g$ is

integrable on D.

(b) Let f be integrable on D such that $f(x,y) \neq 0$ for all $(x,y) \in D$. Prove that $1/f$ is integrable on D.

Consider the condition required on the lower bound of $|f|$.

2. **Composite Functions.** Let $u = u(x,y)$ be integrable on D.

   (a) If $f(u)$ is a continuous function of u, prove that the composition $f(u(x,y))$ is integrable on D.

   (b) If $f(u)$ is merely an integrable function of u, must the composition $f(u(x,y))$ be integrable on D? Provide a proof or a counter-example.

3. **Almost Everywhere Equality.** Let f, g be bounded functions on D.

   (a) Suppose f and g differ only on a set of zero area (Jordan content zero). Prove that f is integrable if and only if g is integrable, and in that case, $\iint_D f = \iint_D g$.

   (b) Discuss the situation if f and g differ on a set of Lebesgue measure zero but not necessarily zero area.

4. **Integrability on Closures.** Let f be bounded and integrable on the closure $\overline{D}$, and suppose the boundary $\partial D = \overline{D} \setminus D$ has zero area. We define the integral on D to be the integral on $\overline{D}$. Discuss the integrability of the following functions on the given domains:

   (a) $f(x,y) = \sin\left(\dfrac{x^2-1}{x^2+(y^2-1)^2}\right)$ on $D = [-1,1] \times [-1,1]$.

   (b) $f(x,y) = \arctan\left(\dfrac{1}{y-x^2}\right)$ on $D = [0,1] \times [0,1]$.

5. **Maximum of Functions.** Let f, g be integrable on D. Prove that the function defined by

$$h(x,y) = \max\{f(x,y), g(x,y)\}$$

is integrable on D.

Express $\max\{a,b\}$ using absolute values.

6. **Local Mean Value Limit.** Let f be continuous in a neighborhood of $P_0(x_0,y_0)$. Evaluate the limit:

$$\lim_{\rho \to 0} \frac{1}{\pi\rho^2} \iint_{(x-x_0)^2+(y-y_0)^2 \leq \rho^2} f(x,y)\, dx\, dy.$$

7. **Reduction to Single Integral.**

   (a) Prove the identity:

$$\iint_{|x|+|y|\leq 1} f(x+y)\, dx\, dy = \int_{-1}^{1} f(u)\, du.$$

   (b) Let D be the region in the first quadrant bounded by the hyperbolas $xy = 1$, $xy = 2$ and the lines $y = x$, $y = 4x$. Prove

that:
$$\iint_D f(xy)\,dx\,dy = \ln 2 \int_1^2 f(u)\,du.$$

8. **Order of Integration.**

   (a) Rewrite the following sum of iterated integrals as a single iterated integral with the order of integration reversed (integrating with respect to y then x):
   $$I = \int_0^{R/\sqrt{1+R^2}} dx \int_0^{Rx} f(x,y)\,dy + \int_{R/\sqrt{1+R^2}}^{R} dx \int_0^{\sqrt{R^2-x^2}} f(x,y)\,dy.$$

   (b) Let f be continuous on $[0,1]$. Prove the reduction formula:
   $$\int_0^1 dx \int_x^1 f(t)\,dt = \int_0^1 t f(t)\,dt.$$

   (c) Calculate the integral $\displaystyle\int_0^{\sqrt{3}} dy \int_y^{\sqrt{4-y^2}} \frac{1}{\sqrt{1+x^2+y^2}}\,dx.$

9. **Evaluations on Elementary Regions.** Calculate the following double integrals:

   (a) $\displaystyle\iint_D \frac{\sin x}{x}\,dx\,dy$, where D is the triangle bounded by $y=\pi-x$, $x=\pi$, and $y=\pi$.

   (b) $\displaystyle\iint_D (x^2+y^2)\,dx\,dy$, where D is the region in the first quadrant bounded by $y=0$, $y=x^2$, and $x+y=2$.

   (c) $\displaystyle\iint_D \frac{x^2\sin xy}{y}\,dx\,dy$, where D is bounded by the parabolas $x^2=ay, x^2=by$ and $y^2=px, y^2=qx$ with $0<a<b$ and $0<p<q$.

