Homework 1

1. Apollonius' Theorem. Let ABC be a triangle with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$. Let M be the midpoint of BC.

(a) Let $\mathbf{u} = \mathbf{b} - \mathbf{a}$ and $\mathbf{v} = \mathbf{c} - \mathbf{a}$. Express $\overrightarrow{AM}$ and $\overrightarrow{BC}$ in terms of $\mathbf{u}$ and $\mathbf{v}$.

(b) Apply the Parallelogram Law (Theorem 16, identity (ii)) to $\mathbf{u}$ and $\mathbf{v}$ to prove:

$$|AB|^2 + |AC|^2 = 2|AM|^2 + 2|BM|^2.$$

2. Three is a Crowd. Prove that any set of three vectors in $\mathbb{R}^2$ must be linearly dependent.

Remark. Let the vectors be $\mathbf{a}, \mathbf{b}, \mathbf{c}$. If $\mathbf{a}, \mathbf{b}$ are dependent, the proof is trivial. If they are independent, use Theorem 5 to show that $\mathbf{c}$ can be solved as a combination of $\mathbf{a}$ and $\mathbf{b}$ by solving the resulting system of linear equations.

3. Reflection Operator. Let L be a line passing through the origin with a normal vector $\mathbf{n}$. The reflection of a vector $\mathbf{x}$ across the line L is given by the function:

$$R(\mathbf{x}) = \mathbf{x} - 2\text{proj}_{\mathbf{n}}\mathbf{x}.$$

(a) Draw a diagram to justify this definition geometrically.
(b) Prove that R is an isometry, meaning it preserves magnitudes: $|R(\mathbf{x})| = |\mathbf{x}|$.
(c) Prove that R is a linear map: $R(r\mathbf{x} + s\mathbf{y}) = rR(\mathbf{x}) + sR(\mathbf{y})$.

4. Locus of Angle Bisectors. Consider two non-parallel lines given by normal equations:

$$\mathbf{n}_1 \cdot \mathbf{x} + c_1 = 0 \quad \text{and} \quad \mathbf{n}_2 \cdot \mathbf{x} + c_2 = 0.$$

Prove that the locus of points equidistant from these two lines is given by the two perpendicular lines:

$$\frac{\mathbf{n}_1 \cdot \mathbf{x} + c_1}{|\mathbf{n}_1|} = \pm \frac{\mathbf{n}_2 \cdot \mathbf{x} + c_2}{|\mathbf{n}_2|}.$$

5. The Circumcentre. Let ABC be a triangle with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ relative to an arbitrary origin.

(a) The perpendicular bisector of the side AB is the set of points $\mathbf{x}$ such that $|\mathbf{x} - \mathbf{a}| = |\mathbf{x} - \mathbf{b}|$. Show that this is equivalent to $\mathbf{x} \cdot (\mathbf{b} - \mathbf{a}) = \frac{1}{2}(|\mathbf{b}|^2 - |\mathbf{a}|^2)$.

(b) Suppose the perpendicular bisectors of sides AB and BC intersect at a point O. Show that O must also lie on the perpendicular bisector of AC.

(c) Conclude that the three perpendicular bisectors of a triangle are concurrent at the circumcentre.

6. The Orthocentre. Let ABC be a triangle. Let the position vectors of the vertices be $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

(a) An altitude from vertex A is the line passing through A and perpendicular to BC. Write down the condition for a point $\mathbf{x}$ to lie on this altitude involving the dot product.

(b) Let the origin be the circumcentre of the triangle (so $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = R$). Consider the point H defined by the position vector $\mathbf{h} = \mathbf{a} + \mathbf{b} + \mathbf{c}$.

(c) Compute the dot product $\mathbf{h} \cdot (\mathbf{b} - \mathbf{c})$.

(d) Deduce that H lies on the altitude from A. By symmetry, conclude that H lies on all three altitudes. This point is called the orthocentre.