Exercises

In the following exercises, vectors are strictly elements of $\mathbb{R}^2$. When asked to prove a statement, explicitly state whether you are relying on the component-wise definition of a vector or the axiomatic properties of the vector space.

1. Let $\mathbf{a} = [8, -2]$, $\mathbf{b} = [3, 4]$, and $\mathbf{c} = [0, 5]$. Compute the following vectors:

(a) $2\mathbf{a} - \mathbf{b}$

(b) $\frac{1}{2}(\mathbf{c} - 3\mathbf{a})$

(c) The vector $\mathbf{x}$ such that $2\mathbf{a} + \mathbf{x} = \mathbf{b} - \mathbf{c}$.

2. Find all real numbers k such that the following pairs of vectors are linearly dependent.

(a) $[1, k]$ and $[k, 4]$

(b) $[k, 1 - k]$ and $[2, 3]$

(c) $[1, 1]$ and $[k^2, k]$

3. Determine the magnitude of the vector $\mathbf{v} = [3, -4]$. Find a scalar r such that $r\mathbf{v}$ is a unit vector pointing in the opposite direction to $\mathbf{v}$.

4. **Collinearity.** Use the conditions for linear dependence to determine if the points $A(-1, -2)$, $B(2, 4)$, and $C(5, 10)$ are collinear.

Remark. Consider the vectors $\mathbf{u}$ representing the segment $\overrightarrow{AB}$ and $\mathbf{v}$ representing $\overrightarrow{AC}$. Are they dependent?

5. Express the vector $[5, 2]$ as a linear combination of:

(a) The standard basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$.

(b) The vectors $\mathbf{u} = [1, 1]$ and $\mathbf{v} = [1, -1]$.

Part II: Axiomatic Structure

The following problems should be solved using *only* the axioms provided in **??** and the subsequent theorems proved in the text. Do not use components.

6. **The Cancellation Law.** Prove that for any vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, if $\mathbf{a} + \mathbf{b} = \mathbf{a} + \mathbf{c}$, then $\mathbf{b} = \mathbf{c}$.

7. **Uniqueness of the Additive Identity.** Prove that the zero vector $\mathbf{0}$ is unique. That is, if there exists a vector $\mathbf{z}$ such that $\mathbf{a} + \mathbf{z} = \mathbf{a}$ for all $\mathbf{a}$, prove that $\mathbf{z} = \mathbf{0}$.

8. **Uniqueness of the Additive Inverse.** Prove that for a given vector $\mathbf{a}$, the vector $-\mathbf{a}$ is unique. That is, if $\mathbf{a} + \mathbf{b} = \mathbf{0}$ and $\mathbf{a} + \mathbf{c} = \mathbf{0}$, then $\mathbf{b} = \mathbf{c}$.

9. Prove that if $\mathbf{a} \neq \mathbf{0}$ and $r\mathbf{a} = s\mathbf{a}$, then $r = s$.

10. Prove that $-(\mathbf{a} + \mathbf{b}) = (-\mathbf{a}) + (-\mathbf{b})$.

Part III: Independence and Bases

11. **Uniqueness of Representation.** Let $\mathbf{a}$ and $\mathbf{b}$ be linearly independent vectors. Prove that if

$$r\mathbf{a} + s\mathbf{b} = p\mathbf{a} + q\mathbf{b},$$

then $r = p$ and $s = q$.

Remark. This result allows us to treat the coefficients (r, s) as "coordinates" relative to the basis $\{\mathbf{a}, \mathbf{b}\}$.

12. Let $\mathbf{a}$ and $\mathbf{b}$ be linearly independent vectors. Determine whether the following pairs of vectors are linearly dependent or independent.

(a) $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$.

(b) $\mathbf{a} - 2\mathbf{b}$ and $3\mathbf{a} - 6\mathbf{b}$.

(c) $\mathbf{a} + 2\mathbf{b}$ and $2\mathbf{a} + \mathbf{b}$.

13. The Determinant Map. Let $D(\mathbf{a}, \mathbf{b}) = a_1 b_2 - a_2 b_1$. Prove the following algebraic properties of the determinant:

(a) *Anti-commutativity:* $D(\mathbf{b}, \mathbf{a}) = -D(\mathbf{a}, \mathbf{b})$.

(b) *Linearity in the first argument:* $D(\mathbf{a} + \mathbf{c}, \mathbf{b}) = D(\mathbf{a}, \mathbf{b}) + D(\mathbf{c}, \mathbf{b})$ and $D(r\mathbf{a}, \mathbf{b}) = rD(\mathbf{a}, \mathbf{b})$.

(c) $D(\mathbf{a}, \mathbf{a}) = 0$.

14. Change of Basis. Suppose $\mathbf{u} = r\mathbf{a} + s\mathbf{b}$ and $\mathbf{v} = p\mathbf{a} + q\mathbf{b}$. Prove that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent if and only if $\mathbf{a}$ and $\mathbf{b}$ are linearly independent and $rq - sp \neq 0$.