Exercises

1. **Affine Collinearity Criterion.** Let A, B, C be points with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$. Suppose O, A, B are not collinear, and that
$$\mathbf{c} = s\mathbf{a} + t\mathbf{b}.$$
Prove that the points A, B, C are collinear if and only if $s + t = 1$.

2. **A Spatial Parallelogram Section.** Let $ABCD$ be a quadrilateral in space. Let E, F, G, H be points on the edges AB, AC, DB, DC respectively such that
$$\frac{|AE|}{|AB|} = \frac{|AF|}{|AC|} = \frac{|DG|}{|DB|} = \frac{|DH|}{|DC|}.$$
Prove that the quadrilateral $EFGH$ is a parallelogram.

3. **Centroids of a Tetrahedron.** Let A, B, C, D be four points in space. Let A' be the centroid of $\triangle BCD$, B' the centroid of $\triangle ACD$, C' the centroid of $\triangle ABD$, and D' the centroid of $\triangle ABC$.
   (a) Prove that the lines AA', BB', CC', DD' are concurrent at the centroid G of the tetrahedron $ABCD$.
   (b) Prove that G is also the centroid of the tetrahedron $A'B'C'D'$.

4. **Vectors at $45°$.**
   (a) Draw a picture of all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ making an angle of $45°$ with the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and identify which of these are unit vectors in your picture.
   (b) Find the unit vectors from part (a) algebraically using dot products.

5. **A Prescribed Angle Computation.** Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ satisfy
$$\|\mathbf{a}\| = 1, \qquad \|\mathbf{b}\| = 2, \qquad \|\mathbf{c}\| = 3.$$
Suppose $\mathbf{a} \perp \mathbf{b}$, the angle between $\mathbf{a}$ and $\mathbf{c}$ is $\pi/3$, and the angle between $\mathbf{b}$ and $\mathbf{c}$ is $\pi/4$. Calculate $\|\mathbf{a} + \mathbf{b} + \mathbf{c}\|$.

6. **Angle Bisectors via Dot Products.** Let $\mathbf{v}$ and $\mathbf{w}$ be non-zero 2-vectors with the same length.
   (a) Use dot products to show that $\mathbf{v} + \mathbf{w}$ bisects the angle between $\mathbf{v}$ and $\mathbf{w}$.
   (b) For $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$, find a non-zero vector $\mathbf{u} \in \mathbb{R}^2$ that bisects the angle between $\mathbf{v}$ and $\mathbf{w}$.
   (c) For $\mathbf{v}' = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and $\mathbf{w}' = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, find a non-zero vector $\mathbf{u}' \in \mathbb{R}^2$ that bisects the angle between $\mathbf{v}'$ and $\mathbf{w}'$.

7. **Projection onto a Line.** Let $P = \begin{bmatrix} x \\ y \end{bmatrix}$ and let ℓ be the line through the origin in the direction of the non-zero vector
$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$
Let P' be the foot of the perpendicular from P to ℓ.
   (a) Explain why P' has the form $\lambda\mathbf{v}$ for some scalar λ.
   (b) Show that
$$\lambda = \frac{\mathbf{v} \cdot P}{\|\mathbf{v}\|^2}.$$
   (c) Deduce the projection formula
$$\operatorname{proj}_{\mathbf{v}} P = \frac{\mathbf{v} \cdot P}{\|\mathbf{v}\|^2}\mathbf{v}.$$

8. **Orthogonality of Opposite Edges.** Prove that in a tetrahedron $ABCD$, if $AB \perp CD$ and $AC \perp BD$, then $AD \perp BC$.

9. Determining a Plane. Find the equation of the plane satisfying the following conditions:

(a) passing through $(5, 1, 4)$ and parallel to the plane $x + y - 2z = 0$;
(b) passing through $(2, 3, -1)$ and containing the line of intersection of the planes $x - y + z = 1$ and $x + y - z = 1$.

10. Ratio of Division by a Plane. Let Π be the plane defined by

$$Ax + By + Cz + D = 0.$$

Let M_1 and M_2 be points not on the plane. If the line segment M_1M_2 meets the plane at M such that

$$\overrightarrow{M_1M} = k\,\overrightarrow{MM_2},$$

show that

$$k = -\frac{Ax_1 + By_1 + Cz_1 + D}{Ax_2 + By_2 + Cz_2 + D}.$$

11. Distance to an Intercept Plane. Suppose a plane cuts the coordinate axes at $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$, where $a, b, c \neq 0$. Let p be the perpendicular distance from the origin to this plane. Prove that

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

12. Reflection in a Plane. Let Π be the plane

$$\mathbf{n} \cdot \mathbf{x} + d = 0.$$

(a) Show that the reflection of a point P, with position vector $\mathbf{p}$, in the plane Π is

$$\mathbf{p}' = \mathbf{p} - 2\left(\frac{\mathbf{n} \cdot \mathbf{p} + d}{\|\mathbf{n}\|^2}\right)\mathbf{n}.$$

(b) Find the reflection of the point $(1, 2, 3)$ in the plane $2x - y + 2z - 6 = 0$.