

## Recitation 1

- Let  $\mathbf{a} = [3, 4]$ ,  $\mathbf{b} = [5, -12]$ , and  $\mathbf{c} = [1, -3]$ .
  - Compute the scalar products  $\mathbf{a} \cdot \mathbf{b}$ ,  $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$ , and  $(\mathbf{a} - \mathbf{c}) \cdot \mathbf{b}$ .
  - Find the angle between  $\mathbf{a}$  and  $\mathbf{b}$ .
  - Determine the orthogonal projection of  $\mathbf{c}$  onto  $\mathbf{a}$ .
  - Find a scalar  $k$  such that  $\mathbf{a} + k\mathbf{b}$  is orthogonal to  $\mathbf{a}$ .
- Find the equation of the straight line in the form  $ax + by + c = 0$  satisfying the following conditions:
  - Passing through  $P(-2, 5)$  with normal vector  $\mathbf{n} = [3, -1]$ .
  - Passing through  $A(4, 1)$  and  $B(2, -3)$ .
  - Passing through  $C(1, 2)$  and perpendicular to the line  $2x - 5y + 10 = 0$ .
- Consider the lines  $L_1 : 3x - 4y + 5 = 0$  and  $L_2 : x + y - 2 = 0$ .
  - Calculate the perpendicular distance from the point  $P(2, -1)$  to  $L_1$ .
  - Find the point on  $L_1$  that is closest to  $P$ .
  - Determine the cosine of the acute angle between  $L_1$  and  $L_2$ .
- Find the values of  $k$  such that the vectors  $\mathbf{u} = [k, -2]$  and  $\mathbf{v} = [3, k + 5]$  are:
  - Orthogonal.
  - Parallel.
  - Separated by an angle of  $\pi/3$ .
- Pythagorean Generalisation.** Let  $\mathbf{u}$  and  $\mathbf{v}$  be vectors in  $\mathbb{R}^2$ . Prove that  $|\mathbf{u} + \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2$  if and only if  $\mathbf{u}$  and  $\mathbf{v}$  are orthogonal. Interpret this geometrically.
- The Rhombus Identity.** Let  $\mathbf{u}$  and  $\mathbf{v}$  be vectors of equal magnitude (i.e.,  $|\mathbf{u}| = |\mathbf{v}|$ ).
  - Prove that the vectors  $\mathbf{u} + \mathbf{v}$  and  $\mathbf{u} - \mathbf{v}$  are orthogonal.
  - Interpret this result in terms of the diagonals of a rhombus.
- Equality in Cauchy-Schwarz.** We established that  $|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$ . Prove that equality holds, i.e.,  $|\mathbf{a} \cdot \mathbf{b}| = |\mathbf{a}||\mathbf{b}|$ , if and only if  $\mathbf{a}$  and  $\mathbf{b}$  are linearly dependent.
- Area of a Triangle.** Let a triangle have vertices at the origin  $O$  and points  $A, B$  with position vectors  $\mathbf{a}, \mathbf{b}$ .
  - From elementary geometry, the area is  $\frac{1}{2}|\mathbf{a}||\mathbf{b}|\sin\theta$ . Use the relation between  $\sin\theta$  and  $\cos\theta$  to prove that the area can be written as:
 
$$\text{Area} = \frac{1}{2}\sqrt{|\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2}.$$
  - Verify that if  $\mathbf{a} = [a_1, a_2]$  and  $\mathbf{b} = [b_1, b_2]$ , this expression reduces to  $\frac{1}{2}|a_1b_2 - a_2b_1|$ .
- Vector Angle Bisector.** Let  $\mathbf{u}$  and  $\mathbf{v}$  be non-zero, non-parallel vectors.
  - Show that the vector  $\mathbf{w} = |\mathbf{v}|\mathbf{u} + |\mathbf{u}|\mathbf{v}$  bisects the angle between  $\mathbf{u}$  and  $\mathbf{v}$ .
  - Hence, or otherwise, find the equation of the angle bisector of the lines  $3x - 4y = 0$  and  $5x + 12y = 0$ .