   (d) $\displaystyle\iint_D \sqrt{x^2+y^2}\,dx\,dy$, where D is the first-quadrant region bounded by y-axis, $x^2+y^2=a^2$, and $x^2-2ax+y^2=0$.

   (e) $\displaystyle\iint_D f(\sqrt{x^2+y^2})\,dx\,dy$, where D is the annulus sector $\{(x,y) \mid |y|<|x|\le 1\}$.

   (f) $\displaystyle\iint_D x\,dx\,dy$, where D is bounded by $xy=1$ and $x^2+y^2=4$.

   (g) $\displaystyle\iint_D |x-y^2|\,dx\,dy$, on the rectangle $D=[0,1]\times[-1,1]$.

   (h) $\displaystyle\iint_D \frac{x}{y^2}\,dx\,dy$, where D is the triangle bounded by $y=x, y=1, x=2$.

10. **Volume and Area Calculations.**

    (a) Find the volume of the solid bounded by the surface $(x^2+y^2)^2=a^2(x^2-y^2)$ and the planes $z=x^2-y^2$ and $z=0$.

(b) Find the area of the figure in the xy-plane bounded by the four lines $x+y=p, x+y=q, y=ax, y=bx$, where $0<p<q$ and $0<a<b$.

(c) Find the area of the region bounded by the curve $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1$ and the coordinate axes $x=0, y=0$.

11. Cauchy-Riemann Invariance. Let D be a region in the xy-plane and consider the functional

$$I(f)=\iint_D\left(\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right)dx\,dy.$$

Let $x=x(u,v)$ and $y=y(u,v)$ be a differentiable transformation mapping a region Ω in the uv-plane to D. Prove that if the transformation satisfies the Cauchy-Riemann equations:

$$\frac{\partial x}{\partial u}=\frac{\partial y}{\partial v},\quad \frac{\partial x}{\partial v}=-\frac{\partial y}{\partial u},$$

then the functional is invariant:

$$\iint_D\left(|\nabla_{x,y}f|^2\right)dx\,dy=\iint_\Omega\left(\left(\frac{\partial f}{\partial u}\right)^2+\left(\frac{\partial f}{\partial v}\right)^2\right)du\,dv.$$

12. Simplex Reduction. Prove that for any continuous function f:

$$\iint_D(y-x)^n f(x)\,dx\,dy=\frac{1}{n+1}\int_a^b(b-x)^{n+1}f(x)\,dx,$$

where D is the triangular region defined by $a\le x\le y\le b$.

13. Integral Inequality. Prove the following estimate:

$$\iint_{|x|+|y|\leqslant 1}(\sqrt{|xy|}+|xy|)\,dx\,dy\leqslant\frac{3}{2}.$$

14. Iterated Triple Integration.

(a) Compute the integral:

$$\int_0^1 dx\int_0^{1-x}dz\int_0^{1-x-z}(1-y)e^{-(1-y-z)^2}\,dy.$$

(b) Transform the following iterated integral into cylindrical coordinates:

$$\int_0^1 dx\int_0^{\sqrt{1-x^2}}dy\int_0^1 f(x,y,z)\,dz.$$

15. Volumes of Revolution. Find $\iiint_\Omega(x^2+y^2)\,dx\,dy\,dz$, where Ω is the region bounded by the surface generated by rotating the curve $y^2=2z$ ($z\ge 0$) about the z-axis, and the planes $z=2$ and $z=8$.

16. Intersection of Balls. Find $\iiint_\Omega xyz\,dx\,dy\,dz$, where Ω is the common interior of the spheres $x^2+y^2+z^2 \le 4$ and $x^2+y^2+(z-2)^2 \le 4$, restricted to the first octant ($x \ge 0, y \ge 0$).

17. Newtonian Potential. Find $\iiint_\Omega \frac{dx\,dy\,dz}{r}$, where Ω is a ball of radius R, and r denotes the distance from a fixed point P outside the ball to the variable point of integration.

18. Spherical Coordinate Applications.

(a) Compute the integral $I = \iiint_\Omega \frac{xyz}{x^2+y^2}\,dx\,dy\,dz$, where Ω is the region bounded above by the surface $(x^2+y^2+z^2)^2 = a^2xy$ and below by the plane $z=0$.

