Homework

1. **Skew Lines and a Common Perpendicular.** Consider the lines ℓ_1, ℓ_2 in $\mathbb{R}^3$ given by

$$\ell_1 : \begin{bmatrix} 5+2t \\ 3+3t \\ -t \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \qquad \ell_2 : \begin{bmatrix} -4+t' \\ -5+t' \\ 4-t' \end{bmatrix} = \begin{bmatrix} -4 \\ -5 \\ 4 \end{bmatrix} + t' \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

   (a) Show that these lines have no point in common.
   (b) Show that two lines in $\mathbb{R}^3$ with no common point cannot lie in a common plane unless their direction vectors are scalar multiples of each other.
   (c) Verify in this particular case that there is a unique line perpendicular to both ℓ_1 and ℓ_2 and meeting each of them, and give a parametric form for that line.

2. **Medians of a Triangle in Space.** Let a triangle T in $\mathbb{R}^3$ have vertices with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$.
   (a) Write a parametric form for the median from the vertex $\mathbf{c}$ to the midpoint of the edge joining $\mathbf{a}$ and $\mathbf{b}$. Show that the point

$$P = \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$$

   lies on that median.
   (b) Show that the same point P lies on all three medians of the triangle.

3. **Column–Row Decomposition.** Let

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 4 & 3 \end{bmatrix}.$$

   (a) Compute the product AB directly.
   (b) Write A in terms of its columns $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ and B in terms of its rows $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$, and verify that

$$AB = \mathbf{a}_1\mathbf{r}_1 + \mathbf{a}_2\mathbf{r}_2 + \mathbf{a}_3\mathbf{r}_3.$$

4. **Commuting with an Upper-Shift Matrix.** Determine all matrices

$$X = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

   that commute with

$$N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

5. **Hermitian–Skew Decomposition.** Let

$$A = \begin{bmatrix} 2+i & 1-2i \\ -3+i & 4 \end{bmatrix}.$$

   (a) Compute the conjugate transpose A^*.
   (b) Form

$$H = \frac{1}{2}(A + A^*), \qquad K = \frac{1}{2}(A - A^*).$$

   (c) Verify directly that H is Hermitian, K is skew-Hermitian, and $A = H + K$.

6. **Stochastic and Doubly Stochastic Matrices.** A square matrix $A \in \mathbb{R}^{n\times n}$ is called *stochastic* if every entry of A is nonnegative and the sum of the entries in each row is equal to 1.
   (a) Prove that the product of two stochastic matrices is stochastic.

(b) A square matrix $A \in \mathbb{R}^{n\times n}$ is called *doubly stochastic* if both A and $A^\top$ are stochastic. Prove that the product of two doubly stochastic matrices is doubly stochastic.

7. Trace Identities. The *trace* of a matrix $A \in \mathbb{C}^{n\times n}$, written $\mathrm{tr}(A)$, is defined to be the sum of the entries on its main diagonal.

(a) Prove that

$$\mathrm{tr}(\alpha A + \beta B) = \alpha\ \mathrm{tr}(A) + \beta\ \mathrm{tr}(B)$$

for all $A, B \in \mathbb{C}^{n\times n}$ and $\alpha, \beta \in \mathbb{C}$.

(b) If $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{m\times n}$, prove that

$$\mathrm{tr}(AB) = \mathrm{tr}(BA).$$

(c) If $A = [a_{ij}]_{i,j=1}^n \in \mathbb{C}^{n\times n}$, prove that

$$\mathrm{tr}(AA^*) = \mathrm{tr}(A^*A) = \sum_{i,j=1}^{n} a_{ij}\overline{a_{ij}}.$$

(d) If $A, B \in \mathbb{C}^{n\times n}$ and A is idempotent, prove that

$$\mathrm{tr}(AB) = \mathrm{tr}(ABA).$$

8. A Commutator Cannot Equal the Identity. Prove that if $n \geq 2$, there do not exist matrices $A, B \in \mathbb{C}^{n\times n}$ such that

$$AB - BA = I_n.$$

Hint: use the previous problem.