Exercises

In the following exercises, vectors are elements of $\mathbb{R}^3$ unless otherwise stated.

Core

1. **Vector arithmetic.** Let

$$\mathbf{a} = [2, 2, -1], \qquad \mathbf{b} = [4, 2, 5], \qquad \mathbf{c} = [-1, -3, 2].$$

Compute:

(a) $3(\mathbf{a} + \mathbf{b}) - 2(\mathbf{b} - \mathbf{c})$,
(b) $2(\mathbf{a} + \mathbf{b} + \mathbf{c}) + 3(\mathbf{a} - \mathbf{b} - \mathbf{c})$,
(c) the vector $\mathbf{x}$ such that $\mathbf{a} + 2\mathbf{x} = \mathbf{b} - \mathbf{c} - \mathbf{x}$,
(d) the vector $\mathbf{y}$ such that $2(\mathbf{a} - \mathbf{b} + \mathbf{y}) = 3(\mathbf{c} - \mathbf{y})$.

2. **Triangle geometry in space.** Let $A(1, 2, 3)$, $B(4, 0, 5)$, and $C(3, 6, 4)$.

(a) Determine whether the triangle ABC is isosceles, right-angled, equilateral, or none of these.
(b) Find the coordinates of the point D such that $ABCD$ is a parallelogram.

3. **Sphere equations.**

(a) Determine the equation of the sphere centred at $(-1, 2, 4)$ with radius $\sqrt{7}$.
(b) Determine the equation of the sphere centred at the origin and passing through the points $A(4, 0, 0)$, $B(0, 4, 0)$, and $C(0, 0, 4)$.
(c) Rewrite

$$x^2 + y^2 + z^2 - 4x + 6y - 2z + 5 = 0$$

in standard form, and hence find its centre and radius.

4. **Locus problems.**

(a) Find the equation of the set of points equidistant from $A(1, 2, 3)$ and $B(-1, 4, 1)$. Describe the resulting geometric object.
(b) Find the equation of the set of points whose distance from the origin is twice their distance from the point $(0, 0, 3)$.

5. **Formal identities in $\mathbb{R}^3$.** Using only the definition

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$$

and the axioms of the vector space $\mathbb{R}^3$, prove that:

(a) $\mathbf{a} - (\mathbf{b} + \mathbf{c}) = (\mathbf{a} - \mathbf{b}) - \mathbf{c}$,
(b) $r(\mathbf{a} - \mathbf{b}) = r\mathbf{a} - r\mathbf{b}$.

6. **Testing independence.** Determine whether each of the following sets is linearly dependent or linearly independent.

(a) $\mathbf{u} = [1, -1, 0]$, $\mathbf{v} = [0, 1, -1]$, $\mathbf{w} = [-1, 0, 1]$,
(b) $\mathbf{u} = [1, 1, 0]$, $\mathbf{v} = [1, 0, 1]$, $\mathbf{w} = [0, 1, 1]$.

7. **A change of generating set.** Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be linearly independent vectors, and define

$$\mathbf{a} = 2\mathbf{x} + \mathbf{y}, \qquad \mathbf{b} = \mathbf{y} - \mathbf{z}, \qquad \mathbf{c} = \mathbf{x} + \mathbf{y} + \mathbf{z}.$$

(a) Prove that $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is linearly independent.
(b) Express the vector $\mathbf{d} = \mathbf{x} + 2\mathbf{y} + \mathbf{z}$ as a linear combination of $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

8. A forced linear dependence. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3$, and let $r, s, t \in \mathbb{R}$. Define

$$\mathbf{u} = r\mathbf{x} - s\mathbf{y}, \qquad \mathbf{v} = t\mathbf{y} - r\mathbf{z}, \qquad \mathbf{w} = s\mathbf{z} - t\mathbf{x}.$$

Show that $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly dependent.

9. The determinant condition. Let

$$\mathbf{a} = [a_1, a_2, a_3], \qquad \mathbf{b} = [b_1, b_2, b_3], \qquad \mathbf{c} = [c_1, c_2, c_3].$$

Prove that a necessary condition for $\mathbf{a}, \mathbf{b}, \mathbf{c}$ to be linearly dependent is

$$a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1) = 0.$$

You may assume dependence in the form $\mathbf{a} = r\mathbf{b} + s\mathbf{c}$, substitute into the expression, and simplify.

Supplementary

10. Convex combinations in the plane. Let

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

(a) Show that for $0 \le t \le 1$, the vector $(1-t)\mathbf{v} + t\mathbf{w}$ lies on the line segment joining the tips of $\mathbf{v}$ and $\mathbf{w}$.
(b) Describe geometrically the set of all vectors of the form $r\mathbf{v} + s\mathbf{w}$, where $r, s \ge 0$ and $r + s \le 1$.
(c) Sketch the figure arising in parts (a) and (b).

11. Normalising vectors. For each of the following vectors, compute its magnitude and the unit vector pointing in the same direction.

(a) $\mathbf{v} = \begin{bmatrix} -8 \\ 15 \end{bmatrix}$,
(b) $\mathbf{w} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$.

Challenge

12. $\star$ **Midpoints of a skew quadrilateral.** Let $ABCD$ be a quadrilateral in space; the four vertices need not be coplanar. Let P, Q, R, S be the midpoints of AB, BC, CD, DA, respectively.

(a) Express $\overrightarrow{PQ}$ in terms of $\mathbf{a}, \mathbf{b}, \mathbf{c}$.
(b) Express $\overrightarrow{SR}$ in terms of $\mathbf{a}, \mathbf{d}, \mathbf{c}$.
(c) Prove that $\overrightarrow{PQ} = \overrightarrow{SR}$, and deduce the shape of $PQRS$.
(d) Conclude that even when $ABCD$ is skew, the four midpoints still lie in a common plane.