(b) Find the four-dimensional integral:

$$\iiiint_\Omega \sqrt{\frac{1-x^2-y^2-z^2-a^2}{1+x^2+y^2+z^2+a^2}}\,dx\,dy\,dz\,du,$$

where the domain is likely implied to be the region where the radicand is real and positive (verify the intended domain, typically a hypersphere or spherical shell structure).

19. Differentiation under the Integral. Let f be a continuous function with $f(1)=1$. Define:

$$F(t) = \iiint_{x^2+y^2+z^2 \le t^2} f(x^2+y^2+z^2)\,dx\,dy\,dz.$$

Prove that $F'(1) = 4\pi$.

20. Integral Average. Let $f(x,y,z) = \sqrt{x^2+y^2+z^2}$. Let $\Omega \subset \mathbb{R}^3$ be the region defined by $x \ge \sqrt{x^2+y^2}$ (a cone) and the spherical shell $4 \le x^2+y^2+z^2 \le 16$. Compute the average value of f over Ω:

$$\bar{f} = \frac{1}{\text{Vol}(\Omega)} \iiint_\Omega f(x,y,z)\,dx\,dy\,dz.$$

21. Generalised Change of Variables. Let the region Ω be bounded by the paraboloid $z = x^2+y^2$, the plane $z=0$, and the hyperbolic cylinders $xy=1, xy=2$, and the planes $y=3x, y=4x$. Evaluate the integral:

$$I = \iiint_\Omega x^2y^2z\,dx\,dy\,dz.$$

22. Orthogonal Invariance. Use an orthogonal linear transformation to compute the triple integral

$$\iiint_V \cos(ax+by+cz)\,dx\,dy\,dz,$$

where V is the unit ball $x^2+y^2+z^2 \le 1$, and a,b,c are constants, not all zero.

23. **Convergence of Improper Integrals.** Discuss the convergence of the following improper integrals:

 (a) $\iint_{\mathbb{R}^2} \frac{dx\,dy}{(1+|x|^p)(1+|y|^q)}$

 (b) $\iint_{|x|+|y|\geq 1} \frac{dx\,dy}{|x|^p+|y|^q}$

 (c) $\iint_{x+y>1} \frac{\sin x \sin y}{(x+y)^p}\,dx\,dy$

 (d) $\iint_{x^2+y^2\leq 1} \frac{dx\,dy}{(x^2+xy+y^2)^p}$

 (e) $\iint_{x^2+y^2\leq 1} \frac{dx\,dy}{(1-x^2-y^2)^p}$

24. **Monotone Convergence for Integrals.** Let D be an unbounded region in $\mathbb{R}^2$, and $\{D_n\}$ be a monotonically increasing sequence of closed regions exhausting D (i.e., $\bigcup_{n=1}^{\infty} D_n = D$). Prove that if f is non-negative on D and integrable on each D_n, then:

$$\iint_D f(x,y)\,dx\,dy = \lim_{n\to\infty} \iint_{D_n} f(x,y)\,dx\,dy,$$

 in the sense that both sides either converge to the same value or diverge to infinity.

25. **Calculations on Unbounded Domains.**

 (a) Compute $\iint_D \frac{dx\,dy}{x^2+y^2}$, where $D = \{(x,y) \mid x^2+y^2 \geq 1\}$.

 (b) Discuss the convergence of $\iint_D \frac{dx\,dy}{x^2+y^2}$, where D is defined by $|y| \leq x^2$ and $x^2+y^2 \leq 1$.

 (c) Let $f(x)$ be continuous on $[a,A]$. Discuss the convergence of $\iint_{[a,A]\times[b,B]} \frac{dx\,dy}{|y-f(x)|^p}$.

26. **Logarithmic Integrals.** Compute:

 (a) $\iint_{x^2+y^2\leq 1} \ln \frac{1}{\sqrt{x^2+y^2}}\,dx\,dy$.

 (b) $\iint_D \ln\sin(x-y)\,dx\,dy$, where D is the triangle bounded by $y=0$, $y=x$, and $x=\pi$.

27. **Volumes of Solids.** Calculate the volume of the solid bounded by the following surfaces:

 (a) The planes $a_i x + b_i y + c_i z = \pm h_i$ for $i = 1,2,3$, assuming the normal vectors are linearly independent.

 (b) The surface $(x^2+y^2+z^2)^2 = a^3 z$, with $a > 0$.

 (c) The surface $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^2 = \frac{x^2}{a^2}+\frac{y^2}{b^2}$.

(d) The surface $(x^2 + y^2)^2 + z^4 = z$.

28. Surface Areas. Calculate the area of the following surfaces:

(a) The portion of the sphere $(x^2 + y^2 + z^2)^2 = x^2 - y^2$ (Lemniscate envelope).

(b) The surface $(x^2 + y^2 + z^2)^2 = z^3$.

(c) The surface of revolution generated by rotating a continuous curve $y = f(x) > 0, x \in [a, b]$ about the x-axis.

29. Mass and Gravity.

(a) Find the mass of the paraboloid $z = \frac{1}{2}(x^2 + y^2)$ for $0 \leq z \leq 1$, given the surface density $\rho = z$.

(b) A uniform disk of radius R has density μ. A uniform thin rod of length l and density ρ lies on the axis of the disk, with its near end at distance a from the center. Find the gravitational force exerted by the disk on the rod.

(c) A disk of radius a has a variable density equal to the distance from the center. A hole of radius $a/2$ is cut out, centered at a distance $a/2$ from the original center. Find the centroid of the remaining shape.

(d) Prove that the center of mass of any convex object with continuous density must lie within the object.

30. Generalised Hölder Inequality. Let $u_i \in L^{p_i}(\Omega)$ for $p_i > 0$ and $i = 1, \ldots, m$. If $\sum_{i=1}^{m} \frac{1}{p_i} = 1$, prove:

$$\iint_\Omega |u_1 u_2 \cdots u_m| \, dA \leq \|u_1\|_{p_1} \|u_2\|_{p_2} \cdots \|u_m\|_{p_m}.$$

31. Iterated Cauchy-Schwarz. For a continuous function f, prove:

$$\left\{ \int_a^b \left[\int_c^d f(x, y) \, dy \right]^2 dx \right\}^{1/2} \leq \int_c^d \left[\int_a^b f^2(x, y) \, dx \right]^{1/2} dy.$$

32. Poincaré-Type Inequality. Suppose $f(x, y)$ has continuous second-order partial derivatives in a region D bounded by $y = \varphi(x)$ and $y = \psi(x)$. If f vanishes on the boundary curve $y = \varphi(x)$, prove that there exists a constant $K > 0$ such that:

$$\iint_D f^2(x, y) \, dx \, dy \leq K \iint_D \left(\frac{\partial f}{\partial y} \right)^2 dx \, dy.$$

33. Minkowski's Inequality. Let $u, v \in L^p(\Omega)$ with $p \geq 1$. Using Hölder's inequality, prove:

$$\|u + v\|_p \leq \|u\|_p + \|v\|_p.$$

Discuss the conditions under which equality holds.

34. **High-Order Bounds.** Let $f(x,y)$ be C^4 on the unit square $D = [0,1]^2$. Suppose f vanishes on the boundary ∂D and satisfies $|\partial^4 f/\partial x^2 \partial y^2| \leq B$. Prove:

$$\left| \iint_D f(x,y)\,dx\,dy \right| \leq \frac{B}{144}.$$

35. **Localization Principle.** Prove that if $\iint_D f(x,y)\,dA > 0$, then there exists a closed subregion $U \subset D$ such that $f(x,y) > 0$ for all $(x,y) \in U$.

36. **Mean Value Theorem for Multiple Integrals.** Let $f(\mathbf{x})$ be continuous on a bounded closed region $\Omega \subset \mathbb{R}^n$. Prove there exists $\xi \in \Omega$ such that:

$$\int_\Omega f(\mathbf{x})\,d\mathbf{x} = f(\xi) \cdot \text{Vol}(\Omega).$$

37. **Chebyshev's Integral Inequality.** Let $p(x) \geq 0$ be continuous on $[a,b]$, and let f, g be continuous and monotonically increasing on $[a,b]$. Prove:

$$\left(\int_a^b p(x) f(x)\,dx \right) \left(\int_a^b p(x) g(x)\,dx \right) \leq \left(\int_a^b p(x)\,dx \right) \left(\int_a^b p(x) f(x) g(x)\,dx \right).$$

38. **Ratio Inequality.** Let f be continuous, monotonically decreasing, and positive on $[0,1]$. Prove:

$$\frac{\int_0^1 x f^2(x)\,dx}{\int_0^1 x f(x)\,dx} \leq \frac{\int_0^1 f^2(x)\,dx}{\int_0^1 f(x)\,dx}.$$

39. **Potential Bounds.** Let $I = \iiint_{B_R} \frac{dV}{\|\mathbf{r}-\mathbf{a}\|}$ where B_R is the ball of radius R centered at the origin, and $\|\mathbf{a}\| = A > R$. Prove:

$$\frac{4\pi R^3}{3(A+R)} \leq I \leq \frac{4\pi R^3}{3(A-R)}.$$

40. **Differentiation on Expanding Domains.** Let D_t be the cube $[0,t]^3$. Define $F(t) = \iiint_{D_t} f(xyz)\,dx\,dy\,dz$ for a continuous function f. Prove that:

$$F'(t) = \frac{3}{t} \int_0^{t^3} \frac{g(u)}{u}\,du, \quad \text{where } g(u) = \int_0^u f(s)\,ds.$$

41. **Tubular Neighborhoods.** Let Γ be a simple closed smooth curve in the plane with perimeter L. Parameterise Γ by arc length s. Let $\theta(s)$ be the angle of the tangent vector. Let D be the region of points at distance $t < 1$ from Γ (the "exterior" tubular neighborhood).

(a) Express the coordinates of a point in D as $x(s,t) = f(s) + t\sin\theta(s)$ and $y(s,t) = \varphi(s) - t\cos\theta(s)$ (or similar normal variation).

(b) Verify that the area of D is $L + \pi$ (or $3\pi l^2$ as stated in the source text, verify the radius l vs perimeter L scaling).

42. Young's Inequality. Prove that for $a, b > 0$ and $1/p + 1/q = 1$:

$$ab \leq \frac{\epsilon a^p}{p} + \frac{\epsilon^{-q/p} b^q}{q}.$$

43. Interpolation Inequality. Let $\mu = (1/p - 1/q)/(1/q - 1/r)$ where $0 < p < q < r$. Prove:

$$\|u\|_q \leq \epsilon\|u\|_r + \epsilon^{-\mu}\|u\|_p.$$

44. Norm Limits. Let $\Phi_p(u) = \left(\frac{1}{|\Omega|}\iint_\Omega u^p\, dA\right)^{1/p}$ for a positive function u. Prove:

(a) $\lim_{p\to+\infty} \Phi_p(u) = \max_\Omega u$.

(b) $\lim_{p\to-\infty} \Phi_p(u) = \min_\Omega u$.

(c) $\lim_{p\to 0} \Phi_p(u) = \exp\left(\frac{1}{|\Omega|}\iint_\Omega \ln u\, dA\right)$ (Geometric Mean).

45. Pedal Surface Volume. Let P_0 be a fixed point inside a sphere. For any point Q on the sphere, let P be the foot of the perpendicular from P_0 to the tangent plane at Q. The locus of P forms a pedal surface.

(a) Find the volume enclosed by this surface.

(b) Determine the direction P_0 should move to maximize the rate of change of this volume.

46. Gaussian Error Function Bounds. Prove the inequalities:

$$\frac{\sqrt{\pi}}{2}(1 - e^{-a^2})^{1/2} < \int_0^a e^{-x^2}\, dx < \frac{\sqrt{\pi}}{2}(1 - e^{-4a^2/\pi})^{1/2}.$$

47. Catalan's Formula. Let the level curves of $f(x,y)$ be simple closed curves. Let $S(v_1, v_2)$ be the region between levels v_1 and v_2. Prove:

$$\iint_{S(v_1,v_2)} f(x,y)\, dx\, dy = \int_{v_1}^{v_2} vF'(v)\, dv,$$

where $F(v)$ is the area enclosed by the level curve $f(x,y) = v